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sur le thème

Quelques contributions au contrôle stochastique appliqué à la finance de marché, d'entreprise et à l'assurance

devant le Jury composé de :

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Table des matières

Introduction

Ce document présente une synthèse de mes travaux de recherche depuis l'obtention de ma thèse. Ces travaux sont des tentatives de réponse à des problématiques issues de trois champs d'application : la finance de marché, la finance d'entreprise et l'assurance. Deux aspects fondamentaux des mathématiques appliquées sont abordés : la formulation et la modélisation des problèmes étudiés, suivies du développement d'outils et d'analyse mathématiques pour donner ou améliorer des méthodes numériques de résolution.
Mes travaux se basent sur des modélisations stochastiques, les problèmes étudiés sont donc formulés comme des problèmes de contrôle stochastique. Leurs résolutions mêlent des approches analytiques et probabilistes. J'ai choisi de regrouper mes travaux en quatre chapitres selon leurs champs d'applications : le risque de liquidité en finance de marché, le risque de liquidité dans le cadre d'options réelles, la structure et gestion de capital en finance d'entreprise et la valorisation de produits d'assurance.

Modélisation et gestion de la liquidité dans un marché financier

La première partie de ce chapitre correspond à un article écrit avec Vathana Ly Vath, Mohamed Mnif (ENIT, Tunis) et leur étudiant en thèse M'hamed Gaigi (voir **(13)**). Nous avons cherché à comprendre l'évolution des prix et en particulier du bid-ask spread en nous plaçant du point de vue d'un animateur de marché, unique fournisseur de liquidité sur le marché. La résolution de ce problème passe par une caractérisation mathématique rigoureuse de la fonction valeur comme l'unique solution de viscosité d'un système d'équation d'Hamilton-Jacobi-Bellman (HJB). Beaucoup de difficultés techniques apparaissent dans ce problème, à commencer par une forme de discontinuité de l'opérateur hyperbolique apparaissant dans l'équation. Ces points sont rigoureusement résolus et une méthode numérique pour approcher les stratégies de tenue de marché optimales est présentée avant d'être mise en oeuvre pour obtenir des illustrations numériques de nos résultats.
Le deuxième partie traite d'un travail réalisé avec V. Ly Vath, Alexandre Roch (UQAM, Montréal) et Simone Scotti (Université Paris-Diderot) où l'on adopte cette fois-ci le point de vue d'un market maker opérant dans un marché avec carnet d'ordre. On s'intéresse à la stratégie de liquidation optimale d'un portefeuille. L'objectif est de minimiser le cout d'exécution de la stratégie (appelé slippage par les praticiens). Le problème est formulé comme un problème de contrôle régulier (le niveau de prix des ordres limites) mêlé à du contrôle impulsionnel (les ordres de marchés) sur un processus à sauts. En utilisant le caractère markovien du modèle, nous caractérisons la fonction valeur comme l'unique solution de viscosité d'un système d'inégalités variationnelles. Nous concluons notre étude par la calibration du modèle sur des données réelles.

Risque de liquidité et options réelles.

La première partie, issue d'un travail, réalisé avec V. Ly Vath, A. Roch et S. Scotti (voir **(9)**), traite un problème d'optimisation de stratégie de sortie d'un projet d'investissement (ou de vente d'un actif) qui n'est pas profitable. L'objectif est de trouver la date optimale de vente, l'investisseur ayant le choix entre attendre un acheteur ou liquider le projet immediatement en payant des coûts de liquidités. Dans notre étude, la valeur du projet et les coûts de liquidation sont modélisés par des processus de diffusion subissant des changements de régimes. Notre problème d'optimisation est donc formulé comme un problème d'arrêt optimal bi-dimensionnel de maturité aléatoire avec changements de régimes exogènes. Après avoir caractériser la fonction valeur comme unique solution de viscosité d'un système d'inéquations variationnelles, nous étudions les conditions de liquidation optimale et trouvons des solutions explicites dans les cas où la fonction d'utilité du gestionnaire est une fonction puissance ou logarithmique.

La seconde partie résume un travail commun avec M. Gaigi et V. Ly Vath (voir **(12)**). Nous combinons problème d'investissement optimal avec coûts de liquidité et distribution optimale de dividendes. Nous supposons que les actifs de l'entreprise se divisent en deux catégories : des actifs dits productifs dont la valeur unitaire évolue selon un mouvement brownien géométrique et la réserve de cash. Toujours disponible et infiniment divisible, elle est modélisée par un brownien arithmétique dont les caractéristiques (dérive et volatilité) dépendent de la quantité d'actifs productifs détenue par l'entreprise. L'achat et la vente des actifs productifs sont soumis à des coûts de liquidités fixes, proportionnels avec éventuellement un impact pour les grandes transactions. Nous obtenons donc un problème de contrôle multi-dimensionnel mêlant contrôle singulier (les dividendes) et impulsionnel (les transactions sur les actifs productifs). Nous montrons, via le principe de la programmation dynamique, que la fonction valeur de l'entreprise est l'unique solution de viscosité d'une équation d'Hamilton-Jacobi-Bellman multi-dimensionnelle et proposons une méthode numérique itérative pour évaluer la fonction valeur et ainsi obtenir une approximation des stratégies optimales d'achat, de vente et de distribution de dividendes que doit suivre l'entreprise.

Structure et gestion optimales de capital sous contraintes.

Dans la continuité du chapitre précédent, nous présentons dans le troisième chapitre des problèmes de finance d'entreprise centrés, non plus sur les risques liés à la liquidité, mais sur la structure et la gestion du capital de la firme.

La première partie présente un article écrit avec V. Ly Vath et S. Scotti (voir **(7)**). Nous nous intéressons à un problème combiné de distribution optimale de dividendes et de décisions optimales d'investissements financés par de l'endettement. La formulation mathématique nous amène à énoncer un problème combinant du contrôle singulier sur les dividendes et du contrôle impulsionnel pour les changement de régime d'endettement. Utilisant le caractère markovien de notre modèle, nous caractérisons notre fonction valeur comme unique solution de viscosité d'une équation d'HJB et en déduisons, dans des cas simples, les stratégies optimales associées.

La seconde partie résume un travail en collaboration avec Erhan Bayraktar (Université du Michigan) et V. Ly Vath (voir **(14)**). Nous cherchons à déterminer la structure optimale du capital d'une banque résultant de stratégies optimales d'investissement et de distribution de dividendes sous contraintes de solvabilté et de liquidité. Nous supposons que le capital de la banque pouvant être investi, est constitué des dépôts des clients et des fonds propres apportés par les actionnaires. Le gérant de la banque a la possibilité d'investir soit dans une classe d'actifs risqués soit dans un actif sans risque. Il doit cependant respecter certains critères définis dans les accords de Bâle. Son objectif est donc de maximiser, sous contraintes de solvabilité, la valeur cumulée des dividendes distribués aux actionnaires durant l'existence de la banque. Nous modélisons également la possibilité pour la banque de se recapitaliser. La formulation mathématique de ce problème mélange des problèmes de contrôle stochastique régulier (gestion des actifs de la banque), impulsionnel (recapitalisation) et singulier (distribution de dividendes). En réduisant la dimension de ce problème bi-dimensionnel, nous obtenons des solutions quasi-explicites via une charactérisation de la fonction objectif comme unique solution de viscosité d'une équation d'HJB. Nous en déduisons les stratégies optimales associées.

Valorisation et couverture de produits d'assurance-vie.

Ce dernier chapitre présente mes travaux sur des produits d'assurance-vie appelés contrats à annuités variables. Le détenteur de la police ou du produit, confie un capital initial à la compagnie d'assurance qui se charge de l'investir dans un portefeuille ou fond de référence. En retour, la compagnie d'assurance distribuera, à partir d'une date déterminée, des rentes (annuités) ou un capital dont le montant dépendra de la performance réalisée par le portefeuille de référence et de clauses de garanties, parfois très complexes.

La première partie de ce chapitre est dédiée à la construction d'un modèle général d'étude de ces produits. En raison des risques venant du détenteur du contrat, le marché est incomplet et nous devons choisir une méthode de valorisation adéquate. Nous avons adopté une méthode de valorisation par indifférence d'utilité exponentielle. Enfin, comme ces produits sont de longue maturité, nous avons choisi de ne pas imposer d'hypothèse de markovianité à notre modèle. Par conséquent, nous mettrons en place des méthodes numériques basées sur la discretisation d'équations différentielles stochastiques rétrogrades (EDSR).

Dans la seconde partie, basée sur un travail commun avec Thomas Lim et notre étudiant de thèse Ricardo Romo (voir **(11)**), nous supposons que l'assuré n'a pas nécessairement un comportement rationnel et modélisons ses retraits par un processus aléatoire arbitraire dont la dynamique pourrait être précisée par calibration sur des données réelles. En utilisant des outils de grossissement de filtration et des résultats récents sur les EDSR, nous mettons en place une méthode numérique pour évaluer le prix d'indifférence du produit.

La dernière partie résume un article co-écrit avec T. Lim, Idriss Karroubi et Christophette Blanchet (voir **(8)**), nous adoptons une approche plus robuste en considérant le pire des cas pour l'assureur : l'assuré suit la stratégie de retraits anticipés minimisant l'esperance de l'utilité de l'assureur. Cela nous amène à étudier un problème d'optimisation stochastique de type max-min.

Publications

(1) Critical price near maturity for an American option on a dividend-paying stock in a local volatility model, *Mathematical Finance*, **15**, No 3, 2005.

(2) Optimal early retirement near the expiration of a pension plan, *Finance and Stochastics*, **10**, No 2, 2006.

(3) Bermudean Approximation of the Free Boundary Associated with an American Option, Free Boundary Problems : Theory and Applications, 137-147, 2007.

(4) On the American option value function near its maturity, Progress in Industrial Mathematics at ECMI 2006, 650-655, 2007.

(5) American options, Encyclopedia of Quantitative Finance, editor Rama Cont, 2009.

(6) Exercise boundary near maturity for an American option on several assets, *Journal of Stochastic Analysis and Applications*, Volume 28, Issue 4, 623-647, 2010.

(7) An Optimal Dividend and Investment Control Problem under Debt Constraints, with V. Ly Vath and S. Scotti, *SIAM Journal on Financial Mathematics*, Vol. 4, No. 1, 297-326, 2013.

(8) Max-min optimization problem for variable annuities pricing, with C.Blanchet-Scalliet, I. Kharroubi and T. Lim, *International Journal of Theoritical and Applied Finance*, Vol. 18, No. 08, 2015

(9) Optimal exit strategies for investment projects, with V. Ly Vath, A. Roch and S. Scotti, *Journal of Mathematical Analysis and Applications*, Vol. 425, No. 2, 666-694, 2015.

(10) Optimal Execution Cost for Liquidation Through a Limit Order Market, with V. Ly Vath, A. Roch and S. Scotti, *International Journal of Theoritical and Applied Finance*, Vol. 19, No. 1, 2016

(11) Indifference fees for variable annuities, with T. Lim, et R. Romo Romero, to appear in *Applied Mathematical Finance*

(12) Liquidity risk and optimal investment/disinvestment strategies, with M.Gaigi and V. Ly Vath, to appear in *Mathematics and Financial Economics*.

Travaux soumis ou en cours

(13) Optimal dealing strategies under inventory constraints, with M.Gaigi, V. Ly Vath and M. Mnif.

(14) A capital structure optimization problem under constraints, with E. Bayraktar and V. Ly Vath.

(15) Optimal execution in one-sided order book with stochastic resilience, with V. Ly Vath, S. Pulido and F. Rasamoely.

(16) Optimal stopping for path-dependent problems, with V. Ly Vath and M. Mnif

(17) Optimal dividend and capital injection policy with external audit, with V. Ly Vath and A. Roch.

Chapitre 1

Modelization and management of market liquidity risks

The content of this chapter is based on :

(10) Optimal Execution Cost for Liquidation Through a Limit Order Market, with V. Ly Vath, A. Roch and S. Scotti, *International Journal of Theoritical and Applied Finance*, Vol. 19, No. 1, 2016

(13) Optimal dealing strategies under inventory constraints, with M.Gaigi, V. Ly Vath and M. Mnif, preprint.

The study of market liquidity consists in quantifying the costs incurred by investors trading in markets in which supply or demand is finite, trading counterparties are not continuously available, or trading causes price impacts. Liquidity is a risk when the extent to which these properties are satisfied varies randomly through time.

Liquidity and liquidity risk models vary considerably from one study to the next according to the problem at hand or the paradigm considered. For instance, Back [9] and Kyle [76] construct an equilibrium model for dealers markets with insider trading. Constantinides [36], Davis and Norman [41], and Shreve and Soner [104] study the portfolio selection problem with first order liquidity costs, namely proportional transaction costs arising from a bid-ask spread (see [51] for a model with jumps). There has also been a number of studies on large trader models ([10], [84], [100]), and dynamic supply curves ([3], [33]), with a more recent emphasis on liquidation problems with market orders ([2], [4], [94], [97]). More recently, some studies model the structural events in the order book, like market order arrivals, cancellations or execution of limit orders (see [1] et [37]).

Classical models on financial markets assume that investors are price-takers, i.e. liquidity takers, in the sense that they trade any financial assets at the available prices including a liquidity premium that must be paid for immediacy. The market liquidity crunch of the financial crisis in 2008 created an important need of better understanding, quantifying and managing the liquidity risk.

It is clear from the structure of financial markets that, in addition to the presence of price-takers, there must necessarily exist market participants who are price-setters or liquidity providers. In limit order book markets or order-driven markets such as the NYSE (New York Stock Exchange), traders can post prices and quantities at which they are willing to buy or sell while waiting for a counterparty to engage in that trade. In dealers' markets or quote-driven markets, for instance the Nasdaq or LSE (London Stock Exchange), registered market makers quote bids/offers and serve as counterparties when an investor wishes to buy or sell the securities.

In the first section, we focus on a single market dealer acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset in a quote-driven market. The market dealer may benefit from the bid-ask spread but has the obligation to permanently quote both prices while satisfying some liquidity and inventory constraints. The objective is to maximize the expected utility from terminal liquidation value over a finite horizon and subject to the above constraints. We characterize the value function as the unique viscosity solution to the associated HJB equation and further enrich our study with numerical results. The contributions of our study, as compared to previous studies [7, 64, 88] concern both the modelling aspects and the dynamic structure of the control strategies.

The last section is devoted to a more classical problem, we study how optimally liquidate a large portfolio position in a limit order book market. We allow for both limit and market orders and the optimal solution is a combination of both types of orders. Market orders deplete the order book, making future trades more expensive, whereas limit orders can be entered at more favorable prices but are not guaranteed to be filled. We model the bid-ask spread with resilience by a jump process, and the market order arrival process as a controlled Poisson process. The objective is to minimize the execution cost of the strategy. We formulate the problem as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. We conclude with a calibration of the model on recent market data and a numerical implementation.

1.1 Optimal market dealing strategies under inventory constraints

In this section, we consider an equity quote-driven market with a single risky equity assets. While most liquid equity markets have now migrated to order-driven markets or hybrid markets, liquidity issues mainly arise for illiquid assets, which coincide in general with small and mid-cap stocks. We may refer, for instance, to the Stock Exchange Automated Quotation system (or SEAQ) which is a system for trading small-cap stocks in the LSE. The SEAQ market is a pure quote-driven market and is the type of markets which we are investigating. As such, in terms of practical applications, it is mainly about finding optimal strategies for market makers dealing in small and mid-cap market such as the SEAQ market.

In order to obtain the role of a market maker of an assigned security, a firm has to sign an agreement with the stock exchange which contains many obligations that the firm has to satisfy. The market maker has, in particular, a contractual obligation to permanently quote bid and ask prices for the security and therefore to satisfy any sell and buy market order from investors.

In the trading of equity assets, there are several registered market makers in competition. In the SEAQ market, for instance, there are in general two or more registered market makers. However, in our study, in order to better focus on the understanding of the market making mechanism and of the trade-off between the gains that could be obtained from the bid-ask spread and the potential loss due to inventory, we consider there is only one "representative" registered market maker in the dealing of the assets.

This assumption is inline with the literature on market making/dealing problems, see for instance [7, 64, 88, 58].

In [7, 64], the authors consider a market making problem as described above but within a financial market in which the risky asset has reference or fair price, assumed to follow an arithmetic Brownian process. The market maker quotes her ask and bid prices at some distances to the reference price. In our study, we do not assume the existence of a reference price. The prices are therefore driven by the equilibrium between buy and sell market orders.

The assumption made in [7, 64, 88] stating that the market maker may liquidate her stock inventory at the reference price (or a constant price independent of the inventory) means that the inventory risk is uniquely due to the market risk of the reference price. However, from studies on liquidation costs and price impacts, see for instance [22, 62, 76, 67], it is clear that the degree of ability to liquidate the stock inventory at the reference price or the mid-price should not be neglected. Inline with this literature, we assume that when the market maker has to liquidate her stock inventory, she incurs a liquidity cost and the price per share received (paid) are lower (higher) than the mid-price in the case of a long (short) position.

To further take into account the microstructure of the financial markets, we no longer consider continuous price processes. Bid and ask prices quoted by the market maker are realistically assumed to be discrete prices, and correspond to multiple of a tick value.

The objective of our market maker is to maximize the expected utility of the terminal wealth. Considering that the market maker should avoid, as much as possible, violating the inventory risk constraint imposed by her firm, we introduce, in the objective function, a penalty cost self-imposed by the market maker herself or her firm, in order to reduce the inventory risk. It is worth noticing that this penalty cost, together with some other features such as the presence of the liquidation costs, largely prevent the market maker from being able to manipulate the stock price.

The contributions of our study, as compared to previous studies [7, 64, 88], concern both the modelling aspects and the dynamic structure of the control strategies. Important features and constraints characterizing market making problems are no longer ignored, turning therefore our market making problem into a non-standard control problem under constraints with real modelling and mathematical challenges. We provide rigorous mathematical characterization and analysis to our control problem by proving that our value functions are the unique viscosity solutions to the associated HJB system. It is always a technical challenge when applying viscosity techniques to non-standard control problems under constraints. In the proof of our comparison theorem, a major problem is to circumvent the difficulty arising from the discontinuity of our HJB operator on some parts of the solvency region boundary.

1.1.1 Dealer market model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where T is a finite horizon. We assume that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$. We consider a financial market, in which there is a risky assets, operated as a single market dealer. She has the obligation to permanently quote bid and ask prices and to act as a counterparty to investors' market orders.

Trading orders. We assume that investors, considered as price-takers, may only submit either buy or sell market orders. We denote by $(\theta_i^a)_{i\geq 1}$ (resp. $(\theta_i^b)_{i\geq 1}$) the sequence of non-decreasing $\mathbb{F}$-stopping times corresponding to the arrivals of buy (resp. sell) market orders. We denote by $(\xi_i)_{i\geq 1}$ the sequence of these trading times.

Market making strategies. We define a strategy control as being a $\mathbb{F}$-predictable process $\alpha = (\alpha_t)_{(0\leq t\leq T)} = (\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b)_{0\leq t\leq T}$ where the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ take values in $\{\chi_{min}, .., \chi_{max}\}$, with $-\chi_{min} \in \mathbb{N}$ and $\chi_{max} \in \mathbb{N}^*$.

We assume that when a sell market order arrives at time θ_j^b, the market maker may either keep the bid and ask prices constant or decrease one or both of them by at most χ_{max} ticks or increase one or both of them by at most χ_{min} ticks. Notice the market maker may decide to change the bid/ask prices but transaction prices are assumed to be based on the one quoted before the prices changes. In here, a tick value is denoted by a strictly positive constant δ. On the opposite side, when a buy market order arrives at time θ_k^a, the market maker may either keep the bid and ask prices constant or increase one or both of them by at most χ_{max} ticks or decrease one or both of them by at most χ_{min} ticks .

Bid-Ask spread modelling.

We denote by $P^a = (P_t^a)_{0\leq t\leq T}$ (resp. $P^b = (P_t^b)_{0\leq t\leq T}$) the price quoted by the market maker to buyers (resp. sellers). When a buy (resp. sell) market order arrives at time θ_i^a (resp. θ_j^b), the market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by P^a (resp. P^b). As in [64, 58], we assume here that transactions are of constant size, scaled to 1.
The dynamics of P^w, where $w \in \{a,\ b\}$, evolves according to the following equations

$$\begin{cases} dP_t^w &= 0,\ \xi_i \leq t < \xi_{i+1} \\ P^w_{\theta^b_{j+1}} &= P^w_{\theta^{b-}_{j+1}} - \delta\epsilon^w_{\theta^b_{j+1}} \\ P^w_{\theta^a_{k+1}} &= P^w_{\theta^{a-}_{k+1}} + \delta\eta^w_{\theta^a_{k+1}} \end{cases}$$

where i is the number of transactions before time t, j the number of buy transactions before time t for the market maker, k the number of sell transactions before time t, and δ represents one tick.
We denote by P the mid-price and S the bid-ask spread of the stocks. The dynamics of the process (P, S) is given by

$$\begin{cases} dP_t &= 0,\ \xi_i \leq t < \xi_{i+1} \\ P_{\theta^b_{j+1}} &= P_{\theta^{b-}_{j+1}} - \frac{\delta}{2}(\epsilon^a_{\theta^b_{j+1}} + \epsilon^b_{\theta^b_{j+1}}) \\ P_{\theta^a_{k+1}} &= P_{\theta^{a-}_{k+1}} + \frac{\delta}{2}(\eta^a_{\theta^a_{k+1}} + \eta^b_{\theta^a_{k+1}}), \end{cases} \quad \text{and} \quad \begin{cases} dS_t &= 0,\ \xi_i \leq t < \xi_{i+1} \\ S_{\theta^b_{j+1}} &= S_{\theta^{b-}_{j+1}} - \delta(\epsilon^a_{\theta^b_{j+1}} - \epsilon^b_{\theta^b_{j+1}}) \\ S_{\theta^a_{k+1}} &= S_{\theta^{a-}_{k+1}} + \delta(\eta^a_{\theta^a_{k+1}} - \eta^b_{\theta^a_{k+1}}). \end{cases}$$

Regime switching. We first consider the tick time clock associated to a Poisson process $(R_t)_{0\leq t\leq T}$ with deterministic intensity λ defined on $[0, T]$, and representing the random times where the intensity of the orders arrival jumps.

We define a discrete-time stationary Markov chain $(\hat{I}_k)_{k\in\mathbb{N}}$, valued in the finite state space $\{1, ..., m\}$, with probability transition matrix $(p_{ij})_{1\leq i,j\leq m}$, i.e. $\mathbb{P}[\hat{I}_{k+1} = j|\hat{I}_k = i] = p_{ij}$ s.t. $p_{ii} = 0$, independent of R. We define the process

$$I_t = \hat{I}_{R_t},\ t \geq 0 \tag{1.1.1}$$

$(I_t)_t$ is a continuous time Markov chain with intensity matrix $\Gamma = (\gamma_{ij})_{1\le i,j\le m}$, where $\gamma_{ij} = \lambda p_{ij}$ for $i \neq j$, and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$.

We model the arrivals of buy and sell market orders by two Cox processes N^a and N^b. The intensity rate of N_t^a and N_t^b is given respectively by $\lambda^a(t, I_t, P_t, S_t)$ and $\lambda^b(t, I_t, P_t, S_t)$ where λ^a and λ^b are continuous functions valued in $\mathbb{R}$ and defined on $[0, T] \times \{1, ..., m\} \times \frac{\delta}{2}\mathbb{N} \times \delta\mathbb{N}$.

Remark 1 *The dependency of intensity rate on the assets prices is inspired by [7] and is used by many previous papers, see for instance [58, 15, 59].*

We now define θ_k^a (resp. θ_k^b) as the k^{th} jump time of N^a (resp. N^b), which corresponds to the k^{th} buy (sell) market order.

We introduce the following stopping times $\rho_j(t) = \inf\{u \ge t, I_u = j\}$ and $\rho(t) = \inf\{u \ge t, R_u > R_t\}$ for $0 \le t \le T$.

Stock holdings. The number of shares held by the market maker at time $t \in [0, T]$ is denoted by Y_t, and Y satisfies the following equations

$$\begin{cases} dY_t &= 0, \ \xi_i \le t < \xi_{i+1} \\ Y_{\theta^b_{j+1}} &= Y_{\theta^{b-}_{j+1}} + 1 \\ Y_{\theta^a_{k+1}} &= Y_{\theta^{a-}_{k+1}} - 1, \end{cases}$$

As in [88, 58], we consider that the market maker has the obligation to respect the risk constraint imposed upon her by her company. Concretely, the stock inventory of the market maker is assumed to have upper and lower bounds which could be high enough to allow some trading flexibility to the market maker. Let $y_{min} < 0 < y_{max}$. We are therefore imposing the following inventory constraint

$$y_{min} \le Y_t \le y_{max} \text{ a.s. } 0 \le t \le T.$$

Cash holdings. We denote by $r > 0$ the instantaneous interest rate. The bank account follows the below equation between two trading times

$$dX_t = rX_t dt, \ \xi_i \le t < \xi_{i+1}.$$

When a discrete trading occurs at time θ^b_{j+1} (resp. θ^a_{k+1}), the cash amount becomes

$$X_{\theta^b_{j+1}} = X_{\theta^{b-}_{j+1}} - P^b_{\theta^{b-}_{j+1}} \quad \text{and} \quad X_{\theta^a_{k+1}} = X_{\theta^{a-}_{k+1}} + P^a_{\theta^{a-}_{k+1}}.$$

State process. We define the state process as follows :

$$Z = (X, \ Y, \ P := \frac{P^a + P^b}{2}, \ S := P^a - P^b).$$

Cost of liquidation of the portfolio. If the current mid-price at time $t < T$ is p and the market maker decides to liquidate her portfolio, then we assume that the price she actually gets is

$$Q(t, y, p, s) = (p - \text{sign}(y)\frac{s}{2})f(t, y), \tag{1.1.2}$$

where f is an impact function defined from $[0,T] \times \mathbb{R}$ into $\mathbb{R}_+$ satisfying

Assumption (H1) *The impact function f is non-negative, non-increasing in y, and satisfies the following conditions*

$$\begin{aligned} f(t,y) &\leq f(t,y^{'}) \text{ if } y^{'} \leq y \\ yf(t^{'},y) &\leq yf(t,y) \text{ if } t^{'} \leq t. \end{aligned}$$

Liquidation value and Solvency constraints. A key issue for the market maker is to maximize the value of the net wealth at time T. In our framework, we impose a constraint on the spread i.e.

$$0 < S_t \leq K\delta, \; 0 \leq t \leq T,$$

where K is a positive constant. We also impose that the bid price remains positive, therefore the market maker has to use controls such that

$$P_t - S_t/2 > 0.$$

When the market maker has to liquidate her portfolio at time t, her wealth will be $L(t, X_t, Y_t, P_t, S_t)$ where L is the liquidation function defined as follows

$$L(t,x,y,p,s) = x + yQ(t,y,p,s),$$

with Q as defined in (1.1.2).

Furthermore, we assume that in the case that the cash held by the market maker falls below a negative constant x_{min}, she has to liquidate her position. We may now introduce the following state space

$$\mathbb{S} =]x_{min}, +\infty[\times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, ..., K\}$$

and then the solvency region

$$\mathcal{S} = \{(t,x,y,p,s) \in [0,T] \times \mathbb{S} : \, p - \frac{s}{2} \geq \delta\}.$$

We denote its closure by $cl(\mathcal{S}) = \mathcal{S} \cup \partial_x \mathcal{S}$ where its boundary is defined by

$$\partial_x \mathcal{S} = \Big\{(t,x,y,p,s) \in [0,T] \times cl(\mathbb{S}) : \, x = x_{min} \text{ and } p - \frac{s}{2} \geq \delta\Big\}.$$

Admissible trading strategies. Given $(t,z) := (t,x,y,p,s) \in \mathcal{S}$, we say that the strategy $\alpha = (\epsilon^a_u, \epsilon^b_u, \eta^a_u, \eta^b_u)_{t \leq u \leq T}$ is admissible, if the $\mathbb{F}$-predictable processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{\chi_{min}, ..., \chi_{max}\}$ and for all $u \in [t,T]$, $(u, Z^{t,i,z,\alpha}_u) \in \mathcal{S}$.

Value functions.

We set g a non-negative penalty function defined on $\{y_{min}, ..., y_{\max}\}$. This penalty may be compared to the holding costs function introduced in [88].

We also consider an exponential utility function U i.e. there exists $\gamma > 0$ such that $U(x) = 1 - e^{-\gamma x}$ for $x \in \mathbb{R}$. We set $U_L = U o L$.

As such, we consider the following value functions $(v_i)_{i\in\{1,...,m\}}$ which are defined on $\mathcal{S}$ by

$$v_i(t,z) := \sup_{\alpha\in\mathcal{A}(t,z)} J_i^\alpha(t,z) \tag{1.1.3}$$

where we have set

$$\begin{aligned} J_i^\alpha(t,z) &:= \mathbb{E}^{t,i,z}\left[U_L(T\wedge\tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}) - \int_t^{T\wedge\tau^{t,i,z,\alpha}} g(Y_s^{t,i,y,\alpha})ds\right], \\ \tau^{t,i,z,\alpha} &:= \inf\{u\geq t : X_u^{t,i,x,\alpha}\leq x_{min} \text{ or } Y_u^{t,i,y,\alpha}\in\{y_{min}-1, y_{max}+1\}\}. \end{aligned}$$

1.1.2 Analytical properties and viscosity characterization

We use a dynamic programming approach to derive the system of partial differential equations satisfied by the value functions. First, we state the following Proposition in which we obtain some bounds of our value functions

Proposition 1 *There exist non-negative constants, C_1, C_2 and C_3, depending on the parameters of our problem, such that*

$$1 - C_1 - C_2e^{C_3p} \leq v_i(t,x,y,p,s) \leq 1, \quad \forall (i,t,x,y,p,s)\in\{1,...,m\}\times\mathcal{S}.$$

For control problems, dynamic programming principle was frequently used by many authors and strongly relies on regularity properties of the objective function. We carefully establish the following result.

Proposition 2 *Let $(i,y,p,s)\in\{1,...,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}^*\times\delta\{1,..K\}$ such that $p-\frac{s}{2}>0$. The function $(t,x)\to v_i(t,x,y,p,s)$ is locally uniformly continuous on $[0,T]\times[x_{min},+\infty[$.*

To characterize the objective function as a viscosity solution of an HJB equation, we first define the following set :

$$\begin{aligned} A(p,s) := \Big\{\alpha = (\varepsilon^a,\varepsilon^b,\eta^a,\eta^b)\in\{-\chi_{min},...,\chi_{max}\}^4 \text{ s.t. } p-\frac{s}{2}-\delta\varepsilon^b\geq\delta, \\ \delta\leq s-\delta(\varepsilon^a-\varepsilon^b)\leq K\delta, \text{ and } \delta\leq s+\delta(\eta^a-\eta^b)\leq K\delta\Big\}. \end{aligned}$$

For all $(i,t,x,y,p,s) := (i,t,z)\in\{1,...,m\}\times\mathcal{S}$ and $\alpha := (\varepsilon^a,\varepsilon^b,\eta^a,\eta^b)$ belonging to $A(p,s)$, we introduce the two following operators :

$$\mathcal{A}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{min},p,s) \text{ if } y = y_{min} \\ v_i(t,x+p+\frac{s}{2},y-1,p+\frac{\delta}{2}(\eta^a+\eta^b),s+\delta(\eta^a-\eta^b)) \text{ otherwise.} \end{cases}$$

$$\mathcal{B}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{max},p,s) \text{ if } y = y_{max} \\ U_L(t,z) \text{ if } x < x_{min}+p-\frac{s}{2} \\ U_L(t,z) \text{ if } x = x_{min}+p-\frac{s}{2} < 0 \\ v_i(t,x-p+\frac{s}{2},y+1,p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b),s-\delta(\varepsilon^a-\varepsilon^b)) \text{ otherwise.} \end{cases}$$

Notice here that a discontinuity appears in the operator $\mathcal{B}$. Indeed, for any function ψ defined on $cl(\mathcal{S})$, any $(t,x,y,p,s) := (t,z) \in [0,T] \times \mathcal{S}$ such that $y < y_{max}$, and any $\alpha \in \mathcal{A}(t,z) \setminus \{0\}$, we have

$$\begin{aligned}\lim_{x \downarrow x_{min}+p-\frac{s}{2}} \mathcal{B}\psi(t,z,\alpha) &= U_L(t, x_{min}, y+1, p - \frac{\delta}{2}(\varepsilon^a + \varepsilon^b), s - \delta(\varepsilon^a - \varepsilon^b)) \\ &\neq U_L(t,z) = \lim_{x \uparrow x_{min}+p-\frac{s}{2}} \mathcal{B}\psi(t,z,\alpha).\end{aligned}$$

We want to show that $(v_i)_{1 \le i \le m}$ is a viscosity solution, on the open set $\{1,...,m\} \times \mathcal{S}$, of the following HJB equation :

$$-\frac{\partial v_i}{\partial t} - \mathcal{H}_i(t,z,v_i,\frac{\partial v_i}{\partial x}) = 0, \;\; (t,z) \in \mathcal{S}, \tag{1.1.4}$$

where $\mathcal{H}_i$ is the Hamiltonian associated with state i and such that for a family of smooth functions ψ :

$$\begin{aligned}\mathcal{H}_i(t,z,\psi_i,\frac{\partial \psi_i}{\partial x}) &= rx\frac{\partial \psi_i}{\partial x} + \sum_{j \neq i} \gamma_{ij}\left(\psi_j(t,x,y,p,s) - \psi_i(t,x,y,p,s)\right) - g(y) \\ &+ \sup_{\alpha \in A(p,s)} \left[\lambda_i^a(t,p,s)\left(\mathcal{A}\psi_i(t,x,y,p,s,\alpha) - \psi_i(t,x,y,p,s)\right)\right. \\ &+ \left.\lambda_i^b(t,p,s)\left(\mathcal{B}\psi_i(t,x,y,p,s,\alpha) - \psi_i(t,x,y,p,s)\right)\right] = 0.\end{aligned}$$

We now provide a rigorous characterization for the value function by means of viscosity solutions to the HJB equation (1.1.4) together with the appropriate boundary terminal conditions and dynamic programming principle. The uniqueness property is particularly crucial to numerically solve the associated HJB. The following theorem relates the value function v_i to the HJB (1.1.4) for all $1 \le i \le m$.

Theorem *The family of value functions $(v_i)_{1 \le i \le m}$ is the unique family of functions such that*

i) Growth condition : There exist C_1, C_2 and C_3 positive constants such that

$$1 - C_1 - C_2 e^{C_3 p} \le v_i(t,x,y,p,s) \le 1, \qquad on \quad \{1,..,m\} \times \mathcal{S}.$$

ii) Boundary and terminal conditions :

$$\begin{aligned} v_i(t,x_{min},y,p,s) &= U_L(t,x_{min},y,p,s) \\ v_i(T,x,y,p,s) &= U_L(T,x,y,p,s).\end{aligned}$$

iii) Viscosity solution : $(v_i)_{1 \le i \le m}$ is a viscosity solution of the system of variational inequalities (1.1.4) on $\{1,...,m\} \times \mathcal{S}$.

Assertions $i)$ and $ii)$ follow from Proposition 1 and the the definition of the value function. Proving that $(v_i)_{1 \le i \le m}$ is viscosity solution of variational inequalities (1.1.4) on $\{1,...,m\} \times \mathcal{S}$ has become quite standard, however, in our case, one need to carefully deal with the discontinuity in the operator $\mathcal{B}$ which leads to some technical difficulties, especially for the uniqueness result which is a direct consequence of a comparison Lemma.

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1.1.3 Numerical Results

To solve the HJB equation (1.1.4) arising from the stochastic control problem (1.1.3), one can use either probabilistic or deterministic numerical method. Below, we present some numerical results obtained by using a deterministic method based on a finite difference scheme. The scheme has the monotonicity, consistency and stability properties. It is well known that these properties, added to a comparison result, ensure the convergence of this scheme.

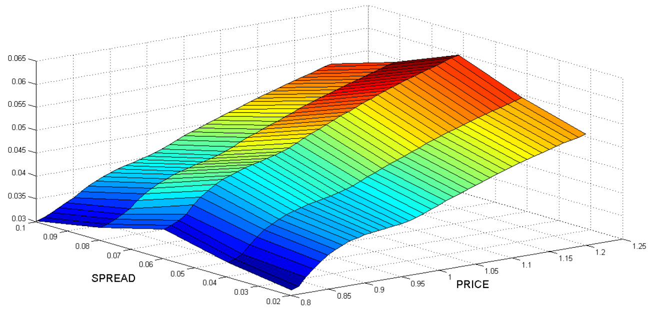


FIGURE 1.1 – *Value function for* $y \geq 0$

Optimal market making strategies

Figure 1.2 describes the optimal control strategies for the market maker when a sell market order arrives and when the market maker's inventory is around zero.

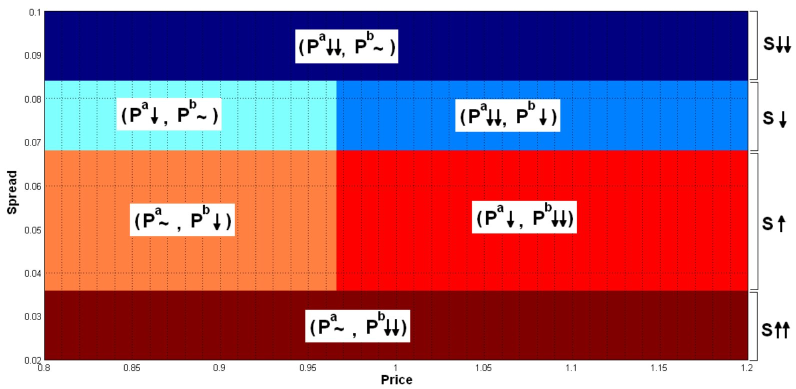


FIGURE 1.2 – *Optimal strategy when a sell market order arrives*

From Figure 1.2, we may make the following observations :

— when the spread is very low, the market maker has to decrease the bid price more than the ask price, see region where the spread value is below 0.07.

— when the spread is high and close to the maximum spread allowed, the market maker should decrease the ask price. She should decrease the spread in order to encourage trades.

Notice that the market maker may make a profit of 3 ticks in the favorable case, i.e., the next market order is a buy order.

1.2 Optimal execution cost for liquidation in a limit order book market

In limit order book markets, any public trader can play the role of liquidity provider by posting prices and quantities at which he is willing to buy or sell while waiting for a counterparty to engage in that trade. Limit orders can be entered at more favorable prices but are not guaranteed to be filled. A market order is filled automatically against existing limit orders, albeit at a less favorable price as it depletes the order book, making additional trades more expensive. It is therefore desirable to consider financial models with an enlarged set of admissible trading strategies by including the possibility of making both limit orders and market orders. In this section, we consider the liquidation problem of a large portfolio position from this perspective.

Many authors have investigated the liquidation and market making problems with limit orders only, in particular [7], [15], [56], [57], [58] and [93]. In these models, the arrival intensity of outside market orders that match the limit orders that are posted is typically a function of the spread between the posted price and a reference price. In a more complex model, Cartea et al. [32] develop a high-frequency limit order trading strategy in a limit order market characterized by feedback effects in market orders and the shape of the order book, and by adverse selection risk due to the presence of informed traders who make influential trades. Kühn and Muhle-Karbe [74] provide an asymptotics analysis for a small investor who sets bid and ask prices and seeks to maximize expected utility when the spread is small.

Recently, some authors consider a limit order market in which both limit and market orders are possible. Guilbaud and Pham [59] determine the optimal trading strategy of a market maker who makes both types of trades and seeks to maximize the expected utility over a short term horizon. Cartea and Jaimungal [30] determine the optimal liquidation schedule in a limit order market in which the liquidity cost of a market order is fixed, and the probability of passing a limit order depends on the spread between the posted price and a reference price, modeled as a Brownian motion plus drift. The investor's value function includes a quadratic penalty defined in terms of a target inventory schedule. We also consider a limit order market in which both limit and market orders are allowed, and study the problem of optimally liquidating a large portfolio position. Our contribution to the above literature is to consider spread dynamics which are impacted by both limit and market order strategies. Market orders that the investor places directly increase the observed bid-ask spread. As a result, past market orders have a direct impact on future liquidity costs. Furthermore, limit orders posted inside the bid-ask spread effectively decrease the observed spread and have an impact on the future probability distributions of its jumps.

We model the bid-ask spread with resilience (mean reversion) and a jump process, and the market order arrival process as a controlled Poisson process (see Section 1.2.1 for a description of the model). The objective is to liquidate a fixed number of shares of a risky asset by minimizing the expected liquidity premium paid. We formulate the problem in Section 1.2.2 as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. In Section 1.2.3, we numerically implement the model and calibrate it to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook*.

1.2.1 The limit order book market model

Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space supporting a random Poisson measure M on $[0,T] \times \mathbb{R}$ with mean measure $\gamma_t \, dt \, m(dz)$ where $\gamma : [0,T] \to (0, \bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$, with $m(\mathbb{R}) < \infty$. We consider a market with a risky asset that can be traded through a limit order book. We consider a large investor whose goal is to liquidate a number $N > 0$ of shares of this risky asset. The investor sets a date T before which the position must be liquidated and attempts to minimize the price impact of the liquidation strategy.

Market orders. The investor can make market orders by controlling the time and the size of his trades. This is modeled by an impulse control strategy $\beta = (\tau_i, \ \xi_i)_{i \le n}$ where the τ_i's are stopping times representing the intervention times of the investor and the ξ_i's are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathbb{N}$ and giving the number of shares sold by a market order at time τ_i.

Limit orders. The investor can also make limit orders. We denote by $\mathcal{A}_0$ a compact subset of $[0, \infty)$ representing the set of possible spreads *below the current best ask price* at which the investor can place a limit order to sell in the order book. We also add the admissibility condition that the limit price is above the current best bid price, otherwise the limit order would in fact be a market order. Since the effect of this new limit order is that the best ask price can now be lower, we call the best ask price *excluding* the investor's limit order the *otherwise best ask price.* The spread *below the current best ask price* is an $\mathcal{A}_0$-valued stochastic control denoted by $\alpha = (\alpha_t)_{t \le T}$.

Investor's control. We define the investor's control as a pair of processes $\delta = (\alpha, \beta)$.

Bid-Ask spread. We denote by X_t the spread between the best bid and the best ask price *excluding* the investor's limit price at time t. Between the investor's market orders, we assume the spread X is impacted by α and follows

$$dX_t^\alpha = \mu(t, X_{t-}^\alpha, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\alpha, \alpha_t, z)\tilde{M}(dt, dz). \tag{1.2.1}$$

Under this construction, the limit order α sends a signal and modifies the distribution of the jumps of X, represented by G. Here $\tilde{M}$ is the compensated random measure of M, and μ is a deterministic and Lipschitz continuous function in the second argument, satisfying a linear growth condition.

Liquidity cost. We summarize the information contained in the order book by a function $S(t, x, n)$ which gives the proceeds obtained for a sale of n shares at time t done through market orders when the spread equals x. In the order book density case, this corresponds to Equation 12 in [2]. Let A_t be a stochastic process representing the best ask price. We may then define the liquidity cost due to a market sell order of size n, denoted by $L(t, x, n)$, in terms of the best ask price as follows

$$L(t, x, n) := nA_t - S(t, x, n). \tag{1.2.2}$$

The slippage of a market order of size n is then defined as a fixed transaction cost,

$k > 0$, plus the liquidity cost, i.e.

$$K(t, x, n) = k + L(t, x, n). \tag{1.2.3}$$

Example. The simplest example is a quadratic model with proportional transaction costs :

$$S(t, x, n) = (A_t - x)n - \zeta_t n^2,$$

with A_t and ζ_t two stochastic processes representing the best ask price and a measure of illiquidity. This model arises from a limit order book with constant density as shown in [101]. In the quadratic model, $L(t, x, n) = xn + \zeta_t n^2$.

We introduce the set of functions from $[0, T] \times \mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, uniformly in the first, and denote it by $\mathcal{P}$. For technical reasons in the proof of a comparison principle, we assume that for all $n \in \{0, .., N\}$, the function $L(\cdot, \cdot, n)$ belongs to $\mathcal{P}$.

Impact on the best bid. During a transaction, the investor's market orders are matched with the existing limit orders in the order book so that the result is a shift in the best bid price to the left by an amount denoted by $I(t, x, n)$.

Dynamics of the controlled bid-ask spread. The dynamic for X^δ (with $\delta = (\alpha, \beta)$) taking into account both α and β is

$$\begin{cases} dX_t^\delta &= \mu(t, X_t^\delta, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\delta, \alpha_{t-}, z)\tilde{M}(dt, dz) \quad \text{if } \tau_n < t < \tau_{n+1} \\ X_{\tau_n}^\delta &= \check{X}_{\tau_n^-}^\delta + I(\tau_n, \check{X}_{\tau_n^-}^\delta, \xi_n), \end{cases} \tag{1.2.4}$$

where $\check{X}_{t^-}^\delta = X_{t^-}^\delta + \Delta X_t^\delta$, ΔX_t^δ is the jump of the measure M at time t. The superscripts in controlled processes will often be omitted to alleviate the notation.

Market orders arrival. We start with a time inhomogeneous Poisson process $\mathcal{N}$, independent of W and M, with intensity $\lambda(t, 0) \geq \underline{\lambda} > 0$, $t \geq 0$. The jumps of this Poisson process are denoted θ_i, $i \geq 1$. For all $x > 0$, we define intensity functions $\lambda(\cdot, x) : [0, T] \to [0, \infty)$, and assume $(\lambda(\cdot, x))_{x>0}$ is an equicontinuous family of functions, bounded below and above by constants $\underline{\lambda}, \overline{\lambda} > 0$. If the investor chooses to place a limit order at a spread α_t below the otherwise best ask price at time t, the likelihood of the execution of this order depends on the observed spread $X_t - \alpha_t$ and arrives with an intensity $\lambda(t, X_t - \alpha_t)$. At the time θ_i, the investor's limit order will go through for a random quantity equal to Y_i, less or equal to n' (the fixed size of the limit order), and independent of $\mathcal{F}_{\theta_i-}$. The fact that the jump intensity is time-dependent is particularly relevant in markets where there is well-known u-shaped trading volume pattern during the day.

Let $\left.\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t^\alpha$ with $Z_0^\alpha = 1$ and

$$dZ_t^\alpha = Z_{t-}^\alpha \left(\frac{\lambda(t, X_t - \alpha_t)}{\lambda(t, 0)} - 1 \right) (d\mathcal{N}_t - \lambda(t, 0)dt) \,.$$

Then a control α changes the distribution of $\mathcal{N}$ under $\mathbb{P}$ to the distribution of $\mathcal{N}$ under $\mathbb{P}^\alpha$, by changing the intensity of $\mathcal{N}$ from $\lambda(t, 0)$ to $\lambda(t, X_t - \alpha_t)$.

The slippage of a limit order that is matched at time θ_i is then given by $\alpha_{\theta_i} Y_i$.

Dynamics of the remaining number of shares to liquidate $N^{\delta,n,t}$.
To keep track of the portfolio through time, we define a pure jump process $N^{\delta,n,t}$ representing the remaining number of shares in the portfolio (taking into consideration transactions through both limit orders and market orders) when the portfolio starts with n remaining shares at time t. The process $N^{\delta,n,t}$ thus starts at $N_t^{\delta,n,t} = n$ at time t, is piecewise constant, and jumps by $-(Y_i \wedge N_{\theta_i-}^{\delta,n,t})$ at time θ_i and by $-(\xi_i \wedge N_{\tau_i-}^{\delta,n,t})$ at time τ_i. This is understood to mean that the process jumps by $-(Y_i + \xi_j) \wedge N_{\theta_i-}^{\delta,n,t}$ if $\theta_i = \tau_j$ for some $i, j \geq 1$.

Admissible control strategies. The limit orders control strategy $\alpha = (\alpha_s)_{0\leq s\leq T}$ is assumed to be a stochastic Markov control such that $\alpha_t < X_{t-}^{\delta}$ for all $t \leq T$. We denote the set of Markov control by $\mathcal{A}$. Let $\mathcal{T}_{t,T}$ be the set of stopping times with values in $[t, T]$. The set of admissible strategies started at time $t \in [0, T]$ when the investor has n shares remaining in the portfolio and that the spread is equal to x is defined as

$$\begin{aligned}\mathcal{AB}(t, n, x) \;=\; & \{\delta = (\alpha, \beta) : \alpha \in \mathcal{A}, \beta = (\tau_i, \xi_i)_{i\leq n}, \tau_i \in \mathcal{T}_{t,T};\ \xi_i \leq n \\ & \text{is an } \mathbb{N}\text{-valued random variable } \mathcal{F}_{\tau_i^-} - \text{measurable s.t. } \tau^{\delta,n,t} \leq T\},\end{aligned}$$

where $\tau^{\delta,n,t} = \inf\{s \geq t :\ N_s^{\delta,n,t} = 0\}$.

The control problem. The investor's goal is to minimize expected slippage by balancing his actions between market orders, which are more expensive due to immediacy, and limit orders, for which the execution time is unknown and random but are executed at more favorable prices. For a strategy $\delta = (\alpha, \beta) \in \mathcal{AB}(t, n, x)$ started at time t, slippage is defined as

$$S_T^{\delta} = \sum_{i=1}^{n} K(\tau_i, \check{X}_{\tau_i^-}^{\delta}, \xi_i) \mathbb{1}_{\tau_i \leq \tau^{\delta}} + \sum_{i\geq 1} \alpha_{\theta_i} Y_i \mathbb{1}_{\theta_i \leq \tau^{\delta}}.$$

For $(t, x, n) \in [0, T] \times [0, +\infty) \times \mathbb{N}$, we define the optimal expected slippage function in the following way :

$$C_n(t, x) = \inf_{\delta\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha} S_T^{\delta}, \tag{1.2.5}$$

with $\mathbb{E}_{t,x,n,\alpha}$ the expectation under $\mathbb{P}^{\alpha}$, given that $N_t = n$ and $X_t = x$. For convenience, we extend this function to negative values of n by letting $C_{-i}(t, x) = 0$ for $i \in \mathbb{N}^*$. We have the following boundary condition :

$$C_n(T, x) = K(T, x, n) \text{ for all } n \in \mathbb{N}^*,$$

which follows readily from the fact that $\tau^{\delta,n,T} = T$, so that the investor must necessarily liquidate the remaining part of his portfolio with a market order at time T.

Penalty function. The maturity T is an urgency parameter. The shorter it is, the more aggressive the strategy and the higher the liquidation cost. However, in order to impose more urgency in the liquidation, it is possible to include a penalty function or a risk aversion parameter in the minimization problem. We may add a penalty function π in terms of the number of remaining shares at time t :

$$C_n(t,x) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha}\left[S_T^\delta + \int_t^T \pi(N_s^\delta, s)ds\right]. \tag{1.2.6}$$

This penalty function can be used to target a specific liquidation schedule as in Cartea and Jaimungal [30], it can be a proxy for the variance of the value of the remaining shares in the portfolio when π is of the quadratic form (see Cartea and Jaimungal [31]), or it can reflect a negative drift in the ask price or “short-term price signal" as suggested by Almgren [4].

1.2.2 Characterization of the slippage function

In this section, we prove that the function C_n is the viscosity solution of an associated quasi-variational inequality. We first introduce the infinitesimal generator of the process $(t, X_t)_{t\geq 0}$ between two market orders :

$$\mathcal{L}^a u(t,x) = \frac{\partial u}{\partial t} + \mu(t,x,a)\frac{\partial u}{\partial x} + \gamma_t \int_{\mathbb{R}} (u(t,x+G(x,a,z)) - u(t,x))m(dz),$$

and the limit orders operator :

$$\Delta_n^a u(t,x) = \lambda(t,x-a)\left[f(a) + \sum_{i=1}^{\infty} p_i C_{n-i}(t,x) - u(t,x)\right],$$

in which $p_i = \mathbb{P}(Y_1 = i)$ $(i \geq 1)$ and $f(a) = a\sum_{i=1}^{\infty} ip_i$, $a \in \mathcal{A}_0$. Finally, define the impulse function for market orders :

$$\mathcal{M}_n(t,x) = \min_{i \in \{1,\dots,n\}} \left[C_{n-i}(t,x+I(t,x,i)) + K(t,i,x)\right].$$

Notice that, for all $(t,x,n) \in [0,T] \times \mathbb{R}_+ \times \mathbb{N}^*$, we deduce from (1.2.5) that

$$0 \leq C_n(t,x) \leq K(t,x,n) = \kappa + L(t,x,n).$$

Therefore, recalling that $\mathcal{P}$ is the set of functions from $[0,T] \times \mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, we have $C_n \in \mathcal{P}$ for all $n \in \mathbb{N}$.

Our main result of this section is the following theorem.

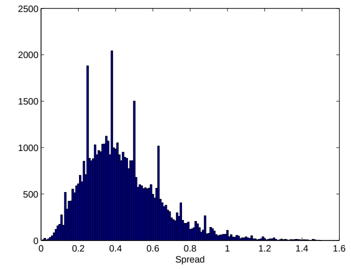
Theorem 1 *For all $n \geq 1$, C_n is the unique continuous viscosity solution in $\mathcal{P}$ of the following variational inequality :*

$$\begin{cases} \min\left(\min_{a \in \mathcal{A}_0} \mathcal{L}^a u + \Delta_n^a u;\ \mathcal{M}_n - u\right) = 0 & \text{on } [0,T) \times [0,\infty), \\ u(T,x) = K(T,n,x) & \text{for } x \geq 0. \end{cases} \tag{1.2.7}$$

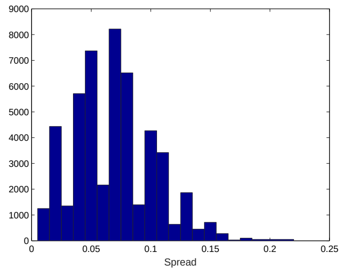
Proof of Theorem 1 : We know that C_0 is continuous and it is the unique viscosity solution of (1.2.7). By induction, suppose C_k is the unique continuous viscosity solutions of (1.2.7) for $k \leq n-1$. We are then able, from a dynamic programming principle, to deduce that C_n is a viscosity solution of (1.2.7) and that a comparison result holds. It follows that C_n is the unique continuous viscosity solution of (1.2.7) . □

1.2.3 Numerical Results

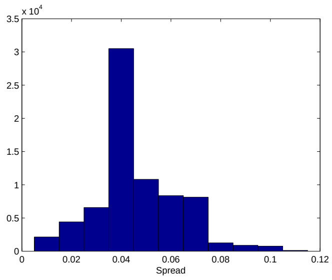
We calibrated the model to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook* from February 28th to March 4th, 2011. The data files obtained from NYXdata.com contains all time-stamped limit orders entered, removed, modified, filled or partially filled on the NYSE ArcaBook platform. The firms considered are Google (GOOG), Air Products & Chemicals Inc. (APD), International Business Machines Corp. (IBM), and J.P. Morgan Chase & Co. (JPM). All four stocks are very liquid and were part of the S&P500 index in 2011. Yet a major difference is that the empirical distribution of their bid-ask spreads differ considerably, as seen in Figure1.3. This is due to the fact that their stock prices are of a different order of magnitude with GOOG at an average price of 606.97, APD at 91.15, IBM at 161.76 and JPM at 45.92 over the five days. In percentage, JPM and IBM have smaller spreads (0.03% of stock price) than GOOG (0.073% of stock price) and APD (0.075%). Since prices are quoted in cents, this offers a large array of values of spreads for GOOG, for which the spread varied from $0.01 to $2.67 during the five trading days considered. The resulting liquidation strategies are very different quantitatively and qualitatively (see figure 1.4).



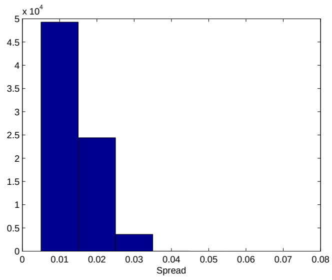
(a) Google



(b) APD

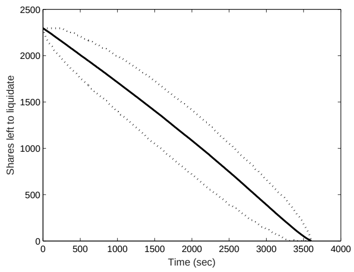


(c) IBM

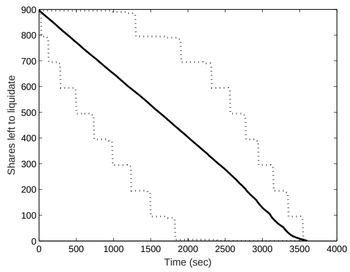


(d) JPM

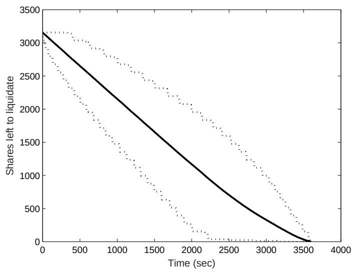
FIGURE 1.3 – Distribution of bid-ask spread



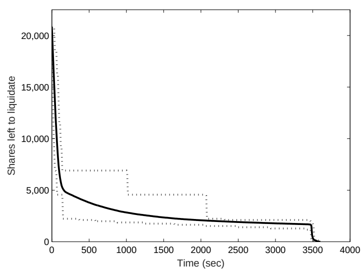
(a) Google



(b) APD



(c) IBM



(d) JPM

FIGURE 1.4 – Average Liquidation Schedule. Dashed lines represent 5th and 95th percentiles.

Chapitre 2

Optimal investment under liquidity risks

The content of this chapter is based on :

(9) Optimal exit strategies for investment projects, with V. Ly Vath, A. Roch and S. Scotti, *Journal of Mathematical Analysis and Applications*, Vol. 425, No. 2, 666-694, 2015.

(12) Liquidity risk and optimal investment/disinvestment strategies, with M.Gaigi et V. Ly Vath, to appear in *Mathematics and Financial Economics.*

The theory of optimal stochastic control problem, developed in the seventies, has over the recent years once again drawn a significance of interest with the main focus on its applications in a variety of fields including economics and finance. For instance, the use of powerful tools from stochastic control theory has provided new approaches and sometimes the first mathematical approaches in solving problems arising from corporate finance. It is mainly about finding the best optimal decision strategy for managers whose firms operate under uncertain environment whether it is financial or operational, see [25] and [45].

The strategy considered may be on firm's investment decisions in stochastic environments, see for instance [24], [44], [81], [87], [95] and [106]. In relation to the first section, Dixit and Pindyck [45] consider various firm's decisions problems with entry, exit, suspension and/or abandonment scenarios in the case of an asset given by a geometric Brownian motion. The firm's strategy can then be described in terms of stopping times given by the time when the value of the assets hits certain threshold levels characterized as free boundaries of a variational problem. Duckworth and Zervos [46], and Lumley and Zervos [82] solve an optimal investment decision problem with switching costs in which the firm controls the production rate and must decide at which time it exits and re-enters production. In the first section, we study the problem of an optimal exit strategy for an investment project which is unprofitable and for which the liquidation costs evolve stochastically. The firm has the option to keep the project going while waiting for a buyer, or liquidating the assets at immediate liquidity and termination costs. The liquidity and termination costs are governed by a mean-reverting stochastic process whereas the rate of arrival of buyers is governed by a regime-shifting Markov process. We formulate this problem as a multidimensional optimal stopping time problem with random maturity. We characterize the objective function as the unique viscosity solution of the associated system of variational HJB equations. We derive explicit solutions and numerical examples in the case of power and logarithmic utility functions when the liquidity premium factor follows a mean-reverting CIR process.

Added to these investment decisions, firm's manager may have to determine strategies to distribute dividends to shareholders. The pionneering works on this subject model dividend strategy as a singular control (see [68]) or a regular control (see [6]) on the firm's cash reserve dynamics which is assumed to follow a drifted brownian motion.
In the second section, we study the problem of determining an optimal control on the dividend and investment policy of a firm operating under uncertain environment and risk constraints. There are a number of research on this corporate finance problem. In [44], Décamps and Villeneuve study the interactions between dividend policy and irreversible investment decision in a growth opportunity and under uncertainty. We may equally refer to [85] for an extension of this study, where the authors relax the irreversible feature of the growth opportunity. In these models, the firm goes into bankruptcy when its cash reserve reaches zero. The underlying financial assumption behind the above model is to consider that the firm's assets may be separated into two types of assets, highly liquid assets which may be assimilated as cash reserve, i.e. cash & equivalents, or infinitely illiquid assets, i.e. producing assets that may not be sold. As such, when the cash reserve gets near the bankruptcy point, the firm manager may not be able to inject any cash by selling parts of its non-liquid assets. The assumptions made in the above models imply that the firm's illiquid assets correspond to producing assets which may be neither increased through investment nor decreased through disinvestment.
We allow the company to make investment decisions by acquiring or selling producing assets whose value is governed by a stochastic process. The firm may face liquidity costs when it decides to buy or sell assets. We formulate this problem as a multi-dimensional mixed singular and multi-switching control problem and use a viscosity solution approach. We numerically compute our optimal strategies and enrich our studies with numerical results and illustrations.

2.1 Optimal exit strategies for investment projects

The firm, we consider must decide between liquidating the assets of an underperforming project and waiting for the project to become once again profitable, in a setting where the liquidation costs and the value of the assets are given by general diffusion processes. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting.

Amongst the large literature on optimal stopping problems, we may refer to some related works including Bouchard, El Karoui and Touzi [20], Carr [29], Dayanik and Egami [42], Dayanik and Karatzas [43], Guo and Zhang [61], Lamberton and Zervos [78]. In [43] and [78], the authors study optimal stopping problems with general 1-dimensional processes. Random maturity in optimal stopping problem was introduced in [29] and [20]. It allowed to reduce the dimension of their problems as well as addressing the numerical issues. We may refer to Dayanik and Egami [42] for a recent paper on optimal stopping time and random maturity. For optimal stopping problem with regime shifting, we may refer to Guo and Zhang [61], where an explicit optimal stopping rule and the corresponding value function in a closed form are obtained.

Our optimal stopping problem combines all the above features, i.e., random maturity and regime shifting, in the bi-dimensional framework. We are able to characterize the value function of our problem and provide explicit solution in some particular cases where we manage to reduce the dimension of our control problem.

In the general bi-dimensional framework, the main difficulty is related to the proof of the continuity property and the PDE characterization of the value function. Since it is not possible to get the smooth-fit property, the PDE characterization may be obtained only via the viscosity approach. To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.

In the particular cases where it is possible to reduce our problem to a one-dimensional problem, we are able to provide explicit solution. Our reduced one-dimensional problem is highly related to previous studies in the literature, see for instance Zervos, Johnson and Alezemi [109] and Leung, Li and Wang [79].

2.1.1 The Investment Project

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times. Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation ρ, i.e. $d[W, B]_t = \rho dt$ for all t.

We consider a firm which owns assets that are currently locked up in an investment project which currently produces no output per unit of time. Because the firm is currently not using the assets at its full potential, it considers two possibilities. The first is to liquidate the assets in a fire sale and recover any remaining value. The cash flow obtained in the latter case is the fair value of the assets minus liquidation and project termination costs. We denote by θ the moment at which the firm decides to take this option. The second option is to wait for the project to become profitable once again, or equivalently, to wait for an investor or another firm who will purchase the assets as a whole at their fair value S_τ where τ is the moment when this happens, so-called the recovery time.

Assets fair value. The fair value of the assets are given by $S = \exp(X)$, in which

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0 \\ X_0 &= x. \end{aligned} \tag{2.1.1}$$

Assume that μ and σ are Lipschitz functions on $\mathbb{R}$ satisfying the following growth condition

$$\lim_{|x|\to\infty} \frac{|\mu(x)| + |\sigma(x)|}{|x|} = 0. \tag{2.1.2}$$

Liquidation and Termination Costs. Should the firm decide to terminate the project operations and liquidate the assets, the resulting cash flow is $S_t f(Y_t)$. where f is strictly decreasing C^2 function defined on $\mathbb{R}^+ \to [0, 1]$, and satisfies the following conditions :

$$f(0) = 1 \text{ and } \exists\, c > 0, \text{ such that } \lim_{y\to\infty} f(y)\exp(y^c) = 0. \tag{2.1.3}$$

The liquidation costs, given by $f(Y_t)$ at time t, is defined in terms of the mean-reverting non-negative process Y which is governed by the following SDE :

$$\begin{aligned} dY_t &= \alpha(Y_t)dt + \gamma(Y_t)dW_t, \\ Y_0 &= y, \end{aligned} \tag{2.1.4}$$

where α is a Lipschitz function on $\mathbb{R}^+$ and, for any $\varepsilon > 0$, γ is a Lipschitz function on $[\varepsilon, \infty)$. We assume that α and γ satisfy a linear growth condition. Furthermore, to insure the mean-reverting property, we assume that there exists $\beta > 0$ such that $(\beta - y)\alpha(y)$ is positive for all $y \geq 0$.

The recovery time. We model the arrival time of a buyer, denoted by τ, or equivalently the time when the project becomes profitable again, by means of an intensity function λ_i depending on the current state i of a continuous-time, time-homogenous, irreducible Markov chain L, independent of W and B, with $m+1$ states. The states of the chain represent liquidity states of the assets. The generator of the chain L under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=0,\ldots m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain L from state i to state j $(0 \leq i, j \leq m)$. Without loss of generality we assume

$$\lambda_0 > \lambda_1 > \ldots > \lambda_m > 0. \tag{2.1.5}$$

Utility function. We denote by U the utility function which satisfies :

Assumption 1 *$U : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing, concave and twice continuously differentiable, and satisfies*

$$\lim_{x \to 0} x\, U'(x) \;<\; +\infty. \tag{2.1.6}$$

Assumption 2 *U is supermeanvalued w.r.t. S, i.e.*

$$U(S_t) \geq \mathbb{E}[U(S_\theta)|\mathcal{F}_t], \quad \textit{for any } \theta \in \mathcal{T}. \tag{2.1.7}$$

The financial interpretation of the supermeanvalued property of U w.r.t. S is as follows : it is always better to accept right away an offer to buy the assets at their fair value then to wait for a later one. For more details on the supermeanvalued property, which is closely related to the concept of superharmonicity, we may refer to Dynkin [48] and Oksendal [96].

Objective function. The objective of the firm is to maximize the expected profit obtained from the sale of the illiquid asset, either through liquidation or at its fair value at the exogenously given stopping time τ. As such, we consider the following value function :

$$v(i,x,y) := \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\, \mathbb{1}_{\theta \leq \tau} + U(e^{X_\tau})\, \mathbb{1}_{\theta > \tau}\right], \quad x \in \mathbb{R},\, y \in \mathbb{R}^+,\, i \in \{0, \ldots, m\}$$

where $h(x,y) = U(e^x f(y))$.

2.1.2 Characterization of the value functions

We first obtain some descriptive properties of these functions including the monotonicity and continuity of the functions v_i. We highlight two main difficulties that need a no-standard treatment. The first one comes from the SDE satisfied by Y (2.1.4) since we do not assume the standard hypothesis of Lipschitz coefficients. We overcome this drawback showing that the local Lipschitz property is satisfied until the smallest optimal exit time from the investment. The second difficulty is related to the bi-dimensional setting where the classical arguments used to show the regularity of value function are not longer available. We then need to show the continuity in term of limits of sequences and to distinguish different sub-sequences with ad-hoc proofs.

The complexity of the proof of the continuity suggests that a direct proof of differentiability, i.e. smooth-fit property, of the value function is probably out of reach in our setting. We will then turn to the viscosity characterization approach to overcome the impossibility to use a verification approach.

Theorem 2.1.1 *The value functions v_i, $i \in \{0, \dots, m\}$, are continuous on $\mathbb{R} \times \mathbb{R}^+$, and constitute the unique viscosity solution on $\mathbb{R} \times \mathbb{R}^+$ with growth condition*

$$|v_i(x, y)| \leq |U(e^x)| + |U(e^x)f(y)|,$$

and boundary condition

$$\lim_{y \downarrow 0} v_i(x, y) = U(e^x),$$

to the system of variational inequalities :

$$\min\Big[-\mathcal{L}v_i(x, y) - \mathcal{G}_i v_{.}(x, y) - \mathcal{J}_i v_i(x, y)\, ,\, v_i(x, y) - U(e^x f(y))\Big] = 0, \qquad \forall\, (x, y) \in \times\mathbb{R} \times \mathbb{R}^+_*, \text{ and } i \in \{0, \dots, n\}, \tag{2.1.8}$$

where the operators $\mathcal{G}_i$ and $\mathcal{J}_i$ are defined as

$$\begin{aligned} \mathcal{G}_i\varphi(., x, y) &= \sum_{j \neq i} \vartheta_{i,j}\left(\varphi(j, x, y) - \varphi(i, x, y)\right) \\ \mathcal{J}_i\varphi(i, x, y) &= \lambda_i\left(e^x - \varphi(i, x, y)\right), \end{aligned}$$

and $\mathcal{L}$ is the second order differential operator associated to the state processes (X, Y).

The uniqueness result relies on a comparison principle. The main difficulty in proving this principle is to deal with the non-linearity of the bounds of the value function when building a strict super-solution

Liquidation and continuation regions

We now prove useful qualitative properties of the liquidation regions of the optimal stopping problem. We introduce the following liquidation and continuation regions :

$$\begin{aligned} \mathcal{LR} &= \left\{(i, x, y) \in \{0, ..., m\} \times \mathbb{R} \times \mathbb{R}^+ \,|\, v(i, x, y) = h(x, y)\right\} \\ \mathcal{CR} &= \{0, ..., m\} \times \mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{LR}. \end{aligned}$$

Clearly, outside the liquidation region $\mathcal{LR}$, it is never optimal to liquidate the assets at the available discounted value. Moreover, the smallest optimal stopping time θ^*_{ixy} verifies

$$\theta^*_{ixy} = \inf\left\{u \geq 0 \,|\, \left(L^i_u, X^x_u, Y^y_u\right) \in \mathcal{LR}\right\}.$$

We define the $(i,x)-$sections for every $(i,x) \in \{0,...,m\} \times \mathbb{R}$ by

$$\mathcal{LR}_{(i,x)} = \{y \geq 0 \,|\, v(i,x,y) = h(x,y)\} \text{ and } \mathcal{CR}_{(i,x)} = \mathbb{R}^+ \setminus \mathcal{LR}_{(i,x)}.$$

Proposition 2.1.2 (Properties of liquidation region)
i) $\mathcal{E}$ is closed in $\{0,...,m\} \times \mathbb{R} \times (0,+\infty)$,
ii) Let $(i,x) \in \{0,...,m\} \times \mathbb{R}$.
- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] = U(e^x)$, then, for all $y \in \mathbb{R}^+$, $v(i,x,y) = U(e^x)$ and $\mathcal{E}_{(i,x)} = \{0\}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] < U(e^x)$, then there exists $x_0 \in \mathbb{R}$ such that $\mathcal{E}_{(i,x_0)} \setminus \{0\} \neq \emptyset$ and $\bar{y}^*(i,x) := \sup \mathcal{E}_{(i,x)} < +\infty$.*

2.1.3 Logarithmic utility

Throughout this section, we assume that the diffusion processes X and Y are governed by the following SDE, which are particular cases of (2.1.1) and (2.1.4)

$$\begin{aligned} dX_t &= \mu dt + \sigma(X_t) dB_t;\; X_0 = x \\ dY_t &= \kappa\,(\beta - Y_t)\,dt + \gamma\sqrt{Y_t} dW_t;\; Y_0 = y \end{aligned}$$

The following theorem shows that in the logarithmic case, we can reduce the dimension of the problem by factoring out the x-variable. For this purpose, we define $\mathcal{T}_{L,W}$ the set of stopping times with respect to the filtration generated by (L,W), and the differential operator $\overline{\mathcal{L}}\phi(y) := \frac{1}{2}\gamma^2 y \frac{\partial^2\phi}{\partial y^2} + \kappa(\beta - y)\frac{\partial\phi}{\partial y} + \mu$, for $\phi \in C^2(\mathbb{R}^+)$.

Theorem 2.1.3 *On $\{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+$, we have $v(i,x,y) = x + w(i,y)$ where*

$$w(i,y) = \sup_{\theta \in \mathcal{T}_{L,W}} \mathbb{E}^{i,y}[\mu(\theta \wedge \tau) + \ln\left(f(Y_\theta)\right) \mathbb{1}_{\{\theta \leq \tau\}}] \text{ on } \{0,...,m\} \times \mathbb{R}^+.$$

Moreover, w is the unique viscosity solution to the system of equations :

$$\min\Big[-\overline{\mathcal{L}}w(i,y) + \lambda_i w(i,y) - \sum_{j\neq i} \vartheta_{i,j}\left(w(j,\,y) - w(i,\,y)\right)\,,\, w(i,y) - g(y)\Big] = 0, \quad (2.1.9)$$

where $g(y) := \ln(f(y))$. The functions $w(i,.)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on $\mathcal{C}_{(i,x)} \cup \mathrm{Int}(\mathcal{E}_{(i,x)})$.

In the logarithmic case, the liquidation region can be characterized in more details.

Proposition 2.1.4 *Let $i \in \{0,...,m\}$ and set*

$$\hat{y}_i = \inf\{y \geq 0 :\; \mathcal{H}_i g(y) \geq 0\} \text{ with } \mathcal{H}_i g(y) = \overline{\mathcal{L}}g(y) - \lambda_i g(y) + \sum_{j\neq i} \vartheta_{i,j}\left(w(j,\,y) - g(y)\right).$$

There exists $y_i^* \geq 0$ such that $[0, y_i^*] = \mathcal{LR}_{(i,.)} \cap [0, \hat{y}_i]$. Moreover, $w(i,\cdot) - g(\cdot)$ is non-decreasing on $[y_i^*, \hat{y}_i]$.

Remark 1 *When $\overline{\mathcal{L}}g(y)$ is non-decreasing in y, the previous result can be specified further. Indeed, in that case, for all $i \in \{0,...,m\}$, $w(i,\cdot) - g(\cdot)$ is non-decreasing on $\mathbb{R}^+$ and we have $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$, with $y_i^* > 0$.*

Explicit solutions in the two regimes case

We assume that there are two regimes (i.e., $m = 1$) and $\vartheta_{0,1}\vartheta_{1,0} \neq 0$. We also assume that, for both $i = 0, 1$, there exists $y_i^* > 0$ such that $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$.
Let Λ be the matrix

$$\Lambda = \begin{pmatrix} \lambda_0 + \vartheta_{0,1} & -\vartheta_{0,1} \\ -\vartheta_{1,0} & \lambda_1 + \vartheta_{1,0} \end{pmatrix}.$$

As $\vartheta_{0,1}\vartheta_{1,0} > 0$ it is easy to check that Λ has two eigenvalues $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1 < \widetilde{\lambda}_0$. Let $\widetilde{\Lambda} = P^{-1}\Lambda P$ be the diagonal matrix with diagonal $(\widetilde{\lambda}_0, \widetilde{\lambda}_1)$. The transition matrix P is denoted by

$$P = \begin{pmatrix} p_0^0 & p_1^0 \\ p_0^1 & p_1^1 \end{pmatrix}.$$

Without loss of generality, we shall assume that $p_0^0 + p_1^0 = 1 = p_0^1 + p_1^1$, indeed $(1, -1)$ is not an eigenvector of Λ as $\lambda_0 > \lambda_1$. With the above assumptions, we obtain $y_0^* \leq y_1^*$ and the value function can be written in terms of the confluent hypergeometric functions.

Proposition 2.1.5 *The function w is given by*

$$w(0,y) = \begin{cases} g(y) & y \in [0, y_0^*] \\ \widehat{c}\Phi\left(\frac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \widehat{d}\Psi\left(\frac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) & y \in (y_0^*, y_1^*] \\ \quad + \mathcal{I}\left(\frac{2\kappa}{\gamma^2}, \beta, -2\frac{\lambda_0 + \vartheta_{0,1}}{\gamma^2}, 2\frac{\vartheta_{0,1}g(\cdot) + \mu}{\gamma^2}\right)(y) & \\ p_0^0\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}x\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^0\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}x\right) + \frac{\mu}{\widetilde{\lambda}_1}\right] & \end{cases}$$

$$w(1,y) = \begin{cases} g(y) & y \in [0, y_1^*] \\ p_0^1\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^1\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_1}\right], & \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind, and $\mathcal{I}$ is a particular solution to the non-homogeneous confluent differential equation. Moreover, $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $w(0, y)$ and $w(1, y)$ belong to $C^1(\mathbb{R}^+)$.

Numerical Simulation

In Figure 2.1, we represent the value functions in the two-regime case, for the cases $\mu = -.05$ and $\mu = -0.3$.

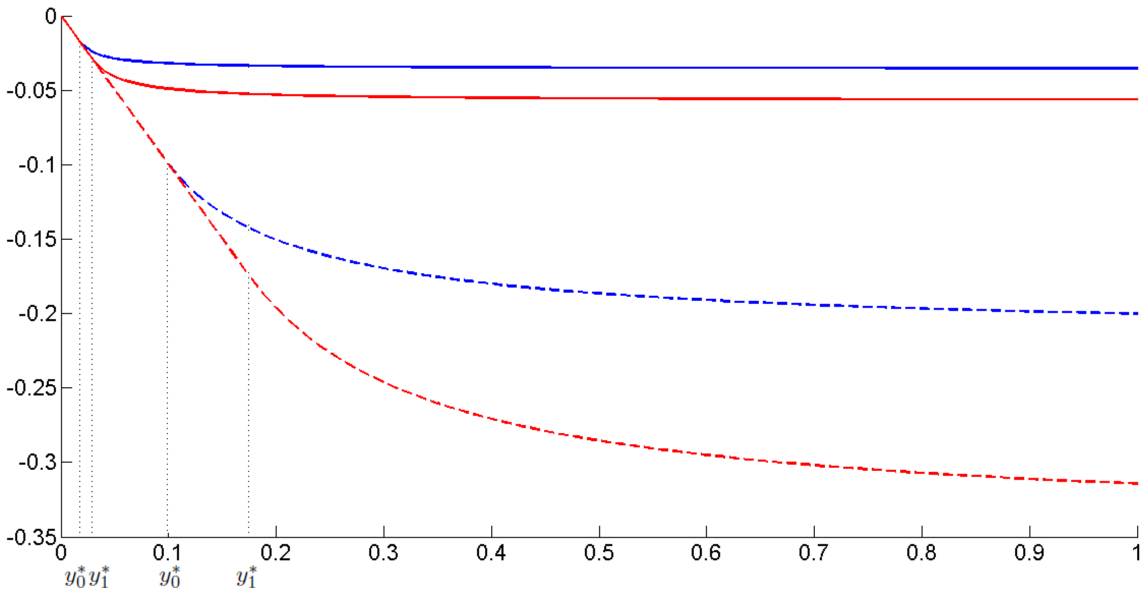


FIGURE 2.1 – Value functions in the two-regime case, for the cases $\mu = -0.05$ (solid line) and $\mu = -0.3$ (dashed line). Regime 0 is presented in blue and regime 1 in red.

2.2 Liquidity risk and optimal investment strategies

In [68], [6], [34], the authors study an optimal dividend problem and consider a single stochastic process which represents the cash reserve of the firm.

We no longer simplify the optimal dividend and investment problem by assuming that firm's assets are either infinitely illiquid or liquid. For the same reason as highlighted in financial market problems, it is necessary to take into account the liquidity constraints. More precisely, investment (for instance acquiring producing assets) and disinvestment (selling assets) should be possible but not necessarily at their fair value. The firm may have to face some liquidity costs when buying or selling assets. While taking into account liquidity constraints and costs has become the norm in recent financial markets problems, it is still not the case in the corporate finance, to the best of our knowledge, in particular in the studies of optimal dividend and investment strategies. In our model, we consider the company's assets may be separated in two categories, cash & equivalents, and risky assets which are subjected to liquidity costs. The risky assets are assimilated to producing assets which may be increased when the firm decides to invest or decreased when the firm decides to disinvest. We assume that the price of the risky assets is governed by a stochastic process. The firm manager may buy or sell assets but has to bear liquidity costs. The objective of the firm manager is to find the optimal dividend and investment strategy maximizing its shareholders' value, which is defined as the expected present value of dividends. Mathematically, we formulate this problem as a combined multidimensional singular and multi-regime switching control problem.

The studies that are most relevant to our problem are the one investigating combined singular and switching control problems [60], [85], and **(7)**. By incorporating uncertainty into illiquid assets value, we no longer have to deal with a uni-dimensional control problem but a bi-dimensional singular and multi-regime switching control problem. In such a setting, it is clear that it will be no longer possible to easily get explicit or quasi-explicit optimal strategies. Consequently, to determine the four regions comprising the continuation, dividend and investment/disinvestment regions, numerical resolutions are required.

2.2.1 Problem Formulation

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space satisfying the usual conditions. Let W and B be two correlated $\mathbb{F}$-Brownian motions, with correlation coefficient c.

We consider a firm which has the ability to make investment or disinvestment by buying or selling producing assets, for instance, factories. We assume that these producing assets are risky assets whose value process S is solution of the following equation :

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right),\ S_0 = s,$$

where μ and σ are positive constants.

We denote by $Q_t \in \mathbb{N}$ the number of units of producing assets owned by the company at time t.

We consider a control strategy : $\alpha = ((\tau_i, q_i)_{i\geq 1}, Z)$ where τ_i are $\mathbb{F}$-stopping times, corresponding to the investment decision times of the manager, and q_i are $\mathcal{F}_{\tau_i}$-measurable variables valued in $\mathbb{Z}$ and representing the number of producing assets units bought (or sold if $q_i \leq 0$) at time τ_i. When q_i is positive, it means that the firm decides to make investment to increase the assets quantity. Each purchase or sale incurs a fixed cost denoted $\kappa > 0$. The non-decreasing càdlàg process Z represents the total amount of dividends distributed up to time t. Starting from an initial number of assets q and given a control α, the dynamics of the quantity of assets held by the firm is governed by :

$$\begin{cases} dQ_t &= 0 \quad \text{for } \tau_i \leq t < \tau_{i+1}, \\ Q_{\tau_i} &= Q_{\tau_i^-} + q_i, \\ Q_0 &= q, \end{cases} \qquad \text{for } i \geq 1.$$

Similarly, starting from an initial cash value x and given a control α, the dynamics of the cash reserve (or more precisely the firm's cash and equivalents) process of the firm is governed by :

$$\begin{cases} dX_t &= rX_t dt + h(Q_t)(bdt + \eta dW_t) - dZ_t, \text{ for } \tau_i \leq t < \tau_{i+1} \\ X_{\tau_i} &= X_{\tau_i^-} - S_{\tau_i} f(q_i) q_i - \kappa, \\ X_0 &= x, \end{cases} \qquad \text{for } i \geq 1.$$

where b, r and η are positive constants and h a non-negative, non-decreasing and concave function satisfying $h(q) \leq H$ with $h(1) > 0$ and $H > 0$. The function f represents the liquidity cost function (or impact function with the impact being temporary) and is assumed to be non-negative, non-decreasing, such that $f(0) = 1$.

We denote by $Y_t^y = (X_t^x, S_t^s, Q_t^q)$ the solution to previous equations with initial condition $(X_0^x, S_0^s, Q_0^q) = (x, s, q) := y$. At each time t, the firm's cash value and number of units of producing assets have to remain non-negative i.e. $X_t \geq 0$ and $Q_t \geq 0$, for all $t \geq 0$.

The bankruptcy time is defined as

$$T := T^{y,\alpha} := \inf\{t \geq 0, X_t < 0\}.$$

We define the liquidation value as $L(x,s,q) := x + (sf(-q)q - \kappa)^+$ and notice that $L \geq 0$ on $\mathbb{R}^+ \times (0,+\infty) \times \mathbb{N}$. We introduce the following notation

$$\mathcal{S} := \mathbb{R}^+ \times (0,+\infty) \times \mathbb{N}.$$

The optimal firm value is defined on $\mathcal{S}$, by

$$v(y) = \sup_{\alpha \in \mathcal{A}(y)} J^\alpha(y), \tag{2.2.1}$$

where $J^\alpha(y) = \mathbb{E}[\int_0^T e^{-\rho u} dZ_u]$, with ρ being a positive discount factor and $\mathcal{A}(y)$ is the set of admissible strategies defined by

$$\begin{aligned}\mathcal{A}(y) \quad = \quad & \{\alpha = ((\tau_i, q_i)_{i\geq 1}, Z) : Z \text{ is a predictable and non-decreasing process,} \\ & (\tau_i)_{i\geq 1} \text{ is an increasing sequence of stopping times such that } \lim_{i\to+\infty} \tau_i = +\infty \\ & \text{ and } q_i \text{ are } \mathbb{F}_{\tau_i} - \text{measurable, and such that } (X_t^{x,\alpha}, Q_t^{q,\alpha}) \in \mathbb{R}^+ \times \mathbb{N}\}.\end{aligned}$$

We may identify the trivial cases where the value function is infinite.

Proposition 2.2.1 *If we have $r > \rho$ or $\mu > \rho$ then $v(y) = +\infty$ on $\mathcal{S}$.*

From this point, we shall assume that the parameters satisfy :

$$\rho \geq \max(r, \mu) \tag{2.2.2}$$

2.2.2 Characterization of auxiliary functions

The aim of this section is to provide an implementable algorithm of our problem. To tackle the stochastic control problem as defined in (2.2.1), one usual way is to first characterize the value function as a unique solution to its associated HJB equation. One would expect here that v is solution of the following HJB equation

$$\min\{\rho v(y) - \mathcal{L}v(y); \frac{\partial v}{\partial x}(y) - 1; v(y) - \mathcal{H}v(y)\} = 0 \text{ on } (0,+\infty)^2 \times \mathbb{N}, \tag{2.2.3}$$

where we have set

$$\mathcal{L}\varphi \quad = \quad \frac{\eta^2 h(q)^2}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 \varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2 \varphi}{\partial s \partial x} + (rx + bh(q))\frac{\partial \varphi}{\partial x} + \mu s\frac{\partial \varphi}{\partial s}.$$

$$\begin{aligned}\mathcal{H}v(x,s,q) \quad &= \quad \sup_{n\in a(x,s,q)} v\left(\Gamma(y,n)\right) \text{ with } \Gamma(y,n) = (x - n(f(n))s - \kappa,\ s,\ q+n), \text{ and} \\ a(x,s,q) \quad &= \quad \left\{n \in \mathbb{Z} : n \geq -q \text{ and } n(f(n)) \leq \frac{x-\kappa}{s}\right\},\end{aligned}$$

with the convention that the supremum of an empty set is equal to -∞.

The second step is to deduce the optimal strategies from smooth-fit properties and more generally from viscosity solution techniques. The optimal strategies may be characterized by different regions of the state-space, i.e. the continuation region, the dividend region as well as the investment and disinvestment regions. In such cases, the solutions may be either of explicit or quasi-explicit nature. However, in a non-degenerate multidimensional setting such as in our problem, getting explicit or quasi-explicit solutions is out of reach.

As such, to solve our control problem, we characterize our value function as the limit of a sequence of auxiliary functions. The auxiliary functions are defined recursively and each one may be characterized as a unique viscosity solution to its associated HJB equation . The use of approximating function will allow us to use a classical dynamic programming principle for optimal stopping problem and to get an implementable algorithm approximating our problem.

An approximating sequence of functions

We recall the notation $y = (x, s, q) \in \mathcal{S}$. We now introduce the following subsets of $\mathcal{A}(y)$:

$$\mathcal{A}_N(y) := \{\alpha = ((\tau_k, \xi_k)_{k\geq 1}, Z) \in \mathcal{A}(y) : \tau_k = +\infty \text{ a.s. for all } k \geq N+1\}$$

and the corresponding value function v_N, which describes the value function when the investor is allowed to make at most N interventions (investments or disinvestments) :

$$v_N(y) = \sup_{\alpha \in \mathcal{A}_N(y)} J^\alpha(y), \ \forall N \in \mathbb{N} \tag{2.2.4}$$

We shall show in Proposition 2.2.6 that the sequence $(v_N)_{N\geq 0}$ goes to v when N goes to infinity, but we first have to carefully study some properties of this sequence.

In the next Proposition, we recall explicit formulas for v_0 and the optimal strategy associated to this singular control problem. This problem is indeed very close to the one solved in the pioneering work of Jeanblanc and Shirayev (see [68]). The only difference in our framework is due to the interest $r \neq 0$ and therefore the cash process X does not follow exactly a Bachelier model. However, proofs and results can easily be adapted to obtain Proposition 2.2.2 and we will skip the proof.

Proposition 2.2.2 *There exists $x^*(q) \in [0, +\infty)$ such that*

$$v_0(x, s, q) := \begin{cases} V_q(x) & \text{if} \quad 0 \leq x \leq x^*(q) \\ x - x^*(q) + V_q(x^*(q)) & \text{if} \quad x \geq x^*(q), \end{cases}$$

where V_q is the $\mathcal{C}^2$ function, solution of the following differential equation

$$\frac{\eta^2 h(q)^2}{2} y'' + (rx + bh(q))y' - \rho y = 0; \quad y(0) = 0, \ y'(x^*(q)) = 1 \text{ and } y''(x^*(q)) = 0.$$

Notice that $x \to v_0(x, s, q)$ is a concave and $\mathcal{C}^2$ function on $[0, +\infty)$ and that if $h(0) = 0$, it is optimal to immediately distribute dividends up to bankruptcy therefore $v_0(x, s, 0) = x$.

We now are able to characterize our impulse control problem as an optimal stopping time problem, defined through an induction on the number of interventions N.

Proposition 2.2.3 *(Optimal stopping)*
For all $(x,s,q,N) \in \mathcal{S} \times \mathbb{N}$, we have

$$v_N(x,s,q) = \sup_{(\tau,Z)\in\mathcal{T}\times\mathcal{Z}} \mathbb{E}[\int_0^{T\wedge\tau} e^{-\rho u}\, dZ_u + e^{-\rho\tau} G_{N-1}(X^x_{\tau^-}, S^s_\tau, q)\,\mathbb{1}_{\{\tau<T\}}], \quad (2.2.5)$$

where $\mathcal{T}$ is the set of stopping times, $\mathcal{Z}$ the set of predictable and non-decreasing càdlàg processes, $G_{-1} = 0$, and, for $N \geq 1$,

$$G_{N-1}(x,s,q) \;:=\; \begin{cases} \sup_{n\in a(x,s,q)} v_{N-1}(\Gamma(y,n)) \;, & \text{if } a(x,s,q) \neq \emptyset \\ -\infty, \;\; \text{if } a(x,s,q) = \emptyset \end{cases}$$

$$\text{with} \qquad a(x,s,q) := \left\{ n \in \mathbb{Z} : n \geq -q \text{ and } nf(n) \leq \frac{x-\kappa}{s} \right\},$$

$$\text{and} \qquad \Gamma(y,n) := (x - nf(n)s - \kappa,\;\; s,\;\; q+n).$$

Bounds and convergence of $(v_N)_{N\geq 0}$

We begin by stating a standard result which says that any smooth function, which is supersolution to the HJB equation, is a majorant of the value function.

Proposition 2.2.4 *Let $N \in \mathbb{N}$ and $\phi = (\phi_q)_{q\in\mathbb{N}}$ be a family of non-negative $\mathcal{C}^2$ functions on $\mathbb{R}^+ \times (0,+\infty)$ such that $\forall q \in \mathbb{N}$ (we may use both notations $\phi(x,s,q) := \phi_q(x,s)$), $\phi_q(0,s) \geq 0$ for all $s \in (0,\infty)$ and*

$$\min\left[\rho\phi(y) - \mathcal{L}^N\phi(y), \phi(y) - G_{N-1}(y), \frac{\partial\phi}{\partial x}(y) - 1\right] \;\geq\; 0 \qquad (2.2.6)$$

for all $y \in (0,+\infty)\times(0,+\infty)\times\mathbb{N}$, where we have set

$$\begin{aligned} \mathcal{L}^N\varphi \;&=\; \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + (rx + bh(q))\frac{\partial\varphi}{\partial x} \\ &\quad + \mathbb{1}_{\{N>0\}}\left[\frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s\partial x} + \mu s\frac{\partial\varphi}{\partial s}\right]. \end{aligned}$$

then we have $v_N \leq \phi$.

Corollary 2.2.5 Bounds *:*
For all $N \in \mathbb{N}$ and $(x,s,q) \in \mathcal{S}$, we have

$$L(x,s,q)\mathbb{1}_{\{N\geq 1\}} + x\mathbb{1}_{\{N=0\}} \leq v_N(x,s,q) \leq x + sq + K \qquad \text{where } \rho K = bH.$$

We are able to conclude on the asymptotic behavior of our approximating sequence of functions. The next Proposition shows that this sequence of functions goes to our value function v when N goes to infinity.

Proposition 2.2.6 *(Convergence) For all $y \in \mathcal{S}$, we have*

$$\lim_{N\to+\infty} v_N(y) = v(y).$$

Viscosity characterization of v_N

Let $N \geq 1$. This subsection is devoted to the characterization of the function v_N as the unique function which satisfies the boundary condition

$$v_N(y) = G_{N-1}(y) \text{ on } \{0\} \times (0, +\infty) \times \mathbb{N}. \tag{2.2.7}$$

and is a viscosity solution of the following HJB equation :

$$\min\{\rho v_N(y) - \mathcal{L} v_N(y); \frac{\partial v_N}{\partial x}(y) - 1; v_N(y) - G_{N-1}(y)\} = 0 \text{ on } (0, +\infty)^2 \times \mathbb{N}, \tag{2.2.8}$$

where we have set

$$\mathcal{L}\varphi = \frac{\eta^2 h(q)^2}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 \varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2 \varphi}{\partial s \partial x} + (rx + bh(q))\frac{\partial \varphi}{\partial x} + \mu s \frac{\partial \varphi}{\partial s}.$$

It relies on the following Dynamic Programming Principle.
Let $\theta \in \mathcal{T}$, $y := (x, s, q) \in \mathcal{S}$ and set $\nu = T \wedge \theta$, we have

$$v_N(y) = \sup_{(\tau, Z) \in \mathcal{T} \times \mathcal{Z}} \mathbb{E}[\int_0^{(\nu\wedge\tau)^-} e^{-\rho s}\, dZ_s + e^{-\rho(\nu\wedge\tau)} v_N\left(X^x_{(\nu\wedge\tau)^-}, S^s_{\nu\wedge\tau}, q\right) \mathbb{1}_{\{\tau<\nu\}}].$$

We are now able to establish the main results of this section.

Theorem 2.2.7 *For all $N \geq 1$ and $q \in \mathbb{N}$, the value function $v_N(\cdot, \cdot, q)$ is continuous on $(0, +\infty)^2$. Moreover v_N is the unique viscosity solution on $(0, +\infty)^2 \times \mathbb{N}$ of the HJB equation (2.2.8) satisfying the boundary condition (2.2.7) and the following growth condition*

$$|v_N(x, s, q)| \leq C_1 + C_2 x + C_3 sq, \quad \forall (x, s, q) \in \mathcal{S},$$

for some positive constants C_1, C_2 and C_3.

2.2.3 Numerical Results

To approximate the solution of the HJB equation (2.2.8) arising from the stochastic control problem (2.2.4), we choose to use a finite difference scheme which leads to the construction of an approximating Markov chain. The convergence of the scheme can be shown using standard arguments as in [75]. We may equally refer to [27], [63], and [70] for numerical schemes involving singular control problems.

We plot the shape of the optimal regions in function of (x, s) for a fixed number of producing assets $q_2 > q_1 > q_0$. We may distinguish four regions : buy, sell, dividend and continuation regions. We may clearly make the following observations

- As the assets price gets higher, the dividend region shrinks in favor of the buy region. Indeed, the firm has to hold sufficient amount of cash in order to be able to invest in more expensive assets.

- However, for very high assets price, the buy region does not exist any more. Financially, it means that for very high assets price, it is no longer optimal to invest in the assets and it is preferable to distribute dividend as if investment opportunities no longer exist.

- The sell region appears as the firm's cash reserve gets close to zero. Indeed, the firm has to make a disinvestment decision in order to inject cash into its balance, therefore avoiding bankruptcy.

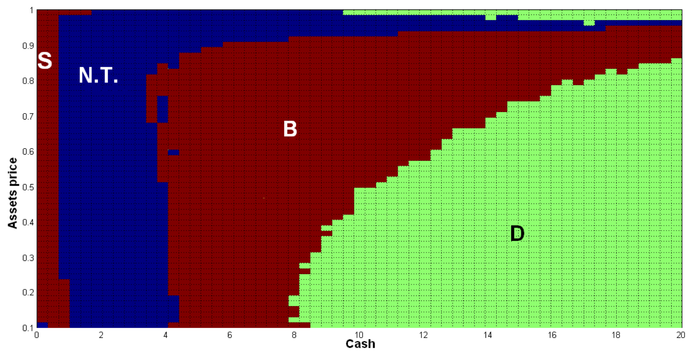


FIGURE 2.2 – *Description of different regions, in (x, s) for a fixed q_0.*

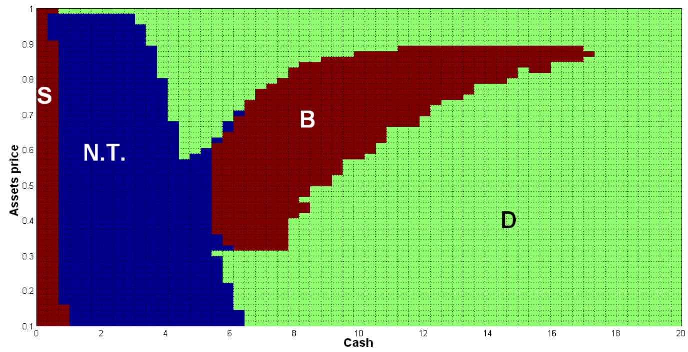


FIGURE 2.3 – *Description of different regions, in (x, s) for $q_1 > q_0$.*

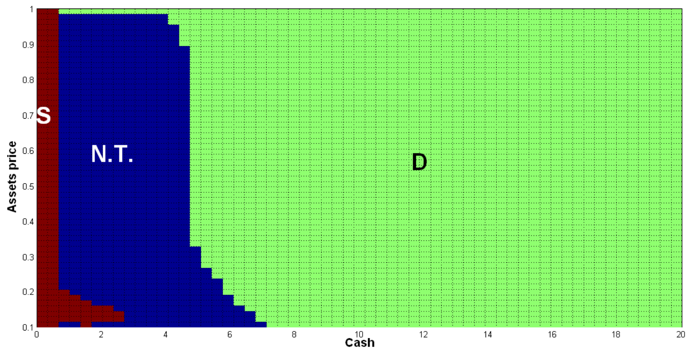


FIGURE 2.4 – *Description of different regions, in (x, s) for $q_2 > q_1$.*

Chapitre 3

Optimal capital structure

The content of this chapter is based on :

(7) An Optimal Dividend and Investment Control Problem under Debt Constraints, with V. Ly Vath and S. Scotti, *SIAM Journal on Financial Mathematics*, Vol. 4, No. 1, 297-326, 2013.

(14) An optimal capital structure control problem under uncertainty, with E. Bayraktar and V. Ly Vath, working paper.

The capital structure of a firm has two main aspects that are interconnected : the firm's liabilities at the one hand and its assets on the other hand. The main components of liabilities are shareholder's equity and debt. Since the pionneering work of Modigliani-Miller (see [92]), a large part of the literature in corporate finance is devoted to optimal capital structure and especially to the management of liabilities. To simplify and summarize the problem, debt value depends on the firm's capital structure and on its dividend policy which have strong impact on shareholder's equity and on the probability of the firm's bankruptcy. However, it is necessary to know the debt value to determine optimal capital structure and dividend policy.
In the first section, we assume that the debt value is exogeneously determined and compute an optimal control on the dividend and investment policy of a firm. We allow the company to make investment by increasing its outstanding indebtedness, which would impact its capital structure and risk profile, thus resulting in higher interest rate debts. Moreover, a high level of debt is also a challenging constraint to any firm as it is no other than the threshold below which the firm value should never go to avoid bankruptcy. It is equally possible for the firm to divest parts of its business in order to decrease its financial debt owed to creditors. In addition, the firm may favor investment by postponing or reducing any dividend distribution to shareholders. We formulate this problem as a combined singular and multi-switching control problem and use a viscosity solution approach to get qualitative descriptions of the solution. We further enrich our studies with a complete resolution of the problem in the two-regime case.
In the second section, we study the capital structure problem for a bank. We assume that the debt, composed of the clients' deposit, has a stochastic dynamic which is not controlled. We no longer neglect assets management to determine the optimal capital structure under solvency constraints. The managers of the bank may invest in either risky assets or in risk-free assets. The objective of the manager is to optimize the bank shareholders' value, ie. the cumulative dividend distributed over the life time of the bank while controlling its solvability. Indeed, the bank is considered to operate

under an uncertain environment and is obliged to respect a number of constraints, in particular solvency ratio constraints as defined under the Basle frameworks. We allow the bank to seek recapitalization or to issue new capital should they fall under financial difficulties. We formulate this problem as a combined impulse control, regular and singular control problem. We will see how this bi-dimensional control problem may be reduced to a one-dimensional one and how quasi-explicit solution may be obtained.

3.1 Optimal dividend and investment under debt constraints

We aim at determining the optimal control on the dividend and investment policy of a firm under debt constraints. As in the Merton model, we consider that firm value follows a geometric Brownian process and more importantly we consider that the firm carries a debt obligation in its balance sheet. However, as in most studies, we still assume that the firm assets is highly liquid and may be assimilated to cash equivalents or cash reserve. We allow the company to make investment and finance it through debt issuance/raising, which would impact its capital structure and risk profile. This debt financing results therefore in higher interest rate on the firm's outstanding debts. More precisely, we model the decisions to raise or redeem some debt obligations as switching decisions controls where each regime corresponds to a specific debt level.

Furthermore, we consider that the manager of the firm works in the interest of the shareholders, but only to a certain extent. Indeed, in the objective function, we introduce a penalty cost P and assume that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy. This penalty cost could represent, for instance, an estimated cost of the negative image upon his/her own reputation due to the bankruptcy under his management leadership. Mathematically, we formulate this problem as a combined singular and multiple-regime switching control problem. Each regime corresponds to a level of debt obligation held by the firm. The studies that are most relevant to our problem are the one investigating combined singular and switching control problems. Recently an interesting connection between the singular and the switching problems was given by Guo and Tomecek [60]. In [85], the authors studied an optimal dividend problem with reversible technology switching investment and used Bachelier process to model the firm's cash reserve. The firm may decide to switch from an old technology to a new technology in order to increases the drift of the cash without affecting the volatility. They proved that the problem can be decoupled in two pure optimal stopping and singular control problems and provided results which are of quasi-explicit nature.

However, none of the above papers on dividend and investment policies, which provides qualitative solutions, has yet moved away from the basic Bachelier model or the simplistic assumption that firms hold no debt obligations. In our model, unlike [86], switching from one regime, i.e. debt level, to another directly impacts the state process itself. Indeed, the drift of the stochastic differential equation governing the firm value would equally switch as the results of the change in interest rate paid on the outstanding debt. A given level of debt is no other than the threshold below which the firm value should never go to avoid bankruptcy. As such, debt level switching also signifies a change of default constraints on the state process in our optimal control problem. Further original contributions in terms of financial studies of our paper include the feature of the conflicts of interest for firm manager through the presence of the penalty cost in the event of bankruptcy. Studying a mixed singular and multi-switching problem combining with the above financial features including debt constraints and penalty cost turns out to be a major mathematical challenge, especially when our objective is to provide quasi-explicit solutions. In addition, it is always tricky to overcoming the combined difficulties of the singular control with those of the switching control, especially when there are multiple regimes, for instance, building a strict supersolution to our HJB system in order to prove the comparison principle.

3.1.1 The model

We consider an admissible control strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, where the nondecreasing càd-làg process Z represents the dividend policy, the nondecreasing sequence of stopping times (τ_n) the switching regime time decisions, and (k_n), which are $\mathcal{F}_{\tau_n}$-measurable valued in $\{1, ..., N\}$, the new value of debt regime at time $t = \tau_n$. Let denote the process $X^{x,i,\alpha}$ as the cash reserve of the firm with initial value of x and initially operating with a debt level D_i and which follow the control strategy α.

We assume that the cash-reserve process, denoted by X when there is no ambiguity, and associated to a strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, is governed by the following stochastic differential equation :

$$dX_t = bX_t dt - r_{I_t} D_{I_t} dt + \sigma X_t dW_t - dZ_t + dK_t \tag{3.1.1}$$

where $I_t = \sum_{n\geq 0} k_n 1_{\tau_n \leq t < \tau_{n+1}}$, $I_{0^-} = i$, $k_n \in \mathbb{I}_N := \{1, ..., N\}$. $(D_i)_{i\in\mathbb{I}_N}$ and $(r_i)_{i\in\mathbb{I}_N}$ represent respectively increasing levels of debt and their associated increasing interest rates paid on those debts.

The process K represnts the cash-flow due to the change in the firm's indebtedness and satisfyies : $K_t = \sum_{n\geq 0} \left(D_{\kappa_{n+1}} - D_{\kappa_n} - g\right) 1_{\tau_{n+1}\leq t}$ where g represents the additional cost associated with the change of firm's level of debt.

For a given control strategy α, the bankruptcy time is represented by the stopping time T^α defined as

$$T^\alpha = \inf\{t \geq 0, X_t^{x,i,\alpha} \leq D_{I_t}\}. \tag{3.1.2}$$

We equally introduce a penalty or a liquidation cost $P > 0$, in the case of a holding company looking to liquidate one of its own affiliate or activity. In the case of the penalty, it mainly assumes that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy.

We define the value functions which the manager actually optimizes as follows

$$v_i(x) \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{(i,x)} \left[\int_0^{T^-} e^{-\rho t} dZ_t - P e^{-\rho T} \right], \quad x \in \mathbb{R},\ i \in \{1, ..., N\}, \tag{3.1.3}$$

where $\mathcal{A}$ represents the set of admissible control strategies, and ρ the discount rate.

3.1.2 Viscosity characterization of the value functions

Using the dynamic programming principle, we obtain the main result

Theorem 3.1.1 *The value functions v_i, $i \in \mathbb{I}_N$, are continuous on $[D_i, \infty)$, and are the unique viscosity solutions on (D_i, ∞) with linear growth condition and boundary data $v_i(D_i) = -P$, to the system of variational inequalities :*

$$\min\left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x) - 1\ ,\ v_i(x) - \max_{j\neq i} v_j(x + D_j - D_i - g)\right] = 0,\ x > D_i. \quad (3.1.4)$$

where $\mathcal{A}_i$ is defined by $\mathcal{A}_i\phi = \mathcal{L}_i\phi - \rho\phi$, and $\mathcal{L}_i\varphi = [bx - r_i D_i]\varphi'(x) + \frac{1}{2}\sigma^2 x^2 \varphi''(x)$.

Actually, we obtain some more regularity results on the value functions.

Proposition 3.1.2 *The value functions v_i, $i \in \mathbb{I}_N$, are C^1 on (D_i, ∞). Moreover, if we set for $i \in \mathbb{I}_N$:*

$$\begin{aligned}
\mathcal{S}_i &= \left\{x \geq D_i\ , v_i(x) = \max_{j\neq i} v_j(x + D_j - D_i - g),\right\} && (3.1.5)\\
\mathcal{D}_i &= \overline{\operatorname{int}\left(\{x \geq D_i\ , v_i'(x) = 1\}\right)}, && (3.1.6)\\
\mathcal{C}_i &= (D_i, \infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i), && (3.1.7)
\end{aligned}$$

then v_i is C^2 on the open set $\mathcal{C}_i \cup \operatorname{int}(\mathcal{D}_i) \cup \operatorname{int}(\mathcal{S}_i)$ of (D_i, ∞), and we have in the classical sense

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i.$$

$\mathcal{S}_i$, $\mathcal{D}_i$, and $\mathcal{C}_i$ respectively represent the switching, dividend, and continuation regions when the outstanding debt is at regime i.

3.1.3 Qualitative results on the dividend and switching regions

For $i,\ j \in \mathbb{I}_N$ and $x \in [D_i, +\infty)$, we introduce some notations :

$$\delta_{i,j} = D_j - D_i, \quad \Delta_{i,j} = (b - r_j)D_j - (b - r_i)D_i \quad \text{and} \quad x_{i,j} = x + \delta_{i,j} - g.$$

We set $x_i^* = \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\}$ for all $i \in \mathbb{I}_N$

We equally define $\mathcal{S}_{i,j}$ as the switching region from debt level i to j.

$$\mathcal{S}_{i,j} = \{x \in (D_i,\ +\infty), v_i(x) = v_j(x_{i,j})\}.$$

From the definition (3.1.5) of the switching regions, we have the following elementary decomposition property :

$$\mathcal{S}_i = \cup_{j\neq i}\mathcal{S}_{i,j},\ i \in \mathbb{I}_N.$$

In the following Lemma, we state that there exists a finite level of cash such that it is optimal to distribute dividends up to this level.

Lemma 3.1.3 *For all $i \in \mathbb{I}_N$, we have $x_i^* := \sup\{x \in [D_i, +\infty) : \; v_i'(x) > 1\} < +\infty$.*

We now establish an important result in determining the description of the switching regions. The following Theorem states that it is never optimal to expand its operation, i.e. to make investment, through debt financing, should it result in a lower "drift" ($(b-r_i)D_i$) regime. However, when the firm's value is low, i.e. with a relatively high bankruptcy risk, it may be optimal to make some divestment, i.e. sell parts of the company, and use the proceedings to lower its debt outstanding, even if it results in a regime with lower "drift". In other words, to lower the firm's bankruptcy risk, one should try to decrease its volatility, i.e. the diffusion coefficient. In our model, this clearly means making some debt repayment in order to lower the firm's volatility, i.e. σX_t.

Theorem 3.1.4 *Let $i, j \in \mathbb{I}_N$ such that $(b-r_j)D_j > (b-r_i)D_i$. We have the following results :*

1) $x_j^* \notin \mathcal{S}_{j,i}$ and $\mathring{\mathcal{D}}_j = (x_j^*, \; +\infty)$.

2) $\mathring{\mathcal{S}}_{j,i} \subset (D_j + g, \; x_j^*)$. Furthermore, if $D_j < D_i$, then $\mathring{\mathcal{S}}_{j,i} = \emptyset$.

From the above Theorem, we may obtain the two following results on the determination of the different strategies. We will see in the next section how, we may deduce from these results, complete description of optimal strategies in the two regime case.

Corollary 3.1.5 *Let $m \in \mathbb{I}_N$ such that $(b - r_m)D_m = \max_{i \in \mathbb{I}_N}(b - r_i)D_i$.*

1) $x_m^* \notin \mathcal{S}_m$ and $\mathring{\mathcal{D}}_m = (x_m^*, \; +\infty)$.

2) For all $i \in \mathbb{I}_N - \{m\}$, we have :

i) If $D_m < D_i$, $\mathring{\mathcal{S}}_{m,i} = \emptyset$.

ii) If $D_i < D_m$, $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, \; x_m^*)$. Furthermore, if $b \geq r_i$, then $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g, \; (a_i^* + \delta_{i,m} + g) \wedge x_m^*)$, where a_i^* is the unique solution of the equation $\rho v_i(x) = (bx - r_i D_i)v_i'(x)$. We further have $a_i^* \neq x_i^*$.

We now turn to the following results ordering the left-boundaries $(x_i^*)_{i \in \mathbb{I}_N}$ of the dividend regions $(\mathcal{D}_i)_{i \in \mathbb{I}_N}$.

Proposition 3.1.6 *Consider $i, j \in \mathbb{I}_N$, such that $(b - r_i)D_i < (b - r_j)D_j$. We always have $x_i^* + \delta_{i,j} - g \leq x_j^*$ unless there exists a regime k such that $(b - r_j)D_j < (b - r_k)D_k$ and $x_i^* \in \mathcal{S}_{i,k}$, then we have $x_j^* - \delta_{i,j} + g < x_i^* < x_k^* - \delta_{i,k} + g$.*

3.1.4 The two regime-case

Throughout this section, we now assume that $N = 2$, in which case, we will get a complete description of the different regions. We will see that the most important parameter to consider is the so-called "drifts" $((b - r_i)D_i)_{i=1,2}$ and in particular their relative positions. To avoid cases with trivial solution, i.e. immediate consumption, we will assume that $-\rho P < (b - r_i)D_i, i = 1, 2$. Throughout Theorem 3.1.7 and Theorem 3.1.8, we provide a complete resolution to our problem in each case.

Theorem 3.1.7 *We assume that $(b - r_2)D_2 < (b - r_1)D_1$.*
We have

$$\mathcal{C}_1 = [D_1, x_1^*), \; \mathcal{D}_1 = [x_1^*, \; +\infty), \; \text{and } \mathring{\mathcal{S}}_1 = \emptyset \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

1) If $\mathcal{S}_2 = \emptyset$ *then we have*

$$\mathcal{C}_2 = [D_2, x_2^*), \text{ and } \mathcal{D}_2 = [x_2^*, \ +\infty) \text{ where } \rho v_2(x_2^*) = b x_2^* - r_2 D_2.$$

2) If $\mathcal{S}_2 \neq \emptyset$ *then there exists* y_2^* *such that* $\mathcal{S}_2 = [y_2^*, \ +\infty)$ *and we distinguish two cases*

a) If $x_2^* + \delta_{2,1} - g < x_1^*$*, then* $y_2^* > x_2^*$*,* $y_2^* = x_1^* + \delta_{1,2} + g$ *and*

$$\mathcal{C}_2 = [D_2, x_2^*), \text{ and } \mathcal{D}_2 = [x_2^*, \ +\infty) \text{ where } \rho v_2(x_2^*) = b x_2^* - r_2 D_2.$$

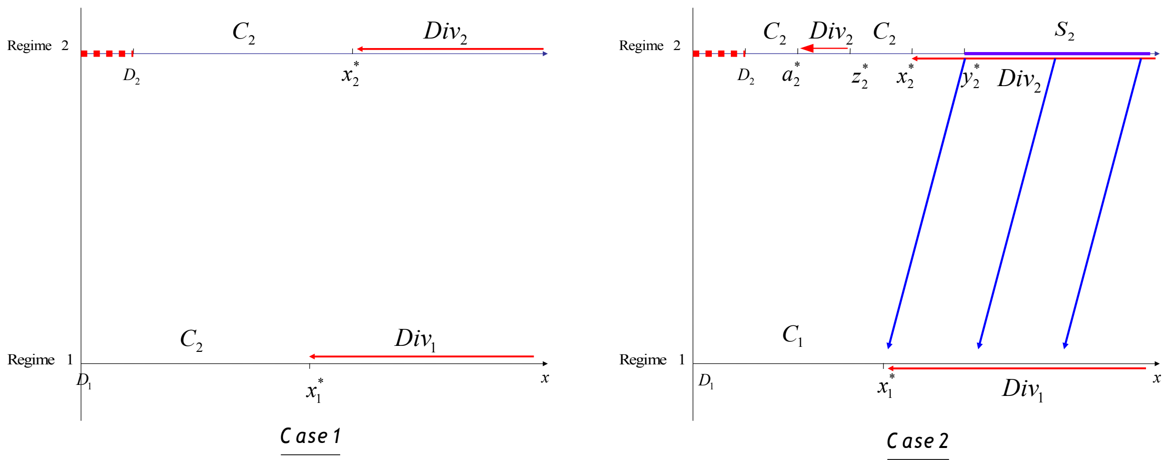
b) If $x_2^* + \delta_{2,1} - g = x_1^*$ *then* $y_2^* \leq x_2^*$*,* $\rho v_2(x_2^*) = b x_2^* - r_2 D_2 + \Delta_{2,1} - bg$*.*
We define a_2^* *as the solution of* $\rho v_2(a_2^*) = b a_2^* - r_2 D_2$ *and have two cases*
i) If $a_2^* \notin \mathcal{D}_2$*, we have*

$$\mathcal{D}_2 = [x_2^*, \ +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2, \ y_2^*).$$

ii) If $a_2^* \in \mathcal{D}_2$*, there exists* $z_2^* \in (a_2^*, y_2^*)$ *such that*

$$\mathcal{D}_2 = [a_2^*, \ z_2^*] \cup [x_2^*, \ +\infty) \quad \text{and} \quad \mathcal{C}_2 = [D_2, \ a_2^*) \cup (z_2^*, \ y_2^*).$$

FIGURE 3.1 – Switching regions : case $(b - r_1)D_1 > (b - r_2)D_2$.



We now turn to the case where $(b - r_1)D_1 < (b - r_2)D_2$.

Theorem 3.1.8 *We assume that* $(b - r_1)D_1 < (b - r_2)D_2$*,*
1) we have

$$\begin{aligned} \mathcal{D}_2 &= [x_2^*, \ +\infty) \text{ where } \rho v_2(x_2^*) = b x_2^* - r_2 D_2 \\ \mathring{\mathcal{S}}_2 &= \emptyset \text{ or there exist } s_2^*, S_2^* \in (D_2 + g, \ x_2^*) \text{ such that } \mathring{\mathcal{S}}_2 = (s_2^*, S_2^*). \end{aligned}$$

2) If $\mathring{\mathcal{S}}_1 = \emptyset$ *then we have*

$$\mathcal{C}_1 = [D_1, x_1^*), \text{ and } \mathcal{D}_1 = [x_1^*, \ +\infty) \text{ where } \rho v_1(x_1^*) = b x_1^* - r_1 D_1.$$

3) If $\mathring{\mathcal{S}}_1 \neq \emptyset$ *there exists* y_1^* *such that* $\mathring{\mathcal{S}}_1 = (y_1^*, \ +\infty)$
a) If $x_1^* + \delta_{1,2} - g < x_2^*$*, then* $y_1^* > x_1^*$*,* $y_1^* = x_2^* + \delta_{2,1} + g$ *and*

$$\mathcal{C}_1 = [D_1, x_1^*), \text{ and } \mathcal{D}_1 = [x_1^*, \ +\infty) \text{ where } \rho v_1(x_1^*) = b x_1^* - r_1 D_1.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$, then $y_1^* \le x_1^*$, $\rho v_1(x_1^*) = bx_1^* - r_1 D_1 + \Delta_{1,2} - bg$. We define a_1^* as the solution of $\rho v_1(a_1^*) = ba_1^* - r_1 D_1$ and have two cases.

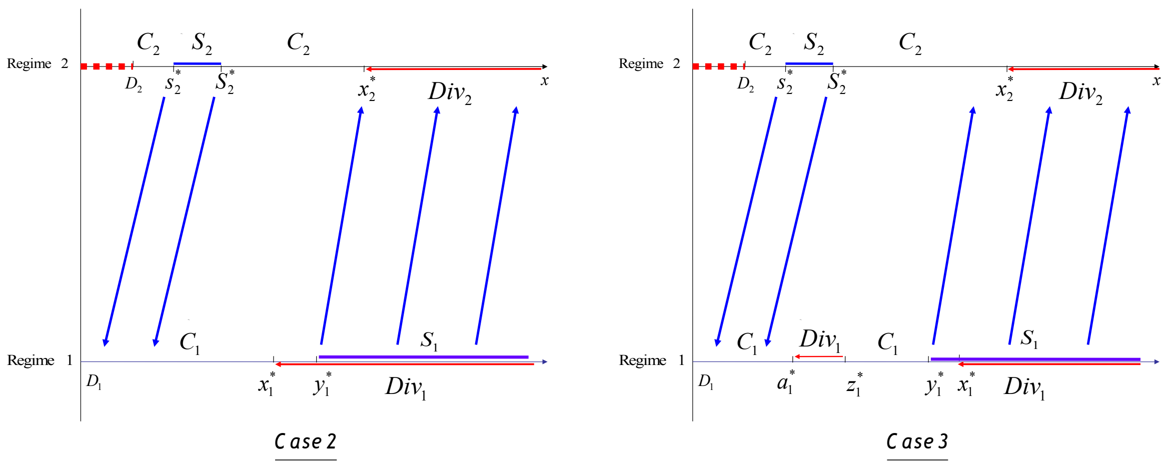
i) If $a_1^* \notin \mathcal{D}_1$, we have

$$\mathcal{D}_1 = [x_1^*,\ +\infty) \quad and \quad \mathcal{C}_1 = [D_1,\ y_1^*).$$

ii) If $a_1^* \in \mathcal{D}_1$, there exists $z_1^* \in (a_1^*, y_1^*)$ such that

$$\mathcal{D}_1 = [a_1^*,\ z_1^*] \cup [x_1^*,\ +\infty) \quad and \quad \mathcal{C}_1 = [D_1,\ a_1^*) \cup (z_1^*,\ y_1^*).$$

FIGURE 3.2 – Switching regions : case $(b - r_1)D_1 < (b - r_2)D_2$.



3.2 Capital structure optimization under constraints

We investigate the problem of determining an optimal control on the capital structure, dividend and investment policy of a bank operating under solvability constraints. We assume that the bank collects deposits from its customers and pays interest on their deposits. We may assume that the bank's liabilities consist of both clients' deposits and shareholders' equity.

The primary objective of the bank is to use customers' deposit and its equity to make investments while controlling its solvency and liquidity risk. We assume that the manager of the bank may invest either in risky assets or in risk-free assets. The bank is considered to operate under an uncertain financial and economic environment and is obliged to respect a number of constraints, in particular solvency constraints as defined under the Basel frameworks. One such constraint is the capital adequacy constraint. A ratio constraint that banks have to satisfy is Tier 1 capital ratio, i.e. the ratio between Tier 1 capital to risk-adjusted assets. Another constraint is the liquidity coverage constraints. The idea is to oblige banks to keep enough cash or equivalent to face its short term financial obligations. The third important constraint worth mentioning is the leverage ratio. The adequate levels of these ratios are under heavy discussion between regulators, in particular under the Basel framework. The on-going rounds of discussion concern the Basel III agreements which are scheduled to be implemented over the next few years. The implementation of stricter financial ratios may drive many financial institutions to seek recapitalization.

Within this regulatory context, in our study, we allow the bank to seek recapitalization or to issue new capital should they fall under financial difficulties. As such, the company will have to either raise new equity or reduce its exposure to risky investment in the case that its financial strengths deteriorate and capital adequacy ratios decrease towards the minimum threshold established by the regulators.

The objective of the manager is to optimize the bank shareholders' value, ie. the cumulative dividend distributed over the life time of the company while controlling its solvability. We formulate this problem as a combined impulse and singular control problem.

3.2.1 Bank capital structure

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation c, i.e. $d[W,B]_t = cdt$ for all t.

We consider a bank and denote by L_t the amount of its liabilities which correspond to customers deposits at time t. We assume that the process L is governed by the following S.D.E.

$$\begin{cases} dL_t &= L_t\left(\mu_{_L} dt + \sigma_{_L} dW_t\right), \\ L_{_0} &= l, \end{cases}$$

where $\sigma_{_L}$ is a positive constant and $\mu_{_L} := \gamma + r_{_L}$, with $\gamma \in \mathbb{R}$ being the growth rate of the bank portfolio and $r_L \geq 0$ the interest rate paid by the bank to its clients. The bank may invest in a risk-free asset with a constant interest rate $r > 0$ or in a risky asset whose value process S is solution to the following S.D.E.

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right), \tag{3.2.1}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$.

We denote by X_t the total wealth of the assets held by the bank at time t, by π_t the proportion of this wealth invested in the risky asset. Obviously $(1-\pi_t)X_t$ is the amount of money invested in the risk-free asset and $\pi_t X_t$ is the amount of money invested in the risky asset by the bank at time t. Notice that from the balance sheet of the bank, we have

$$X_t = F_t + L_t \quad \forall t \geq 0,$$

where F_t correspond to shareholders' equity at time t.
The manager of the bank controls the assets allocation of capital between risk-free and risky assets and she controls bank capital through issues of new capital or dividend payments. We then consider a control strategy $\widehat{\alpha} = ((\tau_n)_{n\in\mathbb{N}^*}, (\widehat{\xi}_n)_{n\in\mathbb{N}^*}, \widehat{Z}, \pi)$, where the $\mathbb{F}$-adapted cád-lág nondecreasing process $\widehat{Z}$ represents the total amount of dividend distributed, with $\widehat{Z}_{0^-} = 0$. The nondecreasing sequence of stopping times (τ_n) represents the decisions time at which the manager decides to issue new capital, and $\widehat{\xi}_n$, which is $\mathcal{F}_{\tau_n^-}$-measurable valued in $(0, +\infty)$, the amount of capital issue at (τ_n). The process π is the proportion of the whole wealth invested in the risky asset. The equity process associated to a control $\widehat{\alpha}$ has then the following dynamic :

$$\begin{cases} dF_t &= -r_{_L} L_t dt - d\widehat{Z}_t + (1-\pi_t)X_t r dt + \pi_t X_t\left(\mu dt + \sigma dB_t\right) \quad \text{for } \tau_i < t < \tau_{i+1} \\ F_{\tau_i} &= (1-\kappa)F_{\tau_i^-} + (1-\kappa')\widehat{\xi}_i \end{cases}$$

where κ', $\kappa > 0$ are fixed proportional costs to issue new capital. More precisely, when issuing capital at time t, we assume that one has to pay a cost proportional to the capital issued and κF_{t^-} is the cost due to compensation for existing (prior to the issue of capital) shareholders (against dilution). We also should assume that $\widehat{\xi}_i$ is big enough to insure that $F_{\tau_i} > F_{\tau_i^-}$ i.e. $\widehat{\xi}_i > \frac{\kappa}{1-\kappa'} F_{\tau_i^-}$. Notice that if it is not the case, the manager had better avoid costs and distribute dividend to the shareholders. It follows that the wealth process X has the following dynamic, for $\tau_i < t < \tau_{i+1}$,

$$dX_t = ((1-\pi_t)rX_t + \pi_t\mu X_t + \gamma L_t)\,dt + \pi_t\sigma X_t dB_t + \sigma_L L_t dW_t - d\widehat{Z}_t.$$

and

$$X_{\tau_i} = X_{\tau_i^-} + (1-\kappa')\widehat{\xi}_i - \kappa F_{\tau_i^-}.$$

We first define the most basic bankruptcy time by

$$T = \inf\{t \geq 0 : \ F_t < 0\}$$

We assume that when the liquidation time is reached, the bank stops its activity and goes bankrupt as there is no equity left. Given an initial liability level $l > 0$ and an initial wealth $x > 0$, the equity value of the bank under policy $\widehat{\alpha}$ to its shareholders may then be defined as

$$J^{\widehat{\alpha}}(l,x) = \mathbb{E}_{l,x}\Big[\int_0^{T^{\widehat{\alpha}}} e^{-\rho t} d\widehat{Z}_t - \sum_{n=1}^{+\infty} e^{-\rho\tau_n}(\widehat{\xi}_n - \kappa F_{\tau_n^-})\mathbb{1}_{\{\tau_n \leq T^{\widehat{\alpha}}\}}\Big],$$

However, in order to take into account the specific characteristics of the banking sector, we now have to introduce some regulatory contraints that the bank has to satisfy. The first constraint is the solvency ratio. The solvency ratio reflects the ability of the bank to bear losses without defaulting on its obligations in term of remuneration and repayment of the collected resources. The solvency ratio is calculated by dividing the bank's capital by the aggregate of its risky assets. In our problem, it corresponds to the ratio between shareholders' equity and its risky investments

$$\frac{F_t}{\pi_t X_t} > a_1 \text{ i.e. } 1 - \frac{1}{Y_t} > a_1\pi_t,$$

where we have set $Y_t = \frac{X_t}{L_t}$
Another important constraint that we are considering in this paper is the Liquidity Coverage Ratio (LCR), which is defined as the ratio between High Quality Liquid Assets (HQLA) and cash outflow during 30 days. In our model, cash outflows comprises two components

- run-off of proportion of retail deposits (around 3 percents)
- the potential loss on risky investments.

$$\frac{(1-\pi_t)X_t}{a_2 L_t + a_3\pi_t X_t} > 1 \text{ for all } t \geq 0, \text{i.e.} 1 - \frac{a_2}{Y_t} > (1+a_3)\pi_t,$$

where a_1, a_2 and a_3 are positive and lower than 1.
At this point, we introduce the function $\overline{\pi}$ defined on $[1, +\infty)$ by

$$\overline{\pi}(y) = \min\left(\frac{1}{a_1}(1-\frac{1}{y});\ \frac{1}{1+a_3}(1-\frac{a_2}{y})\right).$$

The set of admissible policies, denoted by $\widehat{\mathcal{A}}$, is then defined by

$$\widehat{\mathcal{A}} = \{\widehat{\alpha} = ((\tau_n)_{n\in\mathbb{N}^*}, (\widehat{\xi}_n)_{n\in\mathbb{N}^*}, \widehat{Z}, \pi) : \forall 0 \le t \le T^{\widehat{\alpha}},\ 0 \le \pi_t \le \overline{\pi}(\frac{X_t}{L_t}),\ \forall n \ge 0 : \ \widehat{\xi}_n > \frac{\kappa}{1-\kappa'} F_{\tau_n^-}\}.$$

Hence, our objective function is defined by

$$\widehat{v}(l,x) = \sup_{\widehat{\alpha}\in\widehat{\mathcal{A}}} J^{\widehat{\alpha}}(l,x) \quad \text{for } (l,x) \in \mathcal{S} := \{(l,x) \in [0,+\infty)^2 : x \ge l\}. \qquad (3.2.2)$$

We now state a result which transforms our initial constraint bi-dimensional problem into a uni-dimensional control problem.

Theorem 3.2.1 *Let $\alpha := ((\tau_n)_{n\in\mathbb{N}^*}, (\xi_n)_{n\in\mathbb{N}^*}, Z, \pi)$ where $(\tau_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of stopping times going to $+\infty$, $(\xi_n)_{n\in\mathbb{N}^*}$ a sequence of $\mathcal{F}_{\tau_n^-}$-measurable and positive variable and Z an increasing process. We define the process Y^α as a solution of the following stochastic differential equation :*

$$\begin{cases} dY_t^\alpha &= \left(Y_t^\alpha\left[\mu(\pi_t) - \mu_L\right] + \gamma\right) dt \\ & \quad +\pi_t Y_t^\alpha \sigma dB_t + \sigma_L(1 - Y_t^\alpha) dW_t - dZ_t \ \textit{for } \tau_n < t < \tau_{n+1} \\ & \\ Y_{\tau_n} &= (1-\kappa) Y_{\tau_n^-} + (1-\kappa')\xi_n + \kappa, \end{cases}$$

where $\mu(\pi) = (1-\pi)r + \pi\mu$.
We also define the stopping time $T^\alpha = \inf\{t \ge 0 : \ Y_t^\alpha < 1\}$. We have

$$\widehat{v}(l,x) = l v(\frac{x}{l}), \quad \textit{for all } l > 0 \textit{ and } x \ge l,$$

where

$$v(y) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}_y\Big[\int_0^{T^\alpha} e^{-\rho_L t} dZ_t - \sum_{n=1}^{+\infty} e^{-\rho_L \tau_n}(\xi_n - \kappa(Y_{\tau_n^-}^\alpha - 1))\mathbb{1}_{\{\tau_n \le T^\alpha\}}\Big],$$

with $\rho_L = \rho - \mu_L$ and the set $\mathcal{A}$ is defined as follows

$$\mathcal{A} = \{\alpha = ((\tau_n)_{n\in\mathbb{N}^*}, (\xi_n)_{n\in\mathbb{N}^*}, Z, \pi) : \ \forall t \ge 0 : \ 0 \le \pi_t < \overline{\pi}(Y_t^\alpha),\ \forall n \ge 0 : \ \xi_n > \frac{\kappa}{1-\kappa'}(Y_{\tau_n^-}^\alpha - 1)\}.$$

3.2.2 Analytical properties of the value function

The main result of this section is the characterization of the function v as the unique viscosity solution of the following HJB equation :

$$\begin{aligned} 0 &= \min\{\rho_L v(y) - \sup_{0\le\pi\le\overline{\pi}(y)} \mathcal{L}^\pi v(y); v'(y) - 1; v(y) - \mathcal{H}v(y)\}, \\ 0 &= \max\left(v(1),\ v(1) - \mathcal{H}v(1)\right). \end{aligned}$$

where we have set

$$\begin{aligned} \mathcal{L}^\pi \varphi(y) &= \left(\pi^2\sigma^2 y^2 + \pi c \sigma \sigma_L y(1-y) + \sigma_L^2(1-y)^2\right)\varphi'' + \left(y\left[\mu(\pi) - \mu_L\right] + \gamma\right)\varphi' \\ \text{where} \quad & \mu(\pi) = (1-\pi)r + \pi\mu \\ & \\ \mathcal{H}\varphi(y) &= \sup_{\xi > \frac{\kappa}{1-\kappa'}(y-1)} \Big[v((1-\kappa)y + (1-\kappa')\xi + \kappa) - \xi + \kappa(y-1)\Big]. \end{aligned}$$

We start with making assumptions on parameters to avoid trivial cases.

Proposition 3.2.2 *If* $\rho < \max(\mu_L, r, \frac{\mu + a_3 r}{1+a_3})$*, we have* $v(y) = +\infty$ *on* $[1, +\infty)$

Throughout the end of the section, we will assume that

$$\rho > \max(\mu_L, r, \frac{\mu + a_3 r}{1 + a_3}). \tag{3.2.3}$$

Lower and upper bounds for the value function

We first introduce some notations by setting

$$\hat{y} = \frac{1 + a_3 - a_1 a_2}{1 + a_3 - a_1},$$

and the optimal strategy and drift :

$$\pi^*(y) = \overline{\pi}(y) \mathbb{1}_{\{\mu \geq r\}} \text{ and } \mu^*(y) := r + \pi^*(y)(\mu - r) = \max\{(1-\pi)r + \pi\mu : 0 \leq \pi \leq \overline{\pi}(y)\}.$$

Notice that for an initial state $y \geq 1$, one can distribute dividend up to bankruptcy then we obviously have

$$v(y) \geq y - 1, \quad \text{for } y \geq 1. \tag{3.2.4}$$

Now, we will construct an upper bound for v. This will rely on the following result.

Proposition 3.2.3 *Let* $\varphi \in \mathcal{C}^2([1, +\infty))$ *such that* $\max(\varphi(1),\ \varphi(1) - \mathcal{H}\varphi(1)) \geq 0$ *and*

$$\min\left[\rho_L \varphi(y) - \sup_{\pi \in [0, \overline{\pi}(y)]} \mathcal{L}^\pi \varphi(y); \varphi'(y) - 1; \varphi(y) - \mathcal{H}\varphi(y)\right] \geq 0, \quad \text{for any } y > 1 \tag{3.2.5}$$

then we have $v \leq \varphi$ *on* $[1, +\infty)$.

Corollary 3.2.4 *Let* $y \in [1, +\infty)$*. We have*

$$y - 1 \leq v(y) \leq y + \frac{1}{\rho_L} max\left(-\rho_L, A + \gamma, B + \gamma\right),$$

where we have set

$$\begin{aligned} A &:= \left(r + \frac{(\mu - r)^+}{1 + a_3} - \rho\right)\hat{y} - \frac{a_2(\mu - r)^+}{1 + a_3} \\ B &:= \left(r + \frac{(\mu - r)^+}{a_1} - \rho\right)^+ \hat{y} - \left(r + \frac{(\mu - r)^+}{a_1} - \rho\right)^- - \frac{(\mu - r)^+}{a_1} \end{aligned}$$

Especially, if $-\rho_L \geq max(A, B) + \gamma$, $v(y) = y - 1$ *and the optimal policy is to immediately distribute dividends up to bankruptcy.*

At this point, we will assume that the parameters satisfy :

$$\rho > \max(\mu_L, r, \frac{\mu + a_3 r}{1 + a_3}) \quad \text{and} \quad -\rho_L < \max(A, B) + \gamma. \tag{3.2.6}$$

Viscosity characterization of the value function

Theorem 3.2.5 *The value function v is the unique continuous function on $[1,+\infty)$, which satisfies a linear growth condition and is a viscosity solution on $(1,+\infty)$ to the following variational inequality :*

$$\begin{cases} \min\{\rho_L v(y) - \sup_{0\le\pi\le\bar\pi(y)} \mathcal{L}^\pi v(y); v'(y)-1; v(y)-\mathcal{H}v(y)\} = 0, \quad \forall y>1, \\ \max\left(v(1),\ v(1)-\mathcal{H}v(1)\right) = 0 \end{cases}$$

Actually, we obtain some more regularity results on the value functions.

Proposition 3.2.6 *The value function v is C^1 on $(1,+\infty)$. Moreover, if we set :*

$$\begin{aligned} \mathcal{K} &= \{y \ge 1\ , v(y) \ = \ \mathcal{H}v(y)\} && (3.2.7)\\ \mathcal{D} &= \overline{\mathrm{int}\left(\{y\ge 1\ , v'(y)=1\}\right)},, && (3.2.8)\\ \mathcal{C} &= (1,+\infty)\setminus(\mathcal{K}\cup\mathcal{D}) && (3.2.9) \end{aligned}$$

then v is C^2 on the open set $\mathcal{C}\cup\mathrm{int}(\mathcal{D})\cup\mathrm{int}(\mathcal{K})$ of $(1,\infty)$, and we have in the classical sense

$$\rho_L v(y) - \sup_{0\le\pi\le\bar\pi(y)} \mathcal{L}^\pi v(y) \ = \ 0, \quad y\in\mathcal{C}.$$

$\mathcal{K}$, $\mathcal{D}$, and $\mathcal{C}$ respectively represent the capital issuing, dividend, and continuation regions.

3.2.3 Qualitative results on the regions

Proposition 3.2.7 ***Optimal dividend strategy***
The following equation admits a unique solution y^* on $[1,+\infty)$:

$$\rho_L v(y) = \gamma - (\mu_L - \mu^*(y))y. \qquad (3.2.10)$$

We have

$$1 \le y^* < \frac{\rho_L+\gamma}{\rho-\mu^*(y^*)} \qquad \text{and} \qquad \mathcal{D} = [y^*,+\infty).$$

Moreover, v is a concave function on $[1,+\infty)$.

Proposition 3.2.8 ***Optimal capital issuance strategy***

If $\mathcal{K}\neq\emptyset$ then $\mathcal{K}=\{1\}$.

Moreover, if $\mathcal{K}\neq\emptyset$, we have $v'(1^+) > \frac{1}{1-\kappa'}$, then there exists a unique solution ξ^* to the equation $v'(y) = \frac{1}{1-\kappa'}$. We have

$$v(1) = v(\xi^*) - \frac{1}{1-\kappa'}\left(\xi^*-1\right).$$

Chapitre 4

Variable annuities

The content of this chapter is based on :

(8) Max-min optimization problem for variable annuities pricing, with C.Blanchet-Scalliet, I. Kharroubi and T. Lim, *International Journal of Theoritical and Applied Finance,*, Vol. 18, No. 08, 2015

(11) Indifference fees for variable annuities, with T. Lim, et R. Romo Romero, to appear in *Applied Mathematical Finance*

Introduced in the 1970s in the United States (see [105]), variable annuities are equity-linked contracts between a policyholder and an insurance company. The policyholder gives an initial amount of money to the insurer. This amount is then invested in a reference portfolio until a preset date, until the policyholder withdraws from the contract or until he dies. At the end of the contract, the insurance pays to the policyholder or to his dependents a pay-off depending on the performance of the reference portfolio. In the 1990s, insurers included put-like derivatives which provided some guarantees to the policyholder. The most usual are guaranteed minimum death benefits (GMDB) and guaranteed minimum living benefits (GMLB). For a GMDB (resp. GMLB) contract, if the insured dies before the contract maturity (resp. is still alive at the maturity) he or his dependents obtain a benefit corresponding to the maximum of the current account value and of a guaranteed benefit. There exist various ways to fix this guaranteed benefit and we refer to [14] for more details.

These products mainly present three risks for the insurer. First, as the insurer offers a put-like derivative on a reference portfolio to the client, he is considerably exposed to market risk. Moreover, variable annuity policies could have very long maturities so the pricing and hedging errors due to the model choice for the dynamics of the reference portfolio and the interest rates could be very important. The second risk faced by the insurer is the death of his client, this leads to the formulation of a problem with random maturity. Finally, the client may decide at any moment to withdraw, totally or partially, from the contract.

With the commercial success of variable annuities, the pricing and hedging of these products have been studied in a growing literature. Following the pioneering work of Boyle and Schwartz (see [23]), non-arbitrage models allow to extend the Black-Scholes framework to insurance issues. Milvesky and Posner (see [89]) are, up to our knowledge, the first to apply risk neutral option pricing theory to value GMDB. Withdrawal options are studied in [35] and [103], and a general framework to define variable annuities is presented in [14]. Milevsky and Salisbury (see [91]) focus on the links between American put options and dynamic optimal withdrawal policies. This problem is studied in [40] where an HJB equation is derived for a singular control problem where the control is the continuous withdrawal rate. The GMDB pricing problem is described as an impulse control problem in [16]. The authors model the GMDB problem as a stochastic control problem, derive an HJB equation and solve it numerically. The assumptions needed to get these formulations are the Markovianity of the stochastic processes involved and the existence of a risk neutral probability. The variable annuity policies with GMDB and GMLB are long term products therefore models for assets and interest rates have to be as rich as possible. Moreover, as we obviously face an incomplete market model, the price obtained strongly depends on the arbitrary choice of a risk neutral probability.

This chapter attempts to get an answer to these issues. We present a detailled framework, common to our two articles on this subject (see **(8)** and **(11)**), in the first section. In the second section we assume that the insured's withdrawals follow an arbitrary stochastic process. In the last section, we consider the worst case for the insured's withdrawal strategy from the insurer point of view. We begin with a description of our model.

4.1 The model

We shall not make restrictive assumptions on the reference portfolio and the interest rate dynamics. As a result, our problem is not Markovian and we will not be able to derive HJB equations to characterize our value functions. We overcome this difficulty thanks to backward stochastic differential equations (BSDEs) following ideas from [65] and [102].

We shall not use non-arbitrage arguments to price and hedge variable annuities policies. We will assume that the fees, characterized by a preset fee rate, are continuously taken by the insurer from the policyholder's account and we will define an indifference fee rate for the insurer. Indifference pricing is a standard approach in mathematical finance to determine the price of a contingent claim in an incomplete market. This is a utility-based approach that can be summarized as follows. On the one hand, the investor may maximize his expected utility under optimal trading, investing only in the financial market. On the other hand, he could sell the contingent claim, optimally invest in the financial market and make a pay-off at the terminal time. The indifference price of this contingent claim is then the price such that the insurer gets the same expected utility in each case. For more details, we refer to the monograph [28].

Finally, an important risk faced by the seller of a variable annuities contract concerns the characteristics of the buyer. The insurer has to take into account the behavior of the insured, i.e. her withdrawals, and her exit time from the contract, i.e. her death time. Concerning the death time, we allow the death time intensity to be uncertain and to depend, for instance, on fundamental medical breakthroughs or natural disasters. This kind of unpredictable event could impact even large portfolios of policies and therefore this part of mortality risk is not diversifiable (see, for instance, [90]). Moreover, insured's withdrawals strategies may modify the mortality risk profile of the product (see [12] and [13]). Finally, in the case of indifference exponential utility pricing, it has been shown in [19] that diversification may not be consistent. Hence, we model the death time as a random time enlarging the initial filtration related to the market information which is a classical approach in credit risk. As such contracts are generally priced for a class of insured, we suppose that this random time corresponds to the death time of a representative agent in a specific class of clients that satisfy several conditions (age, job, wealth,...). We shall assume that such a class is small enough to be unable to affect the market. From a probabilistic point of view, this justifies that the well known assumption (H) holds true, i.e. any martingale for the initial filtration remains a martingale for the initial filtration enlarged by the exit time. In our case, we then will have to solve BSDEs with random terminal time. For that we apply very recent results on BSDEs with jump (see for example [5] and [72]).

4.1.1 The financial market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F} := (\mathbb{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by B. We consider a financial market on the time interval $[0, T]$ where $T > 0$ corresponds to the expiration date of the variable annuities studied.

Financial assets. We suppose that the financial market is composed by a riskless bond with an interest rate r and a reference portfolio of risky assets underlying the variable annuity policy. The price processes $\hat{S}^0$ of the riskless bond and $\hat{S}$ of a share of the underlying risky portfolio are assumed to be solution of the following linear stochastic differential equations

$$\begin{aligned} d\hat{S}^0_t &= r_t \hat{S}^0_t dt \ , \quad \forall t \in [0,T] \ , \quad \hat{S}^0_0 = 1 \ , \\ d\hat{S}_t &= \hat{S}_t(\mu_t dt + \sigma_t dB_t) \ , \quad \forall t \in [0,T] \ , \quad \hat{S}_0 = s > 0 \ , \end{aligned}$$

where μ, σ and r are bounded and $\mathbb{F}$-adapted processes.
We shall denote by S_t the discounted value of $\hat{S}_t$ at time $t \in [0, T]$, i.e.

$$S_t := e^{-\int_0^t r_s ds} \hat{S}_t \ , \quad \forall t \in [0,T] \ .$$

Insurer's investment strategies and utility function. Assuming that the strategy of the insurer is self-financed and denoting by $X^{x,\pi}_t$ the discounted value of the insurer portfolio at time t with initial capital $x \in \mathbb{R}^+$ and following the strategy π, we have

$$X^{x,\pi}_t = x + \int_0^t \pi_s(\mu_s - r_s)ds + \int_0^t \pi_s \sigma_s dB_s \ , \quad \forall t \in [0,T] \ .$$

If the initial capital is null we denote X^π_t the wealth instead of $X^{0,\pi}_t$.

We consider that the insurer wants to maximize the expected value of the utility of his terminal wealth $U(X^{x,\pi}_T)$ on an admissible strategies set, where

$$U(x) := -\exp(-\gamma x) \text{ with } \gamma > 0.$$

In the following definition, we define the set of admissible strategies for the insurer, making usual restrictions that ensure some integrability properties for the processes involved.

Definition 4.1.1 *($\mathbb{F}$-admissible strategy). Let u and v be two $\mathbb{F}$-stopping times such that $0 \leq u \leq v \leq T$. The set of admissible trading strategies $\mathcal{A}^{\mathbb{F}}[u,v]$ consists of all $\mathbb{F}$-predictable processes $\pi = (\pi_t)_{u\leq t\leq v}$ which satisfy $\mathbb{E}\Big[\int_u^v |\pi_t|^2 dt\Big] < \infty$ and*

$$\Big\{ \exp(-\gamma X^{x,\pi}_\theta),\ \theta \text{ is an } \mathbb{F}\text{-stopping time such that } u \leq \theta \leq v \Big\}$$

is uniformly integrable.

4.1.2 Exit time of the policy

We consider two random times θ^d and θ^w which respectively represent the death time of the insured and the time of early closure of the insured account. We denote by $\tau = \theta^d \wedge \theta^w$. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{G}$ the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time (see e.g. [17, 72]). More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \ := \ \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon} \ ,$$

for all $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathbb{F}_s \vee \sigma(\mathbb{1}_{\tau\leq u} \ , u \in [0,s])$, for all $s \geq 0$.

We impose the following assumptions, which are usual in filtration enlargement theory (see for example [17, Section 6.1.1]).

Hypothesis 4.1.2 ($\mathcal{H}$)-***hypothesis***. *The process B remains a $\mathbb{G}$-Brownian motion.*

In the sequel, we introduce the process H defined by $H = \left(1_{\{\tau\leq t\}}\right)_{0\leq t\leq T}$.

Hypothesis 4.1.3 *The process H admits an $\mathbb{F}$-compensator of the form $\int_0^{\cdot\wedge\tau} \lambda_t dt$, i.e. $H - \int_0^{\cdot\wedge\tau} \lambda_t dt$ is a $\mathbb{G}$-martingale, where λ is a bounded $\mathbb{F}$-adapted process.*

M denotes the $\mathbb{G}$-martingale defined by $M_t := H_t - \int_0^{t\wedge\tau} \lambda_s ds$, for all $t \geq 0$.

If the investment strategy of the insurer depends on this exit time, we shall enlarge the set of admissible strategies through the following definition.

Definition 4.1.4 *($\mathbb{G}$-admissible strategy). Let u and v be two $\mathbb{G}$-stopping times such that $0 \leq u \leq v \leq T$. The set of admissible trading strategies $\mathcal{A}^{\mathbb{G}}[u,v]$ consists of all $\mathbb{G}$-predictable processes $\pi = (\pi_t)_{u\leq t\leq v}$ which satisfy $\mathbb{E}\Big[\int_u^v \left|\pi_t\right|^2 dt\Big] < \infty$ and*

$$\Big\{ \exp(-\gamma X_\theta^{x,\pi}),\ \theta \textit{ is a } \mathbb{G}\textit{-stopping time with values such that } u \leq \theta \leq v \Big\}$$

is uniformly integrable.

4.2 Arbitrary withdrawals process

Throughout this section we shall assume that there is a rate of partial withdrawal that could be stochastic or not but we do not assume that it results from an optimal strategy of the insured as, for example, in [16], [40], [91] or **(9)** which is presented in the next section. In case of total withdrawal, the insured may pay some penalties and will receive the maximum of the account facial value and of a guaranteed benefit

4.2.1 The policy model

Let $\mathbb{T} := (t_i)_{0\leq i\leq n}$ be the set of policy anniversary dates, with $t_0 = 0$ and $t_n = T$.

Discounted account value A^p : The total amount on the account is invested on the market, fees and withdrawals are assumed to be continuously taken from the account therefore the dynamic of the process A^p is as follow

$$dA^p_t = A^p_t\big[(\mu_t - r_t - \xi_t - p)dt + \sigma_t dB_t\big] \;, \quad \forall t \in [0,T] \;,$$

with initial value A_0, p is the fee rate taken by the insurer from the account of the insured and the process ξ is a $\mathbb{G}$-predictable, non-negative and bounded process. ξ_t represents the withdrawal rate chosen by the insured at time $t \in [0,T]$. ξ is then an exogenous process and no additional hypothesis on the policyholder behavior has to be made in this section. We may refer to the [14, Section 3.4] for different policyholder behavior models.

Pay-off of the variable annuities. Let $p \geq 0$, the pay-off is paid at time $T\wedge\tau$ to the insured or his dependents and is equal to the following random variable

$$\hat{F}(p) := \hat{F}^L_T(p)\mathbb{1}_{\{T<\tau\}} + \hat{F}^D_\tau(p)\mathbb{1}_{\{\tau=\theta^d\leq T\}} + \hat{F}^W_\tau(p)\mathbb{1}_{\{\tau=\theta^w<\theta^d\,;\,\tau\leq T\}} \;.$$

$\hat{F}^L_T(p)$ is the pay-off if the policyholder is alive at time T and has not totally withdrawn his money from his account. $\hat{F}^D_\tau(p)$ is the pay-off if the policyholder is dead at time τ. $\hat{F}^W_\tau(p)$ is the pay-off if the policyholder totally withdraws his money from his account at time τ. We suppose that $\hat{F}^L(p)$, $\hat{F}^D(p)$ and $\hat{F}^W(p)$ are bounded, non-negative and $\mathbb{G}$-adapted processes.

Including partial withdrawals in the pay-off, we shall use the following notations

$$F^{D,W}_\tau(p) := e^{-\int_0^\tau r_u du}\Big(\hat{F}^D_\tau(p)\mathbb{1}_{\{\tau=\theta^d\leq T\}} + \hat{F}^W_\tau(p)\mathbb{1}_{\{\tau=\theta^w<\theta^d\,;\,\tau\leq T\}}\Big) + \int_0^\tau \xi_s A^p_s\, ds \;, \tag{4.2.1}$$

$$F^L_T(p) := e^{-\int_0^T r_u du}\hat{F}^L_T(p) + \int_0^T \xi_s A^p_s\, ds \;, \tag{4.2.2}$$

$$F(p) := e^{-\int_0^{T\wedge\tau} r_u du}\hat{F}(p) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds \;. \tag{4.2.3}$$

Usual examples. There exist $\hat{G}^D(p)$ and $\hat{G}^L(p)$ non-negative processes such that, for any $Q \in \{D, L\}$, we have

$$\hat{F}^Q_t(p) = \hat{A}^p_t \vee \hat{G}^Q_t(p) \;, \quad \text{where} \quad \hat{A}^p_t = e^{\int_0^t r_s\, ds}A^p_t \;.$$

The usual guarantee functions used to define GMDB and GMLB are listed below (see [14] for more details).

— Constant guarantee : we have $\hat{G}^Q_t(p) = A_0 - \int_0^t \xi_s \hat{A}^p_s\, ds$ on $[0,T]$, and

$$F(p) = A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds}\Big(A_0 - \int_0^{T\wedge\tau} \xi_s \hat{A}^p_s\, ds\Big) + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds \;,$$

then, setting $A^p_{T\wedge\tau}(0) = A^p_{T\wedge\tau} + \int_0^{T\wedge\tau} \xi_s A^p_s\, ds$ and $\beta_t = 1 - e^{-\int_t^{T\wedge\tau} r_s ds}$ for $t \in [0, T\wedge\tau]$, we get

$$F(p) = A^p_{T\wedge\tau}(0) \vee \Big(e^{-\int_0^{T\wedge\tau} r_s ds}A_0 + \int_0^{T\wedge\tau} \xi_s A^p_s \beta_s\, ds\Big) \;.$$

— Roll-up guarantee : As an interest rate $\eta > 0$ is paid on the guarantee minus the previous withdrawals, we have $\hat{G}^Q_t(p) = (1+\eta)^t \Big(A_0 - \int_0^t \frac{\xi_s \hat{A}^p_s}{(1+\eta)^s} ds \Big)$ on $[0, T]$. We obtain

$$\begin{aligned} F(p) &= A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds} \hat{G}^Q_{T\wedge\tau}(p) + \int_0^{T\wedge\tau} \xi_s A^p_s \, ds \\ &= A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds} (1+\eta)^{T\wedge\tau} \Big(A_0 - \int_0^{T\wedge\tau} \frac{\xi_s \hat{A}^p_s}{(1+\eta)^s} ds \Big) + \int_0^{T\wedge\tau} \xi_s A^p_s \, ds \ , \end{aligned}$$

setting $r^\eta_t = r_t - \ln(1+\eta)$ for all $t \in [0, T]$ and $\beta^\eta_t = 1 - e^{-\int_t^{T\wedge\tau} r^\eta_s ds}$ for $t \in [0, T\wedge\tau]$, we get

$$F(p) = A^p_{T\wedge\tau}(0) \vee \Big(e^{-\int_0^{T\wedge\tau} r^\eta_s \, ds} A_0 + \int_0^{T\wedge\tau} \xi_s A^p_s \beta^\eta_s \, ds \Big) \ . \tag{4.2.4}$$

— Ratchet guarantee : The guarantee depends on the path of A in the following way : $\hat{G}^Q_t(p) = \max(\hat{a}^p_0(t), \ldots, \hat{a}^p_k(t))$ on $[t_k, t_{k+1})$, for all $0 \le k \le n$, where we have set $\hat{a}^p_k(t) = \hat{A}^p_{t_k} - \int_{t_k}^t \xi_s \hat{A}^p_s \, ds$. We get

$$F(p) = A^p_{T\wedge\tau} \vee e^{-\int_0^{T\wedge\tau} r_s ds} \max_{0\le i\le n} \big(\hat{a}^p_i(T\wedge\tau) \mathbb{1}_{\{t_i \le T\wedge\tau\}} \big) + \int_0^{T\wedge\tau} \xi_s A^p_s \, ds \ ,$$

setting $\hat{A}^p_{t_i}(0) = \hat{A}^p_{t_i} + \int_0^{t_i} \xi_s \hat{A}^p_s \, ds$ for all $i \in \{0, .., n\}$, we get that

$$F(p) = A^p_{T\wedge\tau}(0) \vee \Big(\max_{0\le i\le n} \Big[e^{-\int_0^{T\wedge\tau} r_s ds} \hat{A}^p_{t_i}(0) \mathbb{1}_{\{t_i \le T\wedge\tau\}} \Big] + \int_0^{T\wedge\tau} \xi_s A^p_s \beta_s \, ds \Big) \ . \tag{4.2.5}$$

4.2.2 Indifference pricing

The optimal fee rate p^* is defined as the smallest p such that

$$\sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X^{x,\pi}_T \big) \big] = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(X^{x+A_0,\pi}_T - F(p) \big) \big] \ . \tag{4.2.6}$$

A solution of the (4.2.6) will be called an indifference fee rate.
Notice that if there exist solutions to the previous equation, they will not depend on the initial wealth invested by the insurer but only on the initial deposit A_0 made by the insured since $U(y) = -\exp(-\gamma y)$. Therefore, solve (4.2.6) is equivalent to solve

$$V_{\mathbb{F}} := \sup_{\pi\in\mathcal{A}^{\mathbb{F}}[0,T]} \mathbb{E}\big[U\big(X^\pi_T \big) \big] = \sup_{\pi\in\mathcal{A}^{\mathbb{G}}[0,T]} \mathbb{E}\big[U\big(A_0 + X^\pi_T - F(p) \big) \big] := V_{\mathbb{G}}(p) \ .$$

$V_{\mathbb{F}}$ is a classical optimization problem, that has been solved in [65] and [102].

Utility maximization with variable annuities

We study the case in which the insurance company proposes the variable annuity policy and solve the optimal control problem $V_{\mathbb{G}}(p)$. The following lemma allows us to rewrite the problem with a terminal date equal to $T \wedge \tau$.

Lemma 4.2.1 *For any $p \in \mathbb{R}$, we have*

$$V_{\mathbb{G}}(p) \quad = \quad \sup_{\pi \in \mathcal{A}^{\mathbb{G}}[0,T\wedge\tau]} \mathbb{E}\big[- \exp\big(- \gamma\big(X^{A_0,\pi}_{T\wedge\tau} - \mathfrak{H}(p)\big)\big)\big] \ , \tag{4.2.7}$$

with

$$\mathfrak{H}(p) := F(p) + \frac{1}{\gamma} \ln\Big\{ \operatorname*{ess\,inf}_{\pi\in\mathcal{A}^{\mathbb{G}}[T\wedge\tau,T]} \mathbb{E}\big[\exp\big(- \gamma \Delta X^{\pi}_{\tau,T}\big) \big| \mathcal{G}_{T\wedge\tau}\big]\Big\} \ ,$$

where we have set $\Delta X^{\pi}_{\tau,T} := \int_{T\wedge\tau}^{T} \pi_s(\mu_s - r_s)ds + \int_{T\wedge\tau}^{T} \pi_s\sigma_s dB_s$.

We now state a verification theorem which is the main result of this section.

Theorem 4.2.2 *The value function of the optimization problem (4.2.7) is given by*

$$V_{\mathbb{G}}(p) \quad = \quad -\exp(\gamma(Y_0(p) - A_0)) \ ,$$

where $Y_0(p)$ is defined by the initial value of the first component of the solution of the following BSDE

$$\begin{aligned} Y_t(p) \quad &= \quad \mathfrak{H}(p) + \int_{t\wedge\tau}^{T\wedge\tau} \Big(\lambda_s \frac{e^{\gamma U_s(p)} - 1}{\gamma} - \frac{\nu_s^2}{2\gamma} - \nu_s Z_s(p)\Big)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s(p)dB_s \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} U_s(p)dH_s \ , \quad \forall t \in [0,T] \ , \end{aligned} \tag{4.2.8}$$

which admits a solution in $S^{\infty}_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2_{\mathbb{G}}(\lambda)$ given for any $t \in [0,T]$ by

$$\left\{ \begin{array}{rcl} Y_t(p) &=& Y^0_t(p)\mathbb{1}_{t<\tau} + F^{D,W}_{\tau}(p)\mathbb{1}_{\tau\leq t} \ , \\ Z_t(p) &=& Z^0_t(p)\mathbb{1}_{t\leq\tau} \ , \\ U_t(p) &=& \big(F^{D,W}_t(p) - Y^0_t(p)\big)\mathbb{1}_{t\leq\tau} \ , \end{array} \right. \tag{4.2.9}$$

where $(Y^0(p), Z^0(p))$ is the unique solution in $S^{\infty}_{\mathbb{G}} \times L^2_{\mathbb{G}}$ of the following BSDE

$$\left\{ \begin{array}{ll} -dY^0_t(p) &= \Big\{\lambda_t \frac{e^{\gamma(F^{D,W}_t(p) - Y^0_t(p))} - 1}{\gamma} - \frac{\nu_t^2}{2\gamma} - \nu_t Z^0_t(p)\Big\}dt - Z^0_t(p)dB_t \ , \\ Y^0_T(p) &= F^L_T(p) \ . \end{array} \right. \tag{4.2.10}$$

Moreover there exists an optimal strategy $\pi^* \in \mathcal{A}^{\mathbb{G}}[0,T]$ and this one is defined by

$$\pi^*_t \ := \ \frac{\nu_t}{\gamma\sigma_t} + \frac{Z_t(p)}{\sigma_t}\mathbb{1}_{t\leq T\wedge\tau} + \frac{Z^{(\tau)}_t}{\sigma_t}\mathbb{1}_{t>T\wedge\tau} \ , \quad \forall t \in [0,T] \ , \tag{4.2.11}$$

with $Z^{(\tau)})$ defined as component of the solution $(Y^{(\tau)}, Z^{(\tau)})$ of the following BSDE

$$\left\{ \begin{array}{rcl} dY^{(\tau)}_t &=& \Big[\frac{\nu_t^2}{\gamma} + \nu_t Z^{(\tau)}_t\Big]dt + Z^{(\tau)}_t dB_t \ , \\ Y^{(\tau)}_T &=& 0 \ . \end{array} \right.$$

Remark 2 *Existence of a solution to the BSDE* (4.2.8)*, defined by* (4.2.9)*, follows from Theorem 4.3 in [72]. From Theorem 2.1 in [26] and Theorem 1 in [52], we know that there is a unique solution $(Y^0(p), Z^0(p)) \in S^{\infty}_{\mathcal{G}} \times L^2_{\mathcal{G}}$ to the BSDE* (4.2.10)*.*
To apply Theorem 2.1 in [26] or Theorem 1 in [52] and get existence result for a solution of the BSDE (4.2.10)*, the terminal condition $F^L_T(p)$ must be bounded and the process $F^{D,W}(p)$ must be also bounded.*

Indifference fee rate

Our goal is now to determine indifference fee rates It follows from previous results that our problem, stated in equation (4.2.6), can be rewritten in the following way

$$Y_0(p^*) - A_0 \quad = \quad y_0 \ .$$

To study this equation we introduce the function $\psi : \mathbb{R} \to \mathbb{R}$ defined as follows

$$\psi(p) \quad := \quad Y_0(p) - y_0 - A_0 \ , \qquad \forall p \in \mathbb{R} \ .$$

There may exist three cases depending on the coefficients values.

i) For any $p \in \mathbb{R}$, we have $\psi(p) > 0$. That means that, for any fee rate p, we have

$$V_{\mathbb{G}}(p) \quad < \quad V_{\mathbb{F}} \ .$$

Therefore, the insurer's expected utility is always lower if he sells the variable annuities. Thus, he should not sell it.

ii) For any $p \in \mathbb{R}$, we have $\psi(p) < 0$. That means that, for any fee rate p, we have

$$V_{\mathbb{G}}(p) \quad > \quad V_{\mathbb{F}} \ .$$

Therefore, the insurer's expected utility is always higher if he sells the variable annuities. Thus, he should sell it whatever the fees are.

iii) There exist p_1 and p_2 such that $\psi(p_1)\psi(p_2) < 0$. In this case, we prove in the remainder of this section that there exist indifference fee rates thanks to the intermediate value theorem applied to the function ψ.

We now give useful analytical properties of the function ψ.

Proposition 4.2.3 *The function ψ is continuous and non-increasing on $\mathbb{R}$.*

We now consider the cases of usual guarantees.

Corollary 4.2.4 *Ratchet guarantee.*
Let $m > A_0$. Recalling notations of (4.2.5), we assume that

$$F(p) = m \wedge \left[A^p_{T\wedge\tau}(0) \vee \left(\max_{0 \le i \le n} \left[e^{-\int_0^{T\wedge\tau} r_s ds} \hat{A}^p_{t_i}(0) \mathbb{1}_{\{t_i \le T\wedge\tau\}} \right] + \int_0^{T\wedge\tau} \xi_s A^p_s \beta_s \, ds \right) \right] \ .$$

There exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that for $p \ge p^*$ we have $V_{\mathbb{G}}(p) \ge V_{\mathbb{F}}$ and for $p < p^*$ we have $V_{\mathbb{G}}(p) < V_{\mathbb{F}}$.

Corollary 4.2.5 *Roll-up guarantee.*
Let $m > A_0$. Recalling notations of (4.2.4), we assume that

$$F(p) = m \wedge \left[A^p_{T\wedge\tau}(0) \vee \left(e^{-\int_0^{T\wedge\tau} r^\eta_s \, ds} A_0 + \int_0^{T\wedge\tau} \xi_s A^p_s \beta^\eta_s \, ds \right) \right].$$

There exists $\eta_* \ge 0$ such that for any $\eta \in [0, \eta_*]$, there exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that for $p \ge p^*$ we have $V_{\mathbb{G}}(p) \ge V_{\mathbb{F}}$ and for $p < p^*$ we have $V_{\mathbb{G}}(p) < V_{\mathbb{F}}$.

4.2.3 Numerical illustrations

We conclude this section with numerical illustrations of parameters sensibility for indifference fee rates. Figure 4.1 plots the indifference fee rates when the volatility σ ranged from 0.1 to 0.4.

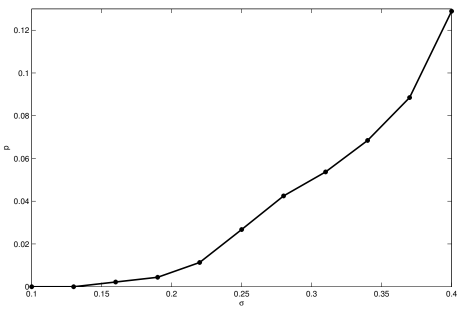


FIGURE 4.1 – Indifference fee rate with respect to σ

The financial interpretation of the monotonicity of the fees w.r.t. market volatility. The bigger is the volatility the more useful are the guarantees, then the fees payed to get these guarantees have to increase. Figure 4.2 plots the indifference fee rates sensibility to λ.

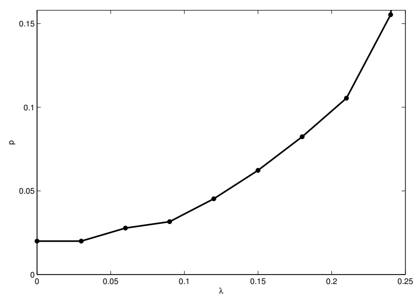


FIGURE 4.2 – Indifference fee rate with respect to λ

Figure 4.3 plots the indifference fee rates when the withdrawal rate ξ is constant and ranged from 0 to 0.3.

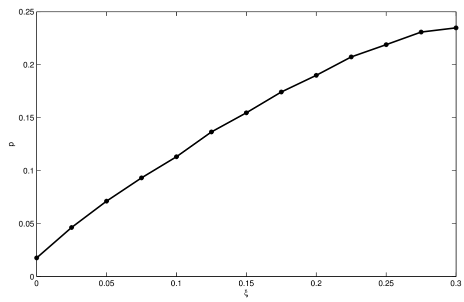


FIGURE 4.3 – Indifference fee rate with respect to ξ

4.3 The worst case of withdrawals

4.3.1 The policy model

We consider a variable annuities product with a maturity $T > 0$. Let $\mathbb{T} := (t_i)_{0\leq i\leq n}$ be the set of policy anniversary dates, with $t_0 = 0$ and $t_n = T$. By convention we set $t_{n+1} = +\infty$. We still denote by A_0 the initial capital invested, by the insured, in the fund related to this product (also called insured account) at time $t = 0$.

Withdrawals. At any date t_i, for $i \in \{1, \ldots, n-1\}$, the insured is allowed to withdraw an amount of money. This should be lower than a bounded non-negative $\mathcal{G}_{t_i}$-measurable random variable $\hat{G}_i$.
We define $\hat{\mathcal{W}}$ as a finite subset of $[0, 1]$ which contains 0 and 1 and introduce the set of admissible withdrawal policies

$$\hat{\mathcal{E}} = \Big\{(\alpha_i\hat{G}_i)_{1\leq i\leq n-1} \;:\; \alpha_i \text{ is a } \mathcal{G}_{t_i}\text{-measurable random variable such that } \alpha_i \in \hat{\mathcal{W}} \quad \text{for all } i \in \{1, \ldots, n-1\}\Big\} .$$

For $\hat{\xi} \in \hat{\mathcal{E}}$ and $i \in \{1, \ldots, n-1\}$, $\hat{\xi}_i$ is the withdrawal made by the insured at time t_i and we introduce the family $(\xi_i)_{1\leq i\leq n-1}$ such that $\xi_i := e^{-\int_0^{t_i} r_s\, ds}\hat{\xi}_i$ is the discounted withdrawal made at time t_i. We define by $\mathcal{E}$ the admissible discounted withdrawal policies with $\xi \in \mathcal{E}$ if and only if the vector $\hat{\xi} \in \hat{\mathcal{E}}$. For any $k \in \{0, \ldots, n-2\}$ and $i \in \{1, \ldots, n-k-1\}$, we also define the set $\mathcal{E}_k^i$ by

$$\mathcal{E}_k^i = \Big\{\xi \in \mathcal{E} \text{ s.t. } \xi_j = 0 \text{ for all } j \notin \{k+1, \ldots, k+i\}\Big\} .$$

$\mathcal{E}_k^i$ is the set of admissible withdrawal policies such that all withdrawals are made between times t_{k+1} and t_{k+i}.

Discounted account value. We denote by A_t^p the discounted value at time t of the fund related to the variable annuities contract sold at fee rate p. If the insured follows the withdrawal policy $\hat{\xi} \in \hat{\mathcal{E}}$, we have

$$\begin{cases} dA_t^p = A_t^p\big[(\mu_t - r_t - p)dt + \sigma_t dB_t\big] , & \text{for } t \notin \mathbb{T} , \\ A_{t_i}^p = \big(A_{t_i^-}^p - f_i\big) \vee 0 , & \text{for } 1 \leq i \leq n-1 , \end{cases} \tag{4.3.1}$$

where f_i is a $\mathcal{G}_{t_i}$-measurable random variable greater than ξ_i for any $i \in \{1, \ldots, n-1\}$ and depending on previous withdrawals, on previous account values and on some guarantees determined in the policy. We give details in the next paragraph.

Penalties and guarantees. We now focus on the dependencies between f_i and $\hat{G}_i$, and begin with introducing two sets of functions defined on $[0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$.
Let $\mathcal{I}$ (resp. $\mathcal{J}$) be the set of bounded, non-negative functions ϕ (resp. ψ) defined on $[0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$ such that for any $i \in \{1, \ldots, n-1\}$ and $(t, x, e) \in [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, the function $y \mapsto \phi(t, x, e_1, \ldots, e_{i-1}, y, e_{i+1}, \ldots, e_{n-1})$ is non-increasing (resp. $y \mapsto \psi(t, x, e_1, \ldots, e_{i-1}, y, e_{i+1}, \ldots, e_{n-1})$ is non-decreasing) and for any $j \in \{1, \ldots, n+1\}$, the function $y \mapsto \phi(t, x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n+1}, e)$ is non-decreasing (resp. $y \mapsto \psi(t, x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_{n+1}, e)$ is non-increasing).

$\hat{G}_i$ is the maximum amount that can be withdrawn at time t_i, hence it decreases with respect to previous withdrawals and increases with previous values of the fund related to the variable annuities contract. We assume that there exists $\hat{g} \in \mathcal{I}$ such that, for any $i \in \{1, \ldots, n-1\}$, we have

$$\hat{G}_i \;=\; \hat{g}(t_i, \hat{A}^p_{t_0}, \ldots, \hat{A}^p_{t_{i-1}}, \hat{A}^p_{t_i^-}, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, 0, \ldots, 0)\;,$$

where $\hat{A}^p_t = e^{\int_0^t r_s\, ds} A^p_t$ for all $t \in [0, T]$.
In the same vein, the random variables $(f_i)_{1 \le i \le n-1}$ corresponds to penalties for early withdrawals. It seems reasonable to assume that they increase with previous withdrawals and, for marketing considerations, decrease with previous values of the fund. We assume that there exists $\hat{f} \in \mathcal{J}$ such that, for any $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned} f_i \;&:=\; f(t_i, \hat{A}^p_{t_0}, \ldots, \hat{A}^p_{t_{i-1}}, \hat{A}^p_{t_i^-}, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_i, 0, \ldots, 0) \\ &=\; e^{\int_0^{t_i} r_s\, ds} \hat{f}(t_i, \hat{A}^p_{t_0}, \ldots, \hat{A}^p_{t_{i-1}}, \hat{A}^p_{t_i^-}, 0, \ldots, 0, \hat{\xi}_1, \ldots, \hat{\xi}_i, 0, \ldots, 0)\;. \end{aligned}$$

We give concrete examples of functions $\hat{g}$ and f in a next paragraph.

Pay off contract. Let $\hat{F}^L$ and $\hat{F}^D$ belong to $\mathcal{I}$, the pay off is paid at time $T \wedge \tau$ to the insured or her dependents, and is equal to the following random variable

$$\hat{F}(p, \hat{\xi}) \;:=\; \hat{F}^L(T, \hat{a}^p, \hat{\xi})\mathbb{1}_{\{T<\tau\}} + \hat{F}^D(\tau, \hat{a}^p, \hat{\xi})\mathbb{1}_{\{\tau \le T\}}\;,$$

where $\hat{a}^p := \big(\hat{A}^p_{t_i \wedge \tau}\big)_{0 \le i \le n}$. $\hat{F}^L$ is the pay-off if the policyholder is alive at time T and $\hat{F}^D$ is the pay-off if the policyholder is dead at time τ. In the following, we denote by $F(p, \hat{\xi})$ the discounted pay-off.

Usual examples. In the usual case of GMDB and GMLB, we may precise guarantees. We introduce $\hat{G}^D$, $\hat{G}^L$ and $\hat{G}^W$ belonging to $\mathcal{I}$ such that, for any $Q \in \{D, L\}$, on $[0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$, we have

$$\hat{F}^Q(t, x, e) \;=\; x_{n+1} \vee \hat{G}^Q(t, x, e)\;,$$

and, on $[0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$,

$$\hat{g}(t, x, e) \;=\; \sum_{i=0}^{n} \Big[x_{i+1} \vee \hat{G}^W(t, x_0, \ldots, x_{i+1}, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0)\Big]\mathbb{1}_{\{t_i \le t < t_{i+1}\}}\;.$$

In that case, the penalty function f is often given by

$$f(t_i, x, e) = \begin{cases} e_i & \text{if } e_i \le G_i\;, \\ G_i + \kappa(e_i - G_i) & \text{if } e_i > G_i\;, \end{cases}$$

where $\kappa > 1$ and $G_i := G^W(t_i, x_0, \ldots, x_{i+1}, 0, \ldots, 0, e_1, \ldots, e_{i-1}, 0, \ldots, 0)$. The insurer takes a fee if the insured withdraws more than the guarantee G_i, this fee is equal to $(\kappa - 1)(e_i - G_i)$.

The usual guarantee functions used to define GMDB and GMLB are listed below (see [14] for more details).

— Constant guarantee. For $i \in \{0, \ldots, n\}$ and $t_i \le t < t_{i+1}$, we set

$$\hat{G}^Q(t, x, e) \;=\; x_1 - \sum_{k=1}^{i} \hat{f}(t_k, x, e) \quad \text{on } [0, T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}\;.$$

Hence, following the withdrawal strategy $\xi \in \mathcal{E}$, the insured will get

$$F(p, \hat{\xi}) \;=\; A^p_{T\wedge\tau} \vee \left(e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \Big(A_0 - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi}) \Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} \right) .$$

— Roll-up guarantee. For $\eta > 0$, $i \in \{0, \ldots, n\}$ and $t_i \le t < t_{i+1}$, we set

$$\hat{G}^Q(t, x, e) \;=\; x_1(1+\eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, x, e)(1+\eta)^{i-k} \quad \text{on } [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1} ,$$

and then if the insured follows the withdrawal strategy $\xi \in \mathcal{E}$, she will get

$$F(p, \hat{\xi}) \;=\; A^p_{T\wedge\tau} \vee \left(e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \Big(A_0(1+\eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi})(1+\eta)^{i-k} \Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} \right) .$$

— Ratchet guarantee. The guarantee depends on the path of A in the following way

$$\hat{G}^Q(t, x, e) \;=\; \sum_{i=0}^{n} \max \Big(x_1 - \sum_{k=1}^{i} \hat{f}(k, x, e), \ldots, x_i - \hat{f}(t_i, x, e), x_{i+1} \Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} ,$$

for any $(t, x, e) \in [0,T] \times \mathbb{R}^{n+1} \times \mathbb{R}^{n-1}$. The insured will get

$$F(p, \hat{\xi}) \;=\; A^p_{T\wedge\tau} \vee \left(e^{-\int_0^{T\wedge\tau} r_s\,ds} \sum_{i=0}^{n} \max \Big(\hat{a}^p_0 - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi}), \ldots, \hat{a}^p_i \Big) \mathbb{1}_{\{t_i \le T\wedge\tau < t_{i+1}\}} \right) .$$

4.3.2 Indifference pricing

The optimal fee rate p^* is then the smallest p such that

$$\begin{aligned} V^0 \;&:=\; \sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}\big[U\big(X^\pi_T\big) \big] \\ &\le\; \sup_{\pi \in \mathcal{A}[0,T]} \inf_{\xi \in \mathcal{E}} \mathbb{E}\Big[U\Big(A_0 + X^\pi_T - \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{t_i \le \tau} - F(p, \hat{\xi}) \Big) \Big] \;:=\; V(p) . \end{aligned}$$

The quantity V^0 corresponds to the maximal expected utility at time T when the insurance company has not sold the variable annuities policy. We can characterize this value function V^0 and the optimal strategy π^* by mean of BSDEs as done by [65]. To this end we define the following spaces.

— $S^\infty_{\mathbb{F}}$ (resp. $S^\infty_{\mathcal{G}}$) is the set of càdlàg $\mathbb{F}$ (resp. $\mathcal{G}$)-adapted essentially bounded processes.

— $L^2_{\mathbb{F}}$ (resp. $L^2_{\mathcal{G}}$) is the set of $\mathcal{P}(\mathbb{F})$ ($\mathcal{P}(\mathcal{G})$)-measurable processes z such that $\mathbb{E}\int_0^T |z_s|^2 ds < \infty$.

— $L^2(\lambda)$ is the set of $\mathcal{P}(\mathcal{G})$-measurable processes u such that $\mathbb{E}\int_0^{T\wedge\tau} \lambda_s |u_s|^2 ds < \infty$.

We then have the following result which is a consequence of Theorem 7 in [65].

Proposition 4.3.1 *The value function* $V^0 := \sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}\big[U\big(X^\pi_T\big)\big]$ *is given by*

$$V^0 \;=\; -\exp(\gamma y_0) ,$$

where (y, z) is the solution in $S^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ to the BSDE

$$\begin{cases} dy_t &= \Big(\frac{\theta_t^2}{2\gamma} + \theta_t z_t\Big)dt + z_t dB_t \;, \\ y_T &= 0 \;. \end{cases} \tag{4.3.2}$$

Moreover, the optimal strategy associated to this problem is defined by

$$\pi_t^* \;:=\; \frac{\theta_t}{\gamma\sigma_t} + \frac{z_t}{\sigma_t} \;, \quad \forall t \in [0, T] \;.$$

In the usual indifference pricing setting, we can isolate p and get a semi-explicit formula for the indifference price. A difficulty with our approach is that fees are continuously payed by the insured and that the fee rate p appears in the pay-off $F(p, \hat{\xi})$. Therefore, one cannot use algebraic properties of the utility function to get semi-explicit formula for indifference fees. Nevertheless, we can prove some monotonicity results on the value function V which will be used to prove that the indifference fee rate exists or not, and to compute it.

Proposition 4.3.2 *The value function V is non-decreasing on $\mathbb{R}$.*

This monotonicity property of the function V allows to conclude the existence of indifference fees.

- If $V(-\infty) < V^0 < V(+\infty)$, then there exists p^* such if $p < p^*$, the insurance company has no interest to sell the contract, and if $p \geq p^*$ then the company has interest to sell the contract.
- If $V(-\infty) > V^0$, the insurance should always sell the contract.
- If $V(+\infty) < V^0$, the insurance should never sell the contract.

The asymptotic behavior of V is then studied for usual guarantees.

Proposition 4.3.3 (Ratchet guarantee) *Let $m > A_0$. We assume that*

$$F(p, \hat{\xi}) \;=\; m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\, ds} \sum_{i=0}^{n} \max\Big(\hat{a}^p_0 - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi}), \ldots, \hat{a}^p_i\Big) \mathbb{1}_{\{t_i \leq T\wedge\tau < t_{i+1}\}}\Big)\Big]$$

for $(p, \xi) \in \mathbb{R} \times \mathcal{E}$. Then, there exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that $V(p) \geq V^0$ for all $p \geq p^*$ and $V(p) < V^0$ for all $p < p^*$.

Proposition 4.3.4 (Roll-up guarantee) *Let $m > A_0$ and $\eta \geq 0$. Assume that*

$$F(p, \hat{\xi}) \;=\; m \wedge \Big[A^p_{T\wedge\tau} \vee \Big(e^{-\int_0^{T\wedge\tau} r_s\, ds} \sum_{i=0}^{n} \Big(A_0(1+\eta)^i - \sum_{k=1}^{i} \hat{f}(t_k, \hat{a}^p, \hat{\xi})(1+\eta)^{i-k}\Big) \mathbb{1}_{\{t_i \leq T\wedge\tau < t_{i+1}\}}\Big)\Big] \;,$$

for all $(p, \xi) \in \mathbb{R} \times \mathcal{E}$. There exists $\eta_* \geq 0$ such that for all $\eta \in [0, \eta_*]$, there exists $p^* \in \mathbb{R} \cup \{-\infty\}$ such that $V(p) \geq V^0$ for all $p \geq p^*$ and $V(p) < V^0$ for all $p < p^*$.

In order to find the indifference fees, we shall compute the following quantities

$$V^0 \;:=\; \sup_{\pi \in \mathcal{A}[0,T]} \mathbb{E}\big[U\big(X_T^\pi\big)\big] \;, \tag{4.3.3}$$

and

$$\begin{aligned} V(p) &:= \sup_{\pi\in\mathcal{A}[0,T]}\inf_{\xi\in\mathcal{E}}\mathbb{E}\Big[U\Big(A_0+X_T^\pi-\sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\le\tau}-F(p,\hat{\xi})\Big)\Big] \\ &= -e^{-\gamma A_0}w(p)\ , \qquad p\in\mathbb{R}\ , \end{aligned} \tag{4.3.4}$$

where w is defined for any $p\in\mathbb{R}$ by

$$w(p) \ := \ \inf_{\pi\in\mathcal{A}[0,T]}\sup_{\xi\in\mathcal{E}}\mathbb{E}\Big[u\Big(X_T^\pi-\sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\le\tau}-F(p,\hat{\xi})\Big)\Big]\ , \tag{4.3.5}$$

with $u(y):=e^{-\gamma y}$ for all $y\in\mathbb{R}$.

4.3.3 Min-Max optimization problem

We determine now a numerical procedure to obtain $w(p)$ defined by $V(p) = -e^{-\gamma A_0}w(p)$. In the sequel we use the following notations. For $x\in\mathbb{R}^n$ and $1\le k\le n$ we denote by $x^{(k)}$ the vector of $\mathbb{R}^k$ defined by $x^{(k)}:=(x_1,\ldots,x_k)$. For $y\in\mathbb{R}^k$ we denote by $\hat{y}$ the vector $\hat{y}:=\big(y_1e^{\int_0^{t_1}r_sds},\ldots,y_ke^{\int_0^{t_k}r_sds}\big)$.

Sequential utility maximization

Proposition 4.3.5 (Initialization) *For any $p\in\mathbb{R}$, we have*

$$w(p) \ = \ \inf_{\pi\in\mathcal{A}[0,T\wedge\tau]}\sup_{\xi\in\mathcal{E}}\mathbb{E}\Big[u\big(X_{T\wedge\tau}^\pi-\sum_{i=1}^{n-1}\xi_i\mathbb{1}_{t_i\le\tau}-H(p,\hat{\xi})\big)\Big]\ ,$$

with

$$H(p,\hat{\xi}) \ := \ F(p,\hat{\xi})+\frac{1}{\gamma}\mathcal{L}og\Big[\operatorname*{ess\,inf}_{\pi\in\mathcal{A}[T\wedge\tau,T]}\mathbb{E}\big[u\big(X_T^{T\wedge\tau,\pi}\big)\big|\mathcal{G}_{T\wedge\tau}\big]\Big]\ ,$$

where $X_T^{T\wedge\tau,\pi}$ is the wealth at time T when we follow the strategy π by starting at time $T\wedge\tau$ with the wealth 0.

We now decompose the initial problem in n subproblems.

Theorem 4.3.6 *The value function w is given by*

$$w(p) \ = \ \inf_{\pi\in\mathcal{A}[0,t_1\wedge\tau]}\mathbb{E}\big[u\big(X_{t_1\wedge\tau}^\pi\big)v(1)\big]\ ,$$

where

— *$v(i,\xi^{(i-1)})$ is defined recursively for any $i\in\{2,\ldots,n\}$ and $\xi\in\mathcal{E}$ by*

$$\begin{cases} v(n,\xi^{(n-1)}) & := \ e^{\gamma H(p,\hat{\xi}^{(n-1)})}\ , \\ v(i,\xi^{(i-1)}) & := \ \operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_{i-1}^1}\ \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_i\wedge\tau,t_{i+1}\wedge\tau]} J(i,\pi,\xi^{(i-1)},\zeta)\ , \end{cases}$$

with for any $i\in\{1,\ldots,n-1\}$, $\pi\in\mathcal{A}[t_i\wedge\tau,t_{i+1}\wedge\tau]$ and $\zeta\in\mathcal{E}_{i-1}^1$

$$J(i,\pi,\xi^{(i-1)},\zeta) \ := \ \mathbb{E}\Big[u\Big(X_{t_{i+1}\wedge\tau}^{t_i\wedge\tau,\pi}-\zeta\,\mathbb{1}_{t_i<\tau}\Big)v\big(i+1,(\xi^{(i-1)},\zeta)\big)\Big|\mathcal{G}_{t_i\wedge\tau}\Big]\ ,$$

— $v(1):=\operatorname*{ess\,sup}_{\zeta\in\mathcal{E}_0^1}\ \operatorname*{ess\,inf}_{\pi\in\mathcal{A}[t_1\wedge\tau,t_2\wedge\tau]}\mathbb{E}\Big[u\Big(X_{t_2\wedge\tau}^{t_1\wedge\tau,\pi}-\zeta\,\mathbb{1}_{t_1<\tau}\Big)v\big(2,\zeta\big)\Big|\mathcal{G}_{t_1\wedge\tau}\Big]$.

Optimal investment and worst withdrawals for the insurer

In the following result, we provide the withdrawal ξ_i^* and the investment strategy $\pi^{*,i}$ that attain the value functions $v(i,.)$ for any $i \in \{1,\dots,n\}$.

Proposition 4.3.7 *For any $i \in \{1,\dots,n-1\}$, there exists a strategy $\pi^{*,i} \in \mathcal{A}[t_i \wedge \tau, t_{i+1} \wedge \tau]$, a withdrawal $\xi_i^* \in \mathcal{E}_{i-1}^1$, and a map $y^{(i),*}$ from $\hat{\mathcal{W}}^{i-1}$ to $L^\infty(\Omega, \mathcal{G}_{t_i \wedge \tau}, \mathbb{P})$ such that*

$$\begin{aligned} v(i, \xi^{(i-1)}) &= \mathbb{E}\Big[u\big(X_{t_{i+1}\wedge\tau}^{t_i\wedge\tau,\pi^{*,i}} - \xi_i^* \mathbb{1}_{t_i<\tau}\big)v\big(i+1, (\xi^{(i-1)}, \xi_i^*)\big)\Big|\mathcal{G}_{t_i\wedge\tau}\Big] \\ &= \exp\big(\gamma y^{(i),*}(\hat{\xi}^{(i-1)})\big)\ . \end{aligned}$$

Moreover there exists $y^{(0)}$ such that the value function v of the initial problem (4.3.5) *is given by $w(p) = \exp(\gamma y^{(0)})$.*

4.3.4 Numerical resolution

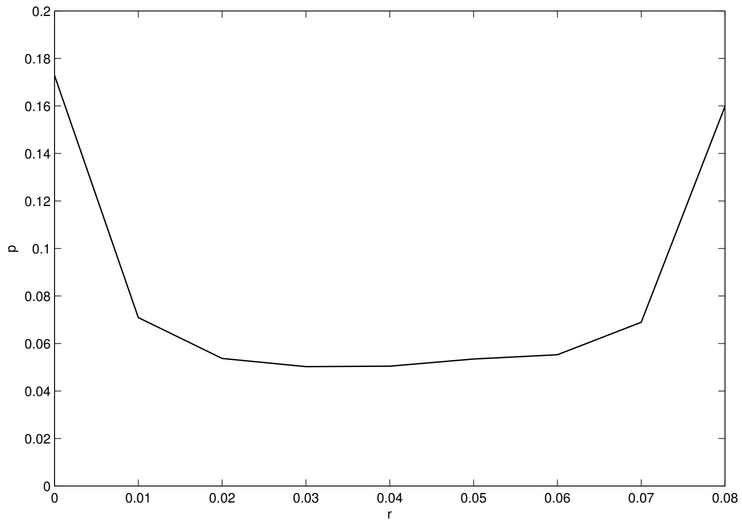
Max-min problem. We first propose a scheme to solve the problem $w(p)$ by using Theorem 4.3.6. We may describe the procedure in an inductive way. The step 0 corresponds to the initialization given by Proposition 4.3.5 and we may characterize $H(p, \hat{\xi})$ thanks to a linear BSDE. Step i corresponds to the computation of the function $v(n-i,.)$, the optimal strategy $\pi^{*,n-i}$ and the worst withdrawal ξ_{n-i}^* once the previous steps have been done. This computation will rely on characterization of $v(n-i,.)$ throught solutions to BSDE's with random terminal date. We would then be able to give optimal strategies and worst withdrawal processes.

Indifference price. As, we now know how to calculate V^0 and $V(p)$ for any $p \in \mathbb{R}$ and V is continuous, we may approximate p^* by bisection or dichotomy method.

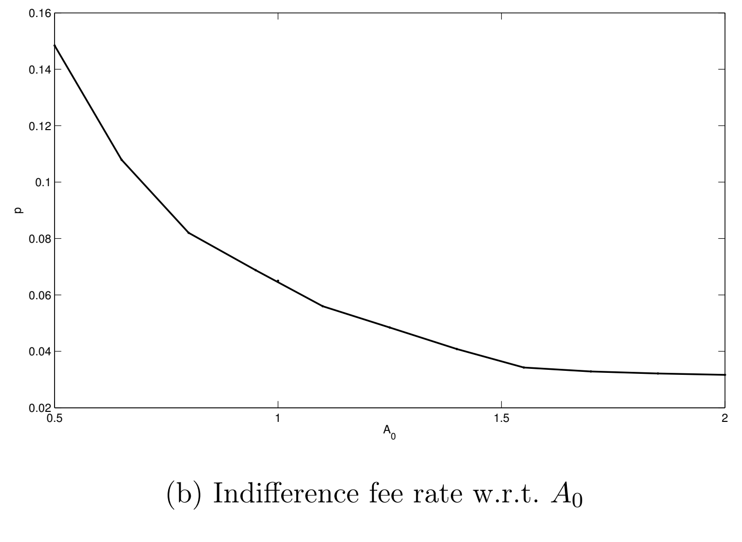
Simulations In this section we present numerical illustrations of parameter sensitivity for indifference fee rates. We use the method described in [71] to decompose BSDEs with a jump in a recursive system of two Brownian BSDEs. Brownian BSDEs involved are then simulated thanks to the discretization scheme studied in [21]. For the computation of the conditional expectations, we use a parametric regression method with polynomial basis.

Dependence with respect to the interest rate is complex. We notice that indifference fee rates increase when the absolute value of the difference between drift and interest rate increases. On the one hand, if r is smaller than μ, when it increases the discounting on future payments make them to become worth less and the price of guarantees decreases. On the other hand, when r is greater than the drift, an exponential utility maximizer should not expose its portfolio to the market volatility. Hence, she should receive a bigger compensation to do so. If the insurer sell its product, she has to hedge her portfolio against volatility and to have a non-zero position on the risky assets. She will sell her product at a bigger price if interest rate is greater than μ and increases.

Since the utility function is an exponential one, indifference fee rates will not depend on the initial wealth invested by the insurer but strongly on the initial deposit A_0 made by the insured (see inequality (4.2.6)). As fees are proportional, the more the insured invests, the more the insurer will get from the contract. Therefore, indifference fee rates will decrease when the initial deposit A_0 increases.



(a) Indifference fee rate w.r.t. r



(b) Indifference fee rate w.r.t. A_0

Chapitre 5

Research projects

5.1 Optimal liquidation in order book with stochastic resilience

With Sergio Pulido, V. Ly Vath and our Ph. D. student Florian Rasamoely (see **(15)**), we study the recent problem of optimal liquidation of several assets before a maturity (see, for instance, [2], [4], [94],[97]). We assume that there exists a Limit Order Book (LOB) in which we can trade, restraining our strategies to market selling orders. In [76], it is asserted that the liquidity in LOB markets has the following characteristics : depth, resilience and tightness. Tightness is the distance between ask and bid prices and induces proportionnal liquidity costs. As we restrict our trading strategy to only one side of the LOB, tightness is irrelevant for our problem. Depth of LOB is the current shape of the LOB which will induce price impact and therefore non-proportionnal costs. Our framework will correspond to the general model of LOB shape studied in [97]. The resilience could be defined as the dynamic of trades impacts on the LOB shape. We assume that resilience is a mean-reverting process following an Ornstein-Uhlenbeck dynamic type. We formulate a stochastic singular control in a bi-dimensionnal setting and characterize optimal values and policies throught HJB equations and viscosity techniques.

Probability space.

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness. We assume that this filtered probability space supports a standard $\mathbb{P}$-Brownian motion W and M a random Poisson measure on $\mathbb{R}^+ \times \mathbb{R}$ with mean measure $\gamma_t dtm(dz)$ where $\gamma : [0,T] \to (0,\bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$.

Reference price and initial limit order book.

Let (A_t) be the reference price of the assets, which we assume to be a continuous $\mathbb{P}$-martingale A_t. In our model, we assume that, in the absence of trading, the number of available shares at time t in the price interval $[A_t, A_t + a)$ is $F(a)$. F is a nondecreasing and left-continuous function.

Investor's strategies. A financial agent wants to buy $\overline{X}$ shares of an illiquid asset over the time interval $[0,T]$. Without loss of generality we will assume that all quantities are discounted.The strategies of the agent are given by nondecreasing right-continuous adapted processes $(X_t)_{0\le t\le T}$ with $X_T = \overline{X}$. We assume that $X_{0-} = 0$ and denote by $\Delta X_t = X_t - X_{t-}$ the jump at time t.

The dynamics of the volume effect process (Y_t)**.** We assume that the strategy of our investor has impact on the price. When the financial agent follows strategy X, we assume that at time t, the ask price is no longer the reference price but is given by $A_t + D_t$ where $D_t := \psi(Y_t)$, with Y_t representing the dynamics of the volume effect process

$$dY_t = dX_t - h(Y_{t-})dt + \sigma(Y_{t-})dW_t + \int_{\mathbb{R}} Y_{t-}q(Y_{t-}, z)\bar{M}(dt, dz); Y_{0-} = y. \qquad (5.1.1)$$

and the left-continuous function ψ given by follows

$$\psi(y) := \sup\{a \ge 0 | F(a) < y\}, \text{ for } y > 0 \text{ and } \psi(0) := 0. \qquad (5.1.2)$$

We denote

$$\check{Y}_{t^-} := Y_{t^-} + \Delta_M Y_t$$

where $\Delta_M Y_t$ is the jump of the mesure M at time t.

Liquidation cost of a strategy. We now can write the cost of the strategy $X = (X_t)_{0\le t\le T}$ as

$$\begin{aligned} C(X) := \quad & \int_0^T (A_t + \check{D}_{t^-})dX_t^c + \sum_{0\le t\le T} [A_t \Delta X_t + (\Phi(Y_t) - \Phi(\check{Y}_{t-}))], \\ = \quad & \int_0^T \psi(\check{Y}_{t-})dX_t^c + \sum_{0\le t\le T} (\Phi(Y_t) - \Phi(\check{Y}_{t-})) + \int_0^T A_t dX_t. \end{aligned}$$

Control Problem. For $0 \le t < T$ and $x \in [0, \overline{X}]$ and $y \ge 0$, after an integration by part, we define our control problem as the infimum on all strategies of the expected cost :

$$v(t, x, y) = \inf_{X\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^T \psi(\check{Y}_{s-}^{t,y,X})dX_s^c + \sum_{t\le s\le T} (\Phi(Y_s^{t,y,X}) - \Phi(\check{Y}_{s-}^{t,y,X}))\right],$$

where $Y_s^{t,y,X}$ for $t \le s \le T$ denotes the solution of (5.1.1) with $Y_{t-}^{t,y,X} = y$ and the set of admissible controls $\mathcal{A}(t, x)$ is given by

$$\mathcal{A}(t, x) := \{X : X \nearrow;\ X_{t-} = x;\ X_T = \overline{X}\}. \qquad (5.1.3)$$

The value function at terminal time T is given by

$$v(T, x, y) = \Phi(y + \overline{X} - x) - \Phi(y). \qquad (5.1.4)$$

Notice that

$$v(t, \overline{X}, y) = 0. \qquad (5.1.5)$$

HJB characterization. We define the infinitesimal generator $\mathcal{L}$ of the process Y

$$\mathcal{L}u := \frac{\sigma^2(y)}{2}\frac{\partial^2 u}{\partial y^2} - h(y)\frac{\partial u}{\partial y} + \gamma_t \int_{\mathbb{R}} (u(t,x,y-q(y,z)) - u(t,x,y))\, m(dz);$$

and get the following characterization of the value function.
The value function v is the unique continuous viscosity solution on $\mathcal{S}$ to the variational inequality :

$$\max\left(-\frac{\partial v}{\partial t} - \mathcal{L}v,\ -\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} - \psi\right) = 0, \tag{5.1.6}$$

satisfying the following growth condition :

$$0 \le v(t,x,y) \le \Phi(y + \overline{X} - x) - \Phi(y) \text{ on } [0,T) \times [0,\overline{X}] \times [0,+\infty),$$

and with boundary data $v(t,\overline{X},y) = 0$ and $v(T,x,y) = \Phi(y + \overline{X} - x) - \Phi(y)$,

5.2 Optimal dividend and capital injection policy with external audit

In this work, written with V. Ly Vath and A. Roch (see **(17)**), we focus our study on the bankruptcy rules for a firm. In the seminal papers [68] and [6], the firm goes to bankruptcy when its cash reserves are below 0. We extended this conditions in **(12)**, **(7)** and **(14)**, respectively taking into account illiquid assets, debt level and capital issuance. In this new study, we want to give to the firm a grace period. The firm can issue capital at any time and pay out dividend when it is not in financial difficulty, situation that is in force when the cash reserve is greater than a level, called debt level. When the firm is in financial difficulty, when its cash reserve is below this level, it can be audited at any time. If it happens, bankruptcy is defined in term of a grace period, denoted δ : the firm is declared bankrupt when its cash reserve has spent a continual period of time δ under the debt level.
Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F})_{t\ge 0}$ satisfying the usual conditions.

Dividend and capital issuance strategies. The cumulative sum that has been injected into the firm at time t is denoted by K_t, whereas the total amount of dividends paid at time t is Z_t.

Audit time. When the firm is in financial difficulty, it can be audited at any time. The probability of being audited in the time interval $[t, t+dt)$ is λdt. Let N be a Poisson process with intensity parameter λ, $\mathbb{F}$-adapted.

Audit and Cash Processes. We define the situation of the firm thanks to the audit process which is equal to 1 when the firm is under audit and 0 else. Its dynamic is given by

$$dI_t = \mathbf{1}_{\{X_t<D\}}(1 - I_t)dN_t - I_t d\mathbf{1}_{\{X_t\ge D\}}$$

in which D is the fixed amount of debt D holded by the firm and X is assumed to satisfy the following stochastic differential equation :

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + (1-\kappa_{I_t})dK_t - (1+\kappa')dZ_t - c_a\mathbf{1}_{\{X_t<D\}}(1-I_t)dN_t,$$

with $\kappa_0 \leq \kappa_1,\ \kappa^{'}$ are constants in $(0,1)$ and $c_a > 0$. The constants κ_0, κ_1 and $\kappa^{'}$ represent proportional transaction costs that must be paid to inject capital in favorable and defavorable cases and to payout dividends. The constant c_a is the fixed penalty cost of being audited during financial distress

Bankruptcy time. We define the time of financial distress under audit process as

$$\tau_t = t - \sup\{s \leq t : I_s = 0\}, \text{ with the convention } \sup \emptyset = 0.$$

The bankruptcy times are then given by

$$\begin{aligned} T_0^x &:= \inf\{t \geq 0 : X_t^x < 0 \text{ or } \tau_t > \delta\} \\ T_1^{(t,x)} &= \inf\{u \geq t : X_u^{(t,x)} < 0 \text{ or } \tau_u > \delta - t\mathbb{1}_{\{\inf_{t\leq s\leq u} I_u = 1\}}\}. \end{aligned}$$

State spaces. We define the state space as $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$, with

$$\mathcal{S}_0 = [0, +\infty) \qquad \text{and} \qquad \mathcal{S}_1 = [0, \delta] \times [0, D].$$

The firm is in financial distress and audited ($I_t = 1$) when $(t, X_t) \in \mathcal{S}_1 \backslash (\{0\} \times [0, D))$. Note that when $(t, X_t) \in \{0\} \times [0, D)$ both $I_t = 1$ and $I_t = 0$ are possible.

Value functions. Value functions of the problem are then defined by

$$\begin{aligned} v_0(x) &:= \sup_{(Z,K)\in\mathcal{A}} \mathbf{E}_{x,0} \int_0^{T_0^x} e^{-\rho s}\, d(Z_s - K_s), \quad x \in \mathcal{S}_0, \\ v_1(t,x) &:= \sup_{(Z,K)\in\mathcal{A}} \mathbf{E}_{t,x,1} \int_t^{T_1^{(t,x)}} e^{-\rho s}\, d(Z_s - K_s), \quad (t,x) \in \mathcal{S}_1, \end{aligned}$$

in which $\rho > 0$ is a discount rate, and

$$\mathcal{A} = \{(Z, K) \in \mathcal{I}_{\mathbb{F}}^2 : \ \forall u \geq 0,\ Z_u - Z_{u^-} \leq (X_{u^-} - D)^+ \text{ and } \int_0^{+\infty} \mathbb{1}_{\{I_u = 1\}} dZ_u = 0\}$$

is the set of admissible dividend and capital injection policies. $\mathcal{I}_{\mathbb{F}}$ denotes the set of non decreasing and $\mathbb{F}$-predictable processes.

Characterization as viscosity solution.

The value functions v_i will be shown be the unique solution of the following system of variational inequalities :

$$\begin{aligned} 0 &= \min\left(-\mathcal{L}v_0 - \mathcal{J}(v_0, v_1), \mathbb{1}_{[0,D)} + \mathbb{1}_{[D,+\infty)}\left[v_0' - \frac{1}{1+\kappa'}\right], \frac{1}{1-\kappa_0} - v_0'\right) \text{ on } \mathring{\mathcal{S}}^0, \\ 0 &= \min\left(-\mathcal{L}v_1 - \frac{\partial v_1}{\partial t}, \frac{1}{1-\kappa_1} - \frac{\partial v_1}{\partial x}\right) \text{ on } \mathring{\mathcal{S}}^1, \\ 0 &= \min\left(v_1, \frac{1}{1-\kappa_1} - \frac{\partial v_1}{\partial x}\right) \text{ on } [0,\delta] \times \{0\}, \\ 0 &= \min\left(v_0(0), \frac{1}{1-\kappa_0} - v_0'(0)\right), \end{aligned}$$

where we have set

$$\mathcal{L}v = -\rho v + \mu(x)\frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2} \quad \text{and} \quad \mathcal{J}(v,w) = \lambda \mathbb{1}_{[0,D]}\Big[w(\cdot - c_a, 0)\mathbb{1}_{[c_a,D]} - v\Big],$$

with the following boundary conditions :

$$\lim_{x\uparrow D} v_1(t,x) = v_0(D) \text{ for } t \in (0,\delta], \tag{5.2.1}$$

$$\lim_{t\uparrow\delta} v_1(t,x) = g(x) := \max(v_0(D) - (D-x)/(1-\kappa_1), 0) \text{ for } x \in [0,D] \tag{5.2.2}$$

$$\lim_{x\uparrow\infty} v_0(x+1) - v_0(x) = \frac{1}{1+\kappa'} \tag{5.2.3}$$

We may deduce from the previous result a description of optimal strategies and compute them thanks to discretization of the Hamilton-Jacobi-Bellman equation.

5.3 Path-dependent American options

To enrich models in market or corporate finance, we would like to have more mathematical tools to build numerical methods for non-markovian control problems. In this context the characterisation of objective functions as solutions of HJB equations was not longer possible. Some recent studies have formalized the links between some stochastic control problems and path-dependent variational inequality (PDVI). In a work, summarized below and made with V. Ly Vath and M. Mnif (see **(16)**), we characterize the value function of an optimal stopping problem as the unique viscosity solution of a Path Dependent Variational Inequality (PDVI) in the class of uniformly bounded and continuous processes in (t,ω). We propose a monotone, stable and consistent numerical scheme. We show that the solution of the numerical scheme converges to the unique viscosity solution of the associated PDVI.
We then provide and analyze a discrete-time approximation scheme for the solution of the PDVI :

$$\min\Big[u(t,\omega) - h(t,\omega);\ -\partial_t u(t,\omega) - \frac{1}{2}\mathrm{Tr}(\sigma\sigma^*\partial^2_{\omega\omega}u)(t,\omega) - \lambda(t,\omega)\partial_\omega u(t,\omega) - f(t,\omega)\Big] = 0,$$

on $[0,T)$, where T is a given terminal time, $\omega \in \Omega$ is a continuous path from $[0,T]$ to $\mathbb{R}^d$ starting from the origin, the diffusion coefficient σ is a mapping from $[0,T]\times\Omega$ to $\mathbb{R}^{d\times d}$ with σ^* denotes its transpose, the drift coefficient λ is a mapping from $[0,T]\times\Omega$ to $\mathbb{R}^d$. The unkown process $\{u(t,w),\ t\in[0,T]\}$ is required to be continuous in (t,ω). Dupire [47] gives the definition of the derivatives $\partial_t u$, $\partial_\omega u$ and $\partial^2_{\omega\omega}u$ which appear in the PDVI and proves a functional Itô's formula. The smoothness requirement of u assumed is not realistic in our case. The derivatives should be interpreted in the viscosity sense. Ekren, Keller, Touzi and Zhang [49] and later Ren, Touzi and Zhang [98] proposed the notion of viscosity solution of path dependent semi-linear PDEs. It is also viewed as viscosity solutions of non-Markovian Backward Stochastic Differential Equations (BSDEs). It is a powerful tool for this type of problems since the derivatives are interpreted in a weaker sense. This theory is an extension of the viscosity solutions in finite dimensional spaces introduced by Crandall and Lions [38]. In the infinite dimensional case, we lose the property of local compacity of $\mathbb{R}$ and the stopping times will play a key role. The set of tested processes is enlarged since we consider all the smooth processes wich are tangent in mean and not pointwise as in

the finite dimensional case.
In our case we relate the PDVI to a Reflected BSDE. It is well-known that the solution of a RBSDE is the value function of an optimal stopping problem which is useful to prove that the value function is a viscosity solution of the associated PDVI. We prove also a comparison result which is easier to obtain than in the finite dimension case since the set of test processes is enlarged. We notice that the optimal stopping problem is a degenerate optimal control problem since we have only the choice between stopping and receiving the pay-off or keeping the system evolving. As a consequence, we don't need to the nonlinear expectations to catch all the possible controls which make the proofs less technical then in Ekren [50] who studied viscosity solutions of obstacle problems for fully nonlinear path dependent PDEs.
The main contribution of our work is to propose an efficient numerical scheme for the path dependent optimal stopping problem, the convergence of which is ensured by the uniqueness result. It is an extension of the convergence theorem of Barles and Souganidis [11]. The main difficulty is that the space Ω is no longer local compact. Recently, convergence of numerical schemes for path dependent PDEs are studied by Zhang and Zhuo [107] and Ren and Tan [99].
In our case, we propose a convergent numerical scheme to price American fixed strike lookback option whose pay-off is path-dependent.
To our knowledge, in the litteraure, we find only numerical procedures which show empirically the convergence of numerical schemes solutions to such path dependent American options. Dai and Kwok [39] and Lai and Lim [77] examine the early exercice policies and pricing behaviours of one asset American option with lookback payoff structures. They give the variational inequality satisfied by the path-dependent value function. Implicitly, they assumed that the classical Itô's calculus and the dynamic programming principle hold as in the Markovian context. Hull and White [66] show how binomial and trinomial tree methods can be extended to value many types of options with path-dependent payoffs. As an example, they determine thanks to their discrete model, the price of a Lookback put option, whose payoff is a function of the maximum stock price realized during the option's life. Babs [8] adapts the binomial scheme to investigate the impact on the value of these options.

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