Enlargements of Filtrations with Finance in view

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These notes are not complete, and have to be considered as a preliminary version of the forthcoming book in Springer brief, co-authored with A. Aksamit [8]. The first version was written to give a support to the lectures given by Monique Jeanblanc and Giorgia Callegaro at University El Manar, Tunis, in March 2010 and later by Monique Jeanblanc in various schools: Jena, in June 2010, Beijing, for the Sino-French Summer institute, June 2011, CREMMA school, ENIT, Tunis, March 2014, Moscow university in April 2014, BICMR summer school Beijing, June 2015 and 2017 and as a main course in Master 2, Marne La Vallée, Winter 2011.

Some solutions of exercices are written by Giorgia Callegaro.

A large part of these notes can be found in the book

Mathematical Methods in Financial Markets,

by Jeanblanc, M., Yor M., and Chesney, M. (Springer), indicated as [3M]. We thank Springer Verlag for allowing us to reproduce part of this volume.

Many examples related to Credit risk can be found in

Credit Risk Modeling

by Bielecki, T.R., Jeanblanc, M. and Rutkowski, M., CSFI lecture note series, Osaka University Press, 2009, indicated as [BJR].

I thank all the participants to the different lectures for their remarks, questions and comments.

Contents

Chapter 1

Theory of Stochastic Processes

In this chapter, we recall some facts on theory of stochastic processes. Proofs can be found for example in Dellacherie [40], Dellacherie and Meyer [44], He, Wang and Yan [67] and Rogers and Williams [119].

1.1 Background

As usual, we start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ is a given filtration satisfying the usual conditions, i.e., $\mathbb{F}$ is continuous on right ($\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$) and $\mathcal{F}_0$ contains all negligeable sets, and $\mathcal{F} = \mathcal{F}_\infty$. A process X is a family of random variables such that $(\omega, t) \to X_t(\omega)$ is $\mathcal{F} \otimes \mathcal{B}$ measurable, where $\mathcal{B}$ is the Borel field on $\mathbb{R}^+$ (one says also measurable process).

1.1.1 Path properties

Definition 1.1.1 *1) A process X is **continuous** if, for almost all ω, the map $t \to X_t(\omega)$ is continuous. A process X is continuous on the right with limits on the left (in short **càdlàg** following the French acronym[1]) if, for almost all ω, the map $t \to X_t(\omega)$ is càdlàg.*
*2) A process A is **increasing** if $A_0 = 0$, A is right-continuous, and $A_s \leq A_t$, a.s. for $s \leq t$. An increasing process $A = (A_t, t \geq 0)$ is integrable if $\mathbb{E}(A_{\infty-}) < \infty$, where $A_{\infty-} = \lim_{t\to\infty} A_t$.*

Sometimes, one has to consider increasing processes defined for $t \in [0, \infty]$ (with a possible jump at $+\infty$). In that case, the process is integrable if $E(A_\infty) < \infty$.

For a (right-continuous) increasing process A, we note $\int_a^b \varphi_s dA_s := \int_{]a,b]} \varphi_s dA_s$ as soon as the integral is well defined. The point here is that the integration is done on the interval $]a, b]$.

Definition 1.1.2 *A process X is $\mathbb{F}$-adapted if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.*

The **natural filtration** $\mathbb{F}^X$ of a stochastic process X is the smallest filtration $\mathbb{F}$ which satisfies the usual hypotheses and such that X is $\mathbb{F}$-adapted. We shall write in short (with an abuse of notation) $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$.

Remark 1.1.3 It is not true in general that if $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ are right-continuous, the filtration $\mathbb{K}$ defined as $\mathcal{K}_t := \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$ is right-continuous. Nevertheless, we shall often write $\mathbb{F} \vee \widetilde{\mathbb{F}}$ (with an abuse of notation) the smallest right-continuous filtration which contains $\mathbb{F}$ and $\widetilde{\mathbb{F}}$.

[1] In French, continuous on the right is **c**ontinu **à** **d**roite, and with limits on the left is admettant des **l**imites **à** **g**auche. We shall also use càd for continuous on the right. The use of this acronym comes from P-A. Meyer.

Exercise 1.1.4 Starting from a non continuous on right filtration $\mathbb{F}^0$, define the smallest right-continuous filtration $\mathbb{F}$ which contains $\mathbb{F}^0$. $\triangleleft$

In all the lecture, we shall write $X \in \mathcal{F}_T$ (resp. $X \in b\mathcal{F}_T$) for X is an $\mathcal{F}_T$-measurable (resp. a bounded $\mathcal{F}_T$-measurable) random variable.

1.1.2 Stopping times

A random variable τ, valued in $[0,\infty]$ is an $\mathbb{F}$-stopping time if, for any $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.

A stopping time τ is **predictable** if there exists an increasing sequence (τ_n) of stopping times such that almost surely

(i) $\lim_n \tau_n = \tau$,

(ii) $\tau_n < \tau$ for every n on the set $\{\tau > 0\}$. If needed, we shall make precise the choice of the filtration, writing that the $\mathbb{F}$-stopping time τ is $\mathbb{F}$-predictable.

A stopping time τ is **totally inaccessible** if $\mathbb{P}(\tau = \vartheta < \infty) = 0$ for any predictable stopping time ϑ (or, equivalently, if for any increasing sequence of stopping times $(\tau_n, n \geq 0)$, $\mathbb{P}(\{\lim \tau_n = \tau\} \cap A) = 0$ where $A = \cap_n \{\tau_n < \tau\}$).

If all $\mathbb{F}$-martingales are continuous, then any $\mathbb{F}$-stopping time is predictable. This is the case in particular if $\mathbb{F}$ is a Brownian filtration

Definition 1.1.5 *If τ is an $\mathbb{F}$-stopping time, the σ-algebra $\mathcal{F}_\tau$ of events prior to τ, and the σ-algebra $\mathcal{F}_{\tau-}$ of events strictly prior to τ are defined as:*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \,:\, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t\}$$

whereas $\mathcal{F}_{\tau-}$ is the smallest σ-algebra which contains $\mathcal{F}_0$ and all the sets of the form $A \cap \{t < \tau\}, t > 0$ for $A \in \mathcal{F}_t$.

For $A \in \mathcal{F}_\tau$, one sets τ_A the stopping time defined as $\tau_A = \tau 1\!\!1_A + \infty 1\!\!1_{A^c}$.

Exercise 1.1.6 Prove that τ_A is a stopping time. $\triangleleft$

Exercise 1.1.7 Show that for a stopping time τ, one has $\tau \in \mathcal{F}_{\tau-}$ and $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$. Find an example where $\mathcal{F}_{\tau-} \neq \mathcal{F}_\tau$ $\triangleleft$

Exercise 1.1.8 Check that if $\mathbb{F} \subset \mathbb{G}$ and τ is an $\mathbb{F}$-stopping time, (resp. $\mathbb{F}$-predictable stopping time) it is a $\mathbb{G}$-stopping time, (resp. $\mathbb{G}$-predictable stopping time). Give an example where τ is a $\mathbb{G}$-stopping time but not an $\mathbb{F}$-stopping time. Give an example where τ is a $\mathbb{G}$-predictable stopping time, and an $\mathbb{F}$-stopping time, but not a predictable $\mathbb{F}$-stopping time. $\triangleleft$

1.1.3 Predictable and optional σ-algebra

If τ and ϑ are two stopping times, the **stochastic interval** $]\!]\vartheta, \tau]\!]$ is the set $\{(\omega, t) \,:\, \vartheta(\omega) < t \leq \tau(\omega)\}$. In the same way, we shall use the notation $[\![\vartheta, \tau]\!]$, as well as for other stochastic intervals.

Proposition 1.1.9 *Let $\mathbb{F}$ be a given filtration.*

- *The* **optional** *σ-algebra $\mathcal{O}$ is the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by càdlàg $\mathbb{F}$-adapted processes (considered as mappings on $\mathbb{R}^+ \times \Omega$). The optional σ-algebra $\mathcal{O}$ is equal to the σ-algebra generated on $\mathcal{F} \otimes \mathcal{B}$ by the stochastic intervals $[\![\tau, \infty[\![$ where τ is an $\mathbb{F}$-stopping time.*

- *The* **predictable** *σ-algebra $\mathcal{P}$ is the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by the $\mathbb{F}$-adapted càg (or continuous) processes. The predictable σ-algebra $\mathcal{P}$ is equal to the σ-algebra generated on $\mathcal{F} \otimes \mathcal{B}$ by the stochastic intervals $]\!]\vartheta, \tau]\!]$ where ϑ and τ are two $\mathbb{F}$-stopping times such that $\vartheta \leq \tau$.*

If necessary, we shall note $\mathcal{P}(\mathbb{F})$ this predictable σ-algebra, to emphasize the rôle of $\mathbb{F}$. A process X is said to be $\mathbb{F}$-**predictable** (resp. $\mathbb{F}$-**optional**) if the map $(\omega, t) \to X_t(\omega)$ is $\mathcal{P}$-measurable (resp. $\mathcal{O}$-measurable).

Example 1.1.10 An adapted càg process is predictable.

The inclusion $\mathcal{P} \subset \mathcal{O}$ holds. These two σ-algebras $\mathcal{P}$ and $\mathcal{O}$ are equal if all $\mathbb{F}$-martingales are continuous. Note that $\mathcal{O} = \mathcal{P}$ if and only if any stopping time is predictable. In general

$$\mathcal{O} = \mathcal{P} \vee \sigma(\Delta M, M \text{ describing the set of } \mathbb{F} \text{ martingales}).$$

If X is a predictable (resp. optional) process and τ a stopping time, then the stopped process $X^\tau = (X^\tau_t = X_{t\wedge\tau}, t \geq 0)$ is also predictable (resp. optional). If X is a càdlàg adapted process, then $(X_{t-}, t \geq 0)$ is a predictable process.

If τ is a stopping time, the (càg) process $\mathbb{1}_{\tau<t}$ is predictable. A stopping time τ is **predictable** if and only if the process $(\mathbb{1}_{\{t<\tau\}} = 1 - \mathbb{1}_{\{\tau\leq t\}}, t \geq 0)$ is predictable, that is if and only if the stochastic interval $[\![0, \tau[\![= \{(\omega, t) : 0 \leq t < \tau(\omega)\}$ is predictable. See Dellacherie [40], Dellacherie and Meyer [42] and Cohen & Elliott [51] for related results.

Definition 1.1.11 *A real-valued process X is **progressively measurable** with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, if, for every t, the map $(\omega, s) \to X_s(\omega)$ from $\Omega \times [0, t]$ into $\mathbb{R}$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t])$-measurable.*

Any càd (or càg) $\mathbb{F}$-adapted process is progressively measurable. An $\mathbb{F}$-progressively measurable process is $\mathbb{F}$-adapted. If X is progressively measurable, then

$$\mathbb{E}\left(\int_0^\infty X_t dt\right) = \int_0^\infty \mathbb{E}(X_t)\, dt,$$

where the existence of one of these expressions implies the existence of the other.
If X is $\mathbb{F}$-progressively measurable and τ an $\mathbb{F}$-stopping time, then the r.v. X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$.

If τ is a **random time** (i.e. a non negative r.v.), the σ-algebra $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau-}$ are defined as

$$\begin{aligned}
\mathcal{F}_\tau &= \sigma(Y_\tau, Y \text{ is an } \mathbb{F} - \text{optional process}) \\
\mathcal{F}_{\tau-} &= \sigma(Y_\tau, Y \text{ is an } \mathbb{F} - \text{predictable process})
\end{aligned}$$

1.1.4 Doob's maximal identity

We present here a result that will be used letter on.

Definition 1.1.12 *An $\mathbb{F}$-local martingale N belongs to the class $(\mathcal{C}_0)$, if it is strictly positive, with no positive jumps, and $\lim_{t\to\infty} N_t = 0$.*

Lemma 1.1.13 *For any $a > 0$, we have:*

$$\mathbb{P}(S_\infty > a) = \left(\frac{x}{a}\right) \wedge 1. \tag{1.1.1}$$

In particular, $\dfrac{x}{S_\infty}$ is a uniform random variable on $(0, 1)$.
For any $\mathbb{F}$-stopping time ϑ, denoting $S^\vartheta = \sup_{u\geq\vartheta} N_u$:

$$\mathbb{P}\left(S^\vartheta > a|\mathcal{F}_\vartheta\right) = \left(\frac{N_\vartheta}{a}\right) \wedge 1, \tag{1.1.2}$$

Hence $\dfrac{N_\vartheta}{S^\vartheta}$ is also a uniform random variable on $(0, 1)$, independent of $\mathcal{F}_\vartheta$.

PROOF: The first part is left as the Exercise 1.6.3. The second part is an application of the first one for the martingale $(N_{\vartheta+t}, t \geq 0)$ and the filtration $(\mathcal{F}_{\vartheta+t}, t \geq 0)$. □

Exercise 1.1.14 et B be a Brownian motion. Prove that $\exp(\lambda B_t - \frac{\lambda^2}{2}t)$ belongs to $(\mathcal{C}_0)$. ◁

1.1.5 Localization

Definition 1.1.15 *An adapted, right-continuous process M is an $\mathbb{F}$-**local martingale** if there exists a sequence of stopping times (τ_n) such that:*

- *The sequence τ_n is increasing and $\lim_n \tau_n = \infty$, a.s.*
- *For every n, the stopped process $M^{\tau_n} 1\!\!1_{\{\tau_n > 0\}}$ is an $\mathbb{F}$-martingale (we recall that $M_t^\tau = M_{t \wedge \tau}$).*

A sequence of stopping times such that the two previous conditions hold is called a localizing or reducing sequence. We also use the following definitions: A local martingale M is locally square integrable if there exists a localizing sequence of stopping times (τ_n) such that $M^{\tau_n} 1\!\!1_{\{\tau_n > 0\}}$ is a square integrable martingale. An increasing process A is locally integrable if there exists a localizing sequence of stopping times such that A^{τ_n} is integrable. By similar localization, we may define locally bounded martingales, local super-martingales, and locally finite variation processes.
If M is a local martingale, it is always possible to choose the localizing sequence $(\tau_n, n \geq 1)$ such that each martingale $M^{\tau_n} 1\!\!1_{\{\tau_n > 0\}}$ is uniformly integrable.
We denote by $\mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ the space of $\mathbb{P}$ local martingales relative to $\mathbb{F}$.

Exercise 1.1.16 Prove that a positive local martingale is a super-martingale. ◁

1.1.6 Doob-Meyer decomposition

An adapted process X is said to be of class[2] (D) if the collection $X_\tau 1\!\!1_{\tau < \infty}$ where τ is a stopping time is uniformly integrable.

If Z is a supermartingale of class (D), there exists a unique increasing, integrable and predictable process A such that $Z_t = \mathbb{E}(A_\infty - A_t | \mathcal{F}_t)$. In particular, any supermartingale of class (D) can be written as $Z = M - A$ where M is a uniformly integrable martingale. The decomposition is unique.

Any supermartingale can be written as $Z = M - A$ where M is a local martingale and A a predictable increasing process. The decomposition is unique.

There are other decompositions of supermartingales, as a sum of a martingale and an optional process which satisfies particular conditions that are useful. We shall comment that later on.

Multiplicative decomposition of positive supermartingales

Lemma 1.1.17 *Let Z be a positive supermartingale of class D. There exists a local martingale N and a predictable decreasing process D such that $Z = ND$.*

PROOF: Assume that the multiplicative decomposition exists. Then, from Yoeurp's lemma 1.2.11 $dZ_t = D_t dN_t + N_{t-} dD_t$, and the Doob-Meyer decomposition of Z is $dZ_t = d\mu_t - dA_t^p$. From uniqueness

$$dA_t^p = -N_{t-} dD_t \tag{1.1.3}$$

[2] Class (D) is in honor of Doob.

which in particular, yields to $\Delta A^p_t = -N_{t-}\Delta D_t$, so that $D_{t-}(1 - \frac{1}{Z_{t-}}\Delta A^p_t) = D_t$. Therefore, using (1.1.3) again, $dD_t = -\frac{D_t}{Z_{t-}-\Delta A^p_t} dA^p_t$, so that

$$D_t = \exp\left(- \int_0^t \frac{1}{Z_{s-} - \Delta A^p_s} dA^p_s \right)$$

Setting $dN_t = \frac{1}{D_t} d\mu_t$, we obtain the existence of the decomposition. If A^p is continuous, $D_t = e^{-\Gamma_t}$ where $\Gamma_t = \int_0^t \frac{1}{Z_{s-}} dA^p_s$. □

1.2 Semi-martingales

1.2.1 Definition

An $\mathbb{F}$-adapted process X is an $\mathbb{F}$-semi-martingale if $X = M + A$ where M is an $\mathbb{F}$-local martingale and A an $\mathbb{F}$-adapted process with finite variation. If there exists a decomposition with a process A which is predictable, the decomposition $X = M + A$ where M is an $\mathbb{F}$-martingale and A an $\mathbb{F}$-predictable process with finite variation is unique and X is called a **special** semi-martingale. If X is continuous, the process A is continuous.

In general, if $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ is a filtration larger than $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, i.e., $\mathcal{F}_t \subset \mathcal{G}_t, \forall t \geq 0$ (we shall write $\mathbb{F} \subset \mathbb{G}$), it is not true that an $\mathbb{F}$-martingale remains a martingale in the filtration $\mathbb{G}$. It is not even true that $\mathbb{F}$-martingales remain $\mathbb{G}$-semi-martingales. One of the goal of this book is to give conditions so that this property holds.

Example 1.2.1 (a) Let $\mathcal{G}_t = \mathcal{F}_\infty$. Then, the only $\mathbb{F}$-martingales which are $\mathbb{G}$-martingales are constants.
(b) An interesting example is Azéma's martingale μ, defined as follows. Let B be a Brownian motion and $g_t = \sup\{s \leq t, B_s = 0\}$. The process

$$\mu_t = (\mathrm{sgn} B_t)\sqrt{t - g_t},\ t \geq 0$$

is a martingale in its own filtration. This discontinuous $\mathbb{F}^\mu$-martingale is not an $\mathbb{F}^B$-martingale, it is not even an $\mathbb{F}^B$-semi-martingale.
(c) Let $\mathbb{F}$ be the filtration generated by a Brownian motion B and $\mathcal{G}_t = \mathcal{F}_{t+\delta}$. The process B is not a $\mathbb{G}$-semimartingale.

Exercise 1.2.2 Let B be a Brownian motion. Prove that $W_t = \int_0^t \mathrm{sgn}(B_s)\, dB_s$ defines an $\mathbb{F}^B$ and an $\mathbb{F}^W$ Brownian motion.
Prove that $\beta_t = B_t - \int_0^t \frac{B_s}{s} ds$ defines a Brownian motion (in its own filtration) which is not a Brownian motion in $\mathbb{F}^B$. ◁

1.2.2 Properties

Proposition 1.2.3 *Let $\mathbb{G}$ be a filtration larger than $\mathbb{F}$, i.e., $\mathbb{F} \subset \mathbb{G}$. If x is a u.i. (uniformly integrable) $\mathbb{F}$-martingale, then there exists a $\mathbb{G}$-martingale X, such that $\mathbb{E}(X_t|\mathcal{F}_t) = x_t$, $t \geq 0$.*

PROOF: The process X defined by $X_t := \mathbb{E}(x_\infty|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale, and

$$\mathbb{E}(X_t|\mathcal{F}_t) = \mathbb{E}\big(\mathbb{E}(x_\infty|\mathcal{G}_t)|\mathcal{F}_t\big) = \mathbb{E}(x_\infty|\mathcal{F}_t) = x_t\ .$$

□

The uniqueness of such a martingale X is not claimed in the above proposition and it is not true in general.

We recall an important (but difficult) result due to Stricker [127].

Proposition 1.2.4 *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. If X is a $\mathbb{G}$-semimartingale which is $\mathbb{F}$-adapted, then it is an $\mathbb{F}$-semimartingale.*

One has also the (obvious) following result (see Exercise 1.2.9)

Proposition 1.2.5 *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. If X is a $\mathbb{G}$-martingale which is $\mathbb{F}$-adapted, then it is also an $\mathbb{F}$-martingale.*

Remark 1.2.6 This result does not extend to local martingales. See Stricker [127] and Föllmer and Protter [61].

Exercise 1.2.7 Let N be a Poisson process (i.e., a process with stationary and independent increments, such that the law of N_t is a Poisson law with parameter λt). Prove that the process M defined as $M_t = N_t - \lambda t$ is a martingale and that the process $M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t$ is also a martingale. Prove that for any $\theta \in [0, 1]$,

$$N_t = \theta(N_t - \lambda t) + (1-\theta)N_t + \theta\lambda t = \mu_t + (1-\theta)N_t + \theta\lambda t$$

is a decomposition of the semi-martingale N, where μ is a martingale. For which decomposition is the finite variation process $(1-\theta)N_t + \theta\lambda t$ a predictable process ? $\triangleleft$

Exercise 1.2.8 Let τ be a random time. Prove that τ is a $\mathbb{H}$-stopping time, where $\mathbb{H}$ is the natural filtration of $H_t = 1\!\!1_{\{\tau \le t\}}$, and that τ is a $\mathbb{G}$ stopping time, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, for any filtration $\mathbb{F}$. $\triangleleft$

Exercise 1.2.9 Prove that, if M is a $\mathbb{G}$-martingale, then $\widehat{M}$ defined as $\widehat{M}_t = \mathbb{E}(M_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. $\triangleleft$

Exercise 1.2.10 Prove that, if $\mathbb{G} = \mathbb{F} \vee \widetilde{\mathbb{F}}$ where $\widetilde{\mathbb{F}}$ is independent of $\mathbb{F}$, then any $\mathbb{F}$ martingale remains a $\mathbb{G}$-martingale.
Prove that, if $\mathbb{F}$ is generated by a Brownian motion W, and if there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\widetilde{\mathbb{F}}$ is independent of $\mathbb{F}$ under $\mathbb{Q}$, then any $(\mathbb{P}, \mathbb{F})$-martingale remains a $(\mathbb{P}, \mathbb{G})$-semi martingale. $\triangleleft$

1.2.3 Stochastic Integration

If $X = M + A$ is a semi-martingale and Y a (bounded) predictable process, we denote $Y_\centerdot X$ the stochastic integral

$$(Y_\centerdot X)_t := \int_0^t Y_s dX_s = \int_0^t Y_s dM_s + \int_0^t Y_s dA_s$$

The process $Y_\centerdot X$ is a semi-martingale. Note that here, for a right-continuous process, the symbol $\int_0^t Y_s dX_s$ stands for $\int_{]0,t]} Y_s dX_s$, i.e., the upper bound t is included in the integration.

1.2.4 Integration by Parts

By definition, any semi-martingale X admits a decomposition as a local martingale M and a finite variation process. The martingale part admits a decomposition as $M = M^c + M^d$ where M^c is continuous and M^d a discontinuous martingale. The process M^c is denoted in the literature as X^c (even if this notation is missleading!). The optional Itô formula is (for f in C^2, with bounded derivatives)

$$\begin{aligned} f(X_t) &= f(X_0) + \textstyle\int_0^t f'(X_{s^-})dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X^c\rangle_s \\ &\quad + \textstyle\sum_{0<s\le t}[f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s]\,. \end{aligned}$$

where $\Delta X_t = X_t - X_{t-}$ [3].

If U and V are two finite variation processes, the Stieltjes integration by parts formula can be written as follows:

$$\begin{aligned} U_t V_t &= U_0 V_0 + \int_{]0,t]} V_s dU_s + \int_{]0,t]} U_{s^-} dV_s \qquad (1.2.1) \\ &= U_0 V_0 + \int_{]0,t]} V_{s^-} dU_s + \int_{]0,t]} U_{s^-} dV_s + \sum_{s \le t} \Delta U_s \, \Delta V_s \,. \end{aligned}$$

As a partial check, one can verify that the jump process of the left-hand side, i.e., $U_t V_t - U_{t^-} V_{t^-}$, is equal to the jump process of the right-hand side, i.e., $V_{t^-} \Delta U_t + U_{t^-} \Delta V_t + \Delta U_t \, \Delta V_t$.

Let X be a continuous local martingale. The **predictable quadratic variation** process of X is the continuous increasing process $\langle X \rangle$ such that $X^2 - \langle X \rangle$ is a local martingale.
Let X and Y be two continuous local martingales. The predictable covariation process is the continuous finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale. The covariation process of continuous martingales does not depend on the filtration.

Let X and Y be two local martingales. The **covariation process** $[X, Y]$ is the finite variation process such that
(i) $XY - [X, Y]$ is a local martingale
(ii) $\Delta [X, Y]_t = \Delta X_t \Delta Y_t$
The process $[X] = [X, X]$ is non-decreasing; if X is continuous, then $[X] = \langle X \rangle$.
The **predictable covariation process** is (if it exists) the predictable finite variation process $\langle X, Y \rangle$ such that $XY - \langle X, Y \rangle$ is a local martingale.
If X is a semi-martingale with respect to $\mathbb{F}$ and to $\mathbb{G}$, then $[X]$ is independent of the filtration.

The integration by parts for semi-martingales is

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X, Y]_t \,. \qquad (1.2.2)$$

For finite variation processes

$$[U, V]_t = \sum_{s \le t} \Delta U_s \, \Delta V_s$$

and, if Y is with finite variation, Yoeurp's formula states that

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_{s-} dX_s \,. \qquad (1.2.3)$$

We recall also Yoeurp [131] lemma (see also [81, Proposition 9.3.7.1]):

Proposition 1.2.11 *Let X be a semi-martingale.*
a) *If A is a bounded variation process*

$$X_t A_t = X_0 A_0 + \int_0^t X_s dA_s + \int_0^t A_{s-} dX_s \qquad (1.2.4)$$

and $[X, A] = \Delta X_{\bullet} A$.
b) *If A is a predictable process with bounded variation*

$$X_t A_t = X_0 A_0 + \int_0^t X_{s-} dA_s + \int_0^t A_s dX_s \qquad (1.2.5)$$

and $[X, A] = \Delta A_{\bullet} X$.

[3] one can prove that, for a semi-martingale X, the sum is well defined.

Exercise 1.2.12 Prove that if X and Y are continuous, $\langle X, Y\rangle = [X, Y]$.
Prove that if M is the compensated martingale of a Poisson process with intensity λ, $[M] = N$ and $\langle M\rangle_t = \lambda t$. $\triangleleft$

1.3 Change of probability and Girsanov's Theorem

1.3.1 Brownian filtration

Let $\mathbb{F}$ be a Brownian filtration, L an $\mathbb{F}$-martingale, strictly positive such that $L_0 = 1$ and define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$. Then,

$$\widetilde{B}_t := B_t - \int_0^t \frac{1}{L_s} d\langle B, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion. If M is an $\mathbb{F}$-martingale,

$$\widetilde{M}_t := M_t - \int_0^t \frac{1}{L_s} d\langle M, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-local martingale.

1.3.2 Doléans-Dade exponential

Let $\mathbb{F}$ be a Brownian filtration and ψ an adapted process satisfying $\int_0^t \psi_s^2 ds < \infty, \forall t$. The solution of $dL_t = L_t\psi_t dW_t$ is the local martingale

$$L_t = L_0 \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) =: L_0 \mathcal{E}(\psi_{\bullet} W)_t$$

If $\mathbb{E}(L_t) = 1$, the process L is a martingale.

If L is a strict local martingale, the positive measure $\mathbb{Q}$ defined as $d\mathbb{Q} = L_t d\mathbb{P}$ is not a probability ($\mathbb{Q}(\Omega) \neq 1$)

For a continuous martingale M, the solution of $dL_t = L_t\psi_t dM_t$ is a positive local martingale

$$L_t = L_0 \exp\left(\int_0^t \psi_s dM_s - \frac{1}{2}\int_0^t \psi_s^2 d\langle M\rangle_s\right) = L_0 \mathcal{E}(\psi_{\bullet} M)_t$$

1.3.3 General case

More generally, let $\mathbb{F}$ be a filtration and L an $\mathbb{F}$-martingale, strictly positive such that $L_0 = 1$ and define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$. Then, if M is an $\mathbb{F}$-martingale,

$$\widetilde{M}_t := M_t - \int_0^t \frac{1}{L_s} d[M, L]_s$$

is a $(\mathbb{Q}, \mathbb{F})$-martingale. If the predictable co-variation process $\langle M, L\rangle$ exists,

$$M_t - \int_0^t \frac{1}{L_{s-}} d\langle M, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-local martingale.

If M is a discontinuous martingale, the solution of $dL_t = L_{t-}\psi_t dM_t$ can take negative values and $\mathbb{Q}$ is a signed measure. The solution of $dL_t = L_{t-} dY_t$ is

$$\mathcal{E}(Y)_t := \exp\left(Y_t - Y_0 - \frac{1}{2}\langle Y^c\rangle_t\right) \prod_{s\le t}(1 + \Delta Y_s)e^{-\Delta Y_s} .$$

The solution of $dL_t = L_{t-}\psi_t dM_t$ is positive if $\psi\Delta M > -1$.

1.3.4 Itô-Kunita-Wentcell formula

We recall here the Itô-Kunita-Wentcell formula (see Kunita [98]). Let $F_t(x)$ be a family of stochastic processes, continuous in $(t,x) \in (\mathbb{R}_+ \times \mathbb{R}^d)$ a.s., and satisfying the following conditions:
(i) for each $t > 0$, $x \to F_t(x)$ is C^2 from $\mathbb{R}^d$ to $\mathbb{R}$,
(ii) for each x, $(F_t(x), t \geq 0)$ is a continuous semimartingale

$$dF_t(x) = \sum_{j=1}^{n} f_t^j(x)\, dM_t^j ,$$

where M^j are continuous semimartingales, and $f^j(x)$ are stochastic processes continuous in (t,x), such that for every $s > 0$, the map $x \to f_s^j(x)$ is C^1, and for every x, $f^j(x)$ is an adapted process. Let $X = (X^1, \cdots, X^d)$ be a continuous semimartingale. Then

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^{n} \int_0^t f_s^j(X_s)\, dM_s^j + \sum_{i=1}^{d} \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)\, dX_s^i \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{n} \int_0^t \frac{\partial f_s}{\partial x_i}(X_s)\, d\langle M^j, X^i\rangle_s + \frac{1}{2}\sum_{i,k=1}^{d} \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k}\, d\langle X^k, X^i\rangle_s . \end{aligned}$$

See Bank and Baum [17] for an extension to processes with jumps.

1.4 Projections and Dual Projections

In this section, after recalling some basic facts about optional and predictable projections, we introduce the concept of a dual predictable (resp. optional) projection, which leads to the fundamental notion of predictable compensators. We recommend the survey paper of Nikeghbali [113].

1.4.1 Definition of Projections

Let X be a bounded (or positive) process, and $\mathbb{F}$ a given filtration (we do not assume that X is $\mathbb{F}$-adapted). The **optional projection** of X is the unique optional process ${}^oX$ which satisfies: for any $\mathbb{F}$-stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^oX_\tau \mathbb{1}_{\{\tau<\infty\}}) . \tag{1.4.1}$$

In case where many filtrations are involved, we shall use the notation ${}^{o,\mathbb{F}}X$ for the $\mathbb{F}$-optional projection. For any $\mathbb{F}$-stopping time τ, let $\Gamma \in \mathcal{F}_\tau$ and apply the equality (1.4.1) to the stopping time $\tau_\Gamma = \tau \mathbb{1}_\Gamma + \infty \mathbb{1}_{\Gamma^c}$. We get the re-inforced identity:

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_\tau) = {}^oX_\tau \mathbb{1}_{\{\tau<\infty\}} .$$

In particular, if A is an increasing process, then, for $s \leq t$:

$$\mathbb{E}({}^oA_t - {}^oA_s|\mathcal{F}_s) = \mathbb{E}(A_t - A_s|\mathcal{F}_s) \geq 0 . \tag{1.4.2}$$

Note that, for any t, $\mathbb{E}(X_t|\mathcal{F}_t) = {}^oX_t$. However, $\mathbb{E}(X_t|\mathcal{F}_t)$ is defined almost surely for any t; thus uncountably many null sets are involved, hence, a priori, $\mathbb{E}(X_t|\mathcal{F}_t)$ is not a well-defined process whereas ${}^oX$ takes care of this difficulty.

Comment 1.4.1 Let us comment the difficulty here. If X is an integrable random variable, the quantity $\mathbb{E}(X|\mathcal{F}_t)$ is defined a.s., i.e., if $X_t = \mathbb{E}(X|\mathcal{F}_t)$ and $\widetilde{X}_t = \mathbb{E}(X|\mathcal{F}_t)$, then $\mathbb{P}(X_t = \widetilde{X}_t) = 1$.

That means that, for any fixed t, there exists a negligible set Ω_t such that $X_t(\omega) = \widetilde{X}_t(\omega)$ for $\omega \notin \Omega_t$. For processes, we introduce the following definition: the process X is a modification (or a **version** of)of Y if, for any t, $\mathbb{P}(X_t = Y_t) = 1$. However, one needs a stronger assumption to be able to compare functionals of the processes. The process X is **indistinguishable from** Y if $\{\omega : X_t(\omega) = Y_t(\omega), \forall t\}$ is a measurable set and $\mathbb{P}(X_t = Y_t, \forall t) = 1$. If X and Y are modifications of each other and are a.s. continuous, they are indistinguishable.
A difficult, but important result (see Dellacherie [39, p.73]) states: Let X and Y two optional (resp. predictable) processes. If for every finite stopping time (resp. predictable stopping time) τ, $X_\tau = Y_\tau$ a.s., then the processes X and Y are indistinguishable.

Likewise, the **predictable projection** of X is the unique predictable process ${}^pX$ such that for any $\mathbb{F}$-predictable stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^pX_\tau \mathbb{1}_{\{\tau<\infty\}}) \,. \tag{1.4.3}$$

As above, this identity reinforces as

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^pX_\tau \mathbb{1}_{\{\tau<\infty\}} \,,$$

for any $\mathbb{F}$-predictable stopping time τ (see Section 1.1 for the definition of $\mathcal{F}_{\tau-}$).

Let τ and ϑ be two stopping times such that $\vartheta \leq \tau$ and X a positive process. If A is an increasing optional process, then,

$$\mathbb{E}\left(\int_\vartheta^\tau X_t dA_t\right) = \mathbb{E}\left(\int_\vartheta^\tau {}^oX_t dA_t\right) \,.$$

If A is an increasing predictable process, then, since $\mathbb{1}_{]\!]\vartheta,\tau]\!]}(t)$ is predictable

$$\mathbb{E}(\int_\vartheta^\tau X_t dA_t) = \mathbb{E}(\int_\vartheta^\tau {}^pX_t dA_t) \,.$$

If A is an increasing integrable (hence optional) adapted process, $\mathbb{E}(\int_{[0,\infty[} X_s dA_s) = \mathbb{E}(\int_{[0,\infty[} {}^oX_s dA_s)$.
If A is an increasing integrable predictable process , $\mathbb{E}(\int_{[0,\infty[} X_s dA_s) = \mathbb{E}(\int_{[0,\infty[} {}^pX_s dA_s)$.

1.4.2 Dual Projections

The notion of interest in this section is that of **dual predictable projection**, which we define as follows:

Proposition 1.4.2 *Let $(A_t, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-predictable increasing process $(A^p_t, t \geq 0)$, called the dual predictable projection of A such that*

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA^p_s\right)$$

for any positive $\mathbb{F}$-predictable process Y.

In the particular case where $A_t = \int_0^t a_s ds$, one has

$$A^p_t = \int_0^t {}^pa_s ds \tag{1.4.4}$$

Proof: See Dellacherie [40, Chapter V], Dellacherie and Meyer [44, Chapter 6, (73), p. 148], or Protter [118, Chapter 3, Section 5]. The integrability condition of $(A^p_t, t \geq 0)$ results from the definition, since for $Y = 1$, one obtains $\mathbb{E}(A^p_{\infty-}) = \mathbb{E}(A_{\infty-})$. □
The dual optional projection is also useful

Proposition 1.4.3 *Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A^o_t, t \geq 0)$, called the dual optional projection of A such that*

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA^o_s\right)$$

for any positive $\mathbb{F}$-optional process Y.

In the particular case where $A_t = \int_0^t a_s ds$, one has

$$A^o_t = \int_0^t {}^o a_s ds \tag{1.4.5}$$

This definition extends to the difference between two integrable increasing processes. The terminology “dual predictable projection" refers to the fact that

$$\mathbb{E}\left(\int_0^\infty Y_s dA^p_s\right) = \mathbb{E}\left(\int_0^\infty {}^p Y_s dA_s\right)$$

for any positive $\mathbb{F}$-measurable process Y. Note that the predictable projection of an increasing process is not necessarily increasing, whereas its dual predictable projection is.

If X is bounded and A (not necessarily adapted) has integrable variation, then

$$\mathbb{E}((X_\bullet A^p)_\infty) = \mathbb{E}(({}^p X_\bullet A)_\infty)\,.$$

This is equivalent to: for $s < t$,

$$\mathbb{E}(A_t - A_s|\mathcal{F}_s) = \mathbb{E}(A^p_t - A^p_s|\mathcal{F}_s)\,. \tag{1.4.6}$$

Hence, if A is $\mathbb{F}$-adapted (not necessarily predictable), then $(A_t - A^p_t, t \geq 0)$ is an $\mathbb{F}$-martingale. In that case, A^p is also called the predictable compensator of A.

Example 1.4.4 If N is a Poisson process, $N^p_t = \lambda t$. If X is a Lévy process with Lévy measure ν and f a positive function with compact support which does not contain 0, the predictable compensator of $\sum_{s\leq t} f(\Delta X_s)$ is $t\int f(x)\nu(dx)$

In a general setting, the predictable projection of an increasing process A is a sub-martingale whereas the dual predictable projection is an increasing process. The predictable projection and the dual predictable projection of an increasing process A are equal if and only if ${}^pA$ is increasing.

Proposition 1.4.5 *If A is increasing, the process ${}^oA$ is a sub-martingale and A^p is the predictable increasing process in the Doob-Meyer decomposition of the sub-martingale ${}^oA$. The process ${}^oA - A^p$ is a martingale.*

PROOF: Apply (1.4.1) and (1.4.6). □

Using that terminology, for two martingales X, Y, the predictable covariation process $\langle X, Y\rangle$ is the dual predictable projection of the covariation process $[X, Y]$. The predictable covariation process depends on the filtration.

Example

We now present an example of computation of dual predictable projection. Let $(B_s)_{s\geq 0}$ be an $\mathbb{F}-$Brownian motion starting from 0 and $B^{(\nu)}_s = B_s + \nu s$. Let $\mathbb{G}^{(\nu)}$ be the filtration generated by the process $(|B^{(\nu)}_s|, s \geq 0)$ (which coincides with the one generated by $(B^{(\nu)}_s)^2$) (note that $\mathbb{G}^{(\nu)} \subset \mathbb{F}$).

We now compute the decomposition of the semi-martingale $(B^{(\nu)})^2$ in the filtration $\mathbb{G}^{(\nu)}$ and the $\mathbb{G}^{(\nu)}$-dual predictable projection of the finite variation process $\int_0^t B_s^{(\nu)} ds$.

Itô's lemma provides us with the decomposition of the process $(B^{(\nu)})^2$ in the filtration $\mathbb{F}$:

$$(B_t^{(\nu)})^2 = 2\int_0^t B_s^{(\nu)} dB_s + 2\nu \int_0^t B_s^{(\nu)} ds + t\,. \tag{1.4.7}$$

To obtain the decomposition in the filtration $\mathbb{G}^{(\nu)}$ we remark that,

$$\mathbb{E}(e^{\nu B_s} | \mathcal{F}_s^{|B|}) = \cosh(\nu B_s) (= \cosh(\nu |B_s|))$$

which leads, thanks to Girsanov's Theorem to the equality:

$$\mathbb{E}(B_s + \nu s | \mathcal{F}_s^{|B|}) = \frac{\mathbb{E}(B_s e^{\nu B_s} | \mathcal{F}_s^{|B|})}{\mathbb{E}(e^{\nu B_s} | \mathcal{F}_s^{|B|})} = B_s \tanh(\nu B_s) = \psi(\nu B_s)/\nu\,,$$

where $\psi(x) = x\tanh(x)$. We now come back to equality (1.4.7). Due to (1.4.4), we have just shown that:

$$\text{The dual predictable projection of } 2\nu \int_0^t B_s^{(\nu)} ds \text{ is } 2\int_0^t ds\, \psi(\nu B_s^{(\nu)})\,. \tag{1.4.8}$$

As a consequence,

$$(B_t^{(\nu)})^2 - 2\int_0^t ds\, \psi(\nu B_s^{(\nu)}) - t$$

is a $\mathbb{G}^{(\nu)}$-martingale with increasing process $4\int_0^t (B_s^{(\nu)})^2 ds$. Hence, there exists a $\mathbb{G}^{(\nu)}$-Brownian motion β such that

$$(B_t + \nu t)^2 = 2\int_0^t |B_s + \nu s| d\beta_s + 2\int_0^t ds\, \psi(\nu(B_s + \nu s)) + t\,. \tag{1.4.9}$$

1.4.3 Compensator of a random time

Let τ be a random time and $H_t := 1\!\!1_{\tau \leq t}$. It will be convenient to introduce the following terminology:

Definition 1.4.6 *We call the $\mathbb{F}$-**predictable compensator** associated with τ the $\mathbb{F}$-dual predictable projection A^p of the increasing process $1\!\!1_{\{\tau \leq t\}}$. This dual predictable projection A^p satisfies*

$$\mathbb{E}(Y_\tau) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^p\right) \tag{1.4.10}$$

for any positive, $\mathbb{F}$-predictable process Y.

In case of possible confusion, we shall denote $A^{p,\tau}$, or even $A^{p,\tau,\mathbb{F}}$ this projection.

In the case where τ is an $\mathbb{F}$-stoping time, the process $1\!\!1_{\{\tau \leq t\}} - A^{p,\tau}$ is an $\mathbb{F}$-martingale.

In what follows (in particular in Chapter 8), a main tool will be the process $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$, which is the optional projection of $1\!\!1_{]0,\tau[}$ and is a right-continuous supermartingale (This process is also called the Azéma supermartingale). Note that the process Z_{t-} is the predictable projection of $1\!\!1_{]0,\tau[}$. (see [41, Chapter XX]).

Proposition 1.4.7 *The Doob-Meyer decomposition of the super-martingale $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$ is*

$$Z_t = \mathbb{E}(A_\infty^p | \mathcal{F}_t) - A_t^p = \mu_t - A_t^p$$

where $\mu_t := \mathbb{E}(A_\infty^p | \mathcal{F}_t)$ is the martingale part of Z.

PROOF: From the definition of the dual predictable projection, for any predictable process Y, one has

$$\mathbb{E}(Y_\tau) = \mathbb{E}\left(\int_0^\infty Y_u dA^p_u\right).$$

Let t be fixed and $F_t \in \mathcal{F}_t$. Then, the process $Y_u = F_t \mathbb{1}_{\{t<u\}}, u \geq 0$ is $\mathbb{F}$-predictable. Then

$$\mathbb{E}(F_t \mathbb{1}_{\{t<\tau\}}) = \mathbb{E}(F_t(A^p_\infty - A^p_t)).$$

It follows that $\mathbb{E}(A^p_\infty|\mathcal{F}_t) = Z_t + A^p_t$. Note that μ is a non-negative martingale. □

Proposition 1.4.8 *Let τ be a totally inaccessible stopping time for a filtration $\mathbb{F}$.*
a) The process $H_t = \mathbb{1}_{\tau \leq t}$ is a submartingale, and there exists a continuous increasing, $\mathbb{F}$-adapted process $C = (C_t), t \geq 0$ such that $H - C$ is an $\mathbb{F}$-martingale.
b) If the process C is absolutely continuous with respect to Lebesgue measure, then the compensator of τ is absolutely continuous in any smaller filtration and in particular $F(t) = P(\tau \leq t)$ is an absolutely continuous function.
c) There exists an event $\Gamma \in \mathcal{G}_\tau$ such that τ_Γ has an absolutely continuous compensator and the compensator of τ_{Γ^c} is not absolutely continuous

Notation: We shall use frequently the two following conditions :
Condition **(A)**: the random time τ avoids the $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for any $\mathbb{F}$-stopping time ϑ
Condition **(C)** : all $\mathbb{F}$-martingales are continuous

Lemma 1.4.9 *Let τ a random time, A^p be the $\mathbb{F}$-dual predictable projection of the process H and let A^o be the $\mathbb{F}$-dual optional projection of H.*
*1) Assume condition **(A)**, then $A^p = A^o$ and these processes are continuous.*
*2) Under conditions **(C)** and **(A)**, $Z_t := \mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous.*

PROOF: Indeed, if ϑ is a jump time of A^p, it is an $\mathbb{F}$-stopping time, hence is predictable, and

$$\mathbb{E}(A^p_\vartheta - A^p_{\vartheta-}) = \mathbb{E}(\mathbb{1}_{\tau=\vartheta}) = 0;$$

the continuity of A^p follows.
See Dellacherie and Meyer [44] or Nikeghbali [113].

Lemma 1.4.10 *Let τ be a finite random time such that its associated Azéma's supermartingale Z is continuous. Then τ avoids $\mathbb{F}$-stopping times.*

PROOF: See Coculescu and Nikeghbali [35].

Comment 1.4.11 It can be proved that the martingale

$$\mu_t := \mathbb{E}(A^p_\infty|\mathcal{F}_t) = A^p_t + Z_t$$

is BMO. We recall that a continuous uniformly integrable martingale M belongs to BMO space if there exists a constant m such that

$$\mathbb{E}(\langle M\rangle_\infty - \langle M\rangle_\tau|\mathcal{F}_\tau) \leq m$$

for any stopping time τ. It can be proved (see, e.g., Dellacherie and Meyer [44, Chapter VII]) that the space BMO is the dual of $\mathbb{H}^1$, the space of martingales such that $\mathbb{E}(\sup_{t\geq 0}|M_t|) < \infty$. Recall that $\mathcal{M}_{loc} = \mathbb{H}^1_{loc}$.

Single Jump Processes, Counting Processes

For τ being an $\mathbb{F}$-stopping time and $X = U 1\!\!1_{]\tau,\infty[}$ where U is a non negative r.v. $\mathbb{F}_\tau$ measure, we study some properties of the dual predictable projection (also called compensator) of X.

If τ is positive and predictable, then $X^p = \mathbb{E}(U|\mathcal{F}_{\tau-}) 1\!\!1_{]\tau,\infty[}$

If τ is totally inaccessible, then X^p is the unique continuous finite variation process such that $e^{-\lambda X^p_t}(1 + \lambda U 1\!\!1_{]\tau,\infty[})$ is a local martingale.

Proposition 1.4.12 *If X is a quasi-left continuous counting process with compensator X^p, then X^p is continuous and $\exp(aX_t - (e^a - 1)X^p_t)$ is a local martingale for any a.*

PROOF: The proof follows by Itô's calculus. In a first step, setting $\alpha = e^a - 1$, one has $d\exp(aX_t) = \alpha \exp(aX_{t-})dX_t$. Then, setting $Y_t = \exp(aX_t - (e^a - 1)X^p_t = \exp(aX_t - \alpha X^p_t)$, one deduces

$$\begin{aligned} dY_t &= e^{-\alpha X^p_t} d(e^{aX_t} - \alpha Y_{t-} dX^p_t \\ &= e^{-\alpha X^p_t} e^{aX_{t-}} dX_t - \alpha Y_{t-} dX^p_t = e^{-\alpha X^p_t} e^{aX_{t-}} dX_t - \alpha Y_{t-} d(X_t - X^p_t) \,. \end{aligned}$$

□

Exercise 1.4.13 Let M a càdlàg martingale. Prove that its predictable projection is M_{t-}. ◁

Exercise 1.4.14 Let X be a measurable process such that $\mathbb{E}(\int_0^t |X_s| ds) < \infty$ and $Y_t = \int_0^t X_s ds$. . Prove that ${}^oY_t - \int_0^t {}^oX_s ds$ is an $\mathbb{F}$-martingale ◁

Exercise 1.4.15 Prove that if X is bounded and Y predictable ${}^p(YX) = Y\,{}^pX$ ◁

Exercise 1.4.16 Prove that, more generally than (1.4.8), the dual predictable projection of $\int_0^t f(B^{(\nu)}_s)ds$ is $\int_0^t \mathbb{E}(f(B^{(\nu)}_s)|\mathcal{G}^{(\nu)}_s)ds$ and that

$$\mathbb{E}(f(B^{(\nu)}_s)|\mathcal{G}^{(\nu)}_s) = \frac{f(B^{(\nu)}_s)e^{\nu B^{(\nu)}_s} + f(-B^{(\nu)}_s)e^{-\nu B^{(\nu)}_s}}{2\cosh(\nu B^{(\nu)}_s)} \,.$$

◁

Exercise 1.4.17 Prove that, if $(\alpha_s, s \geq 0)$ is an increasing $\mathbb{F}$-predictable process and X a positive measurable process, then

$$\left(\int_0^{\cdot} X_s d\alpha_s\right)^p = \int_0^{\cdot} {}^pX_s d\alpha_s$$

In particular

$$\left(\int_0^{\cdot} X_s ds\right)^p = \int_0^{\cdot} {}^pX_s ds$$

◁

Exercise 1.4.18 Give an example of random time τ where $A^{(p)}$ and $A^{(o)}$ are different. ◁

1.5 Arbitrages

We recall some standard definitions on arbitrages (adapted to the case of enlargement of filtration). We assume that the financial market has a savings account with null interest rate and a risky asset, with price S which is an $\mathbb{F}$-adapted semi-martingale and a $\mathbb{G}, \mathbb{F}^{(\tau)}$ semi-martingale

Let $\mathbb{K}$ be one of the filtrations $\{\mathbb{F}, \mathbb{G}, \mathbb{F}^{(\tau)}\}$.

For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{K}}(S)$ is said to be an a-admissible $\mathbb{K}$-strategy if $(\theta \centerdot S)_\infty := \lim_{t\to\infty} (\theta \centerdot S)_t$ exists and $V_t(0,\theta) := (\theta \centerdot S)_t \geq -a$ $\mathbb{P}$-a.s. for all $t \geq 0$. We denote by $\mathcal{A}_a^{\mathbb{K}}$ the set of all a-admissible $\mathbb{K}$-strategies. A process $\theta \in L^{\mathbb{K}}(S)$ is called an *admissible* $\mathbb{K}$*-strategy* if $\theta \in \mathcal{A}^{\mathbb{K}} := \bigcup_{a\in\mathbb{R}_+} \mathcal{A}_a^{\mathbb{K}}$.

1.5.1 Classical arbitrages and NFLVR

An admissible strategy yields an Arbitrage Opportunity if $V(0,\theta)_\infty \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}\big(V(0,\theta)_\infty > 0\big) > 0$. In order to avoid confusions, we shall call these arbitrages *classical arbitrages.* If there exists no such $\theta \in \mathcal{A}^{\mathbb{K}}$ we say that the financial market $\mathcal{M}(\mathbb{K}) := (\Omega, \mathbb{K}, \mathbb{P}; S)$ satisfies the No Arbitrage (NA) condition. No Free Lunch with Vanishing Risk (NFLVR) holds in the financial market $\mathcal{M}(\mathbb{K})$ if and only if there exists an Equivalent Martingale Measure in $\mathbb{K}$, i.e., a probability measure $\mathbb{Q}$, such that $\mathbb{Q} \sim \mathbb{P}$ and the process S is a $(\mathbb{Q},\mathbb{K})$-local martingale. If NFLVR holds, there are no classical arbitrages. In this section, we study another kind of arbitrages. We do not present the full theory (for which we refer the reader to [5, 6, 4] and [1]).

If there exists no such $\theta \in \mathcal{A}^{\mathbb{K}}$ we say that the financial market $\mathcal{M}(\mathbb{K}) := (\Omega, \mathbb{K}, \mathbb{P}; S)$ satisfies the No Arbitrage (NA) condition. No Free Lunch with Vanishing Risk (NFLVR) holds in the financial market $\mathcal{M}(\mathbb{K})$ if and only if there exists an Equivalent Martingale Measure in $\mathbb{K}$, i.e., a probability measure $\mathbb{Q}$, such that $\mathbb{Q} \sim \mathbb{P}$ and the process S is a $(\mathbb{Q},\mathbb{K})$-local martingale. If NFLVR holds, there are no classical arbitrages.

For future use, we state the following (obvious) proposition

Proposition 1.5.1 *Assume that the financial market $(S,\mathbb{F})$ is complete, and that S is a $\mathbb{G}$ semi-martingale. Assume that X is an $\mathbb{F}$-martingale such that $X_0 = 1$ and there exists a with $X_t \geq a$. If, $X_\tau \geq 1$ and $\mathbb{P}(X_\tau > 1) > 0$, then, there is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau, \mathbb{G})$.*

PROOF: From the market completeness, there exists an $\mathbb{F}$-predictable process φ such that $X = 1 + \varphi \centerdot S$. Then, $\varphi 1\!\!1_{t\leq\tau}$ is a $\mathbb{G}$-predictable admissible self-financing strategy with initial value 1 and final value $X_\tau - 1$ satisfying $X_\tau - 1 \geq 0$ a.s. and $\mathbb{P}(X_\tau - 1 > 0) > 0$, so it is a classical arbitrage strategy in $(S^\tau, \mathbb{G})$. □

1.5.2 NUPBR

We present another kind of Arbitrages; Unbounded Profit with Bounded Risk.

A non-negative $\mathcal{K}_\infty$-measurable random variable ξ with $\mathbb{P}(\xi > 0) > 0$ yields an Unbounded Profit with Bounded Risk if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V(x,\theta^x)_\infty := x + (\theta^x \centerdot S)_\infty \geq \xi$ $\mathbb{P}$-a.s. If there exists no such random variable, we say that the financial market $\mathcal{M}(\mathbb{K})$ satisfies the No Unbounded Profit with Bounded Risk (NUPBR) condition (we recall that NFLVR is equivalent to NA and NUPBR).

A strictly positive $\mathbb{K}$-local martingale $L = (L_t)_{t\geq 0}$ with $L_0 = 1$ and $L_\infty > 0$ $\mathbb{P}$-a.s. is said to be a local martingale deflator in $\mathbb{K}$ on the time horizon $[0,\varrho]$ if the process LS^ϱ is a $\mathbb{K}$-local martingale; here ϱ is a $\mathbb{K}$-stopping time. If there exists a deflator, then NUPBR holds.

1.6 Some Important Exercices

Exercise 1.6.1 Let B be a Brownian motion, $\mathbb{F}$ its natural filtration and $B_t^* = \sup_{s\leq t} B_s$. Prove that, for $t < 1$,

$$\mathbb{E}(f(B_1^*)|\mathcal{F}_t) = F(1-t, B_t, B_t^*)$$

with

$$F(s,a,b) \quad = \quad \sqrt{\frac{2}{\pi s}} \left(f(b) \int_0^{b-a} e^{-u^2/(2s)} du + \int_b^{\infty} f(u) \exp\left(-\frac{(u-a)^2}{2s} \right) du \right) .$$

$\triangleleft$

Exercise 1.6.2 Let φ be a C^1 function, B a Brownian motion and $B_t^* = \sup_{s \le t} B_s$. Prove that the process

$$\varphi(B_t^*) - (B_t^* - B_t)\varphi'(B_t^*)$$

is a local martingale. $\triangleleft$

Exercise 1.6.3 A Useful Lemma: Doob's Maximal Identity. (see [107, lemma 0.1]) Let M be a positive continuous martingale such that $M_0 = x$.

(i) Prove that if $\lim_{t\to\infty} M_t = 0$, then

$$\mathbb{P}(\sup M_t > a) = \left(\frac{x}{a}\right) \wedge 1 \tag{1.6.1}$$

and $\sup M_t \stackrel{\text{law}}{=} \dfrac{x}{U}$ where U is a random variable with a uniform law on $[0,1]$.

(ii) Conversely, if $\sup M_t \stackrel{\text{law}}{=} \dfrac{x}{U}$, show that $M_\infty = 0$.

(iii) Let T a stopping time and $S^T = \sup_{s \ge T} M_s$. Prove that M_T/S^T has a uniform law and is independent from $\mathcal{F}_T$

$\triangleleft$

Exercise 1.6.4 Prove that, for any (bounded) process a (not necessarily adapted)

$$M_t := \mathbb{E}(\int_0^t a_u du | \mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u) du$$

is an $\mathbb{F}$-martingale. Extend the result to the case $\int_0^{\cdot} X_s d\alpha_s$ where $(\alpha_s, s \ge 0)$ is an increasing predictable process and X a positive measurable process. $\triangleleft$

Exercise 1.6.5 Show that if $X_n, n \ge 1$ is an integrable sequence of r.vs, viewed as a discrete time process, adapted to some filtration $\mathbb{F}$, then, there exists a martingale M and a predictable process A such that $X_n = M_n + A_n$. $\triangleleft$

Chapter 2

Compensators, Single Default

The $\mathbb{F}$-compensator of a càdlàg $\mathbb{F}$-submartingale X is the càdlàg increasing and $\mathbb{F}$-predictable process A such that $X - A$ is an $\mathbb{F}$-martingale. From Doob-Meyer decomposition, the compensator exists (and is unique) if X is of class (D). Of course, the value of the compensator depends on the underlying filtration, as well on the underlying probability.

An important example is a Poisson process N, with constant intensity λ. In that case, the increasing process N (a sub-martingale) admits $A_t = \lambda t$ as compensator (in its own filtration).

In this chapter, we shall study in more details compensators of some increasing processes (which are obviously submartingales), in particular compensators of $\mathbb{1}_{\tau \leq t}$ for a positive random variable τ, of single jumps processes and of counting processes. Let us note that, if $\mathbb{F}$ is a Brownian filtration and τ an $\mathbb{F}$-stopping time (or more generally, if τ is an $\mathbb{F}$-predictable stopping time), the $\mathbb{F}$-compensator of $\mathbb{1}_{\tau \leq t}$ is $\mathbb{1}_{\tau \leq t}$.

2.1 Compensator of a Random Time

Let τ be a random time (a non-negative random variable) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $(H_t, t \geq 0)$ the right-continuous increasing process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and by $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ its natural filtration. It is proved in Bélanger et al. [21] that the filtration $\mathbb{H}$ is continuous on right. The filtration $\mathbb{H}$ is the smallest filtration which satisfies usual hypothesis, which makes τ a stopping time.

A key point is that any integrable $\mathcal{H}_t$-measurable r.v. K is of the form $K = g(\tau)\mathbb{1}_{\{\tau \leq t\}} + h(t)\mathbb{1}_{\{t < \tau\}}$ where g, h are Borel functions. It is also important (and obvious) to note that $\int_0^t h(u)dH_u = \int_{]0,t]} h(u)dH_u = H_t h(\tau)$, where the first equality is due to the definition of the symbol $\int_0^t \cdot dK_s$ for a continuous on right process K.

We denote by F the (right-continuous) cumulative distribution function of τ, defined as $F(t) = \mathbb{P}(\tau \leq t)$, and by G the survival function $G(t) = 1 - F(t)$.
We first give some elementary tools to compute the conditional expectation w.r.t. $\mathcal{H}_t$, as presented in Brémaud [29], Dellacherie [39, 40], Cohen & Elliott [51]. Note that if the cumulative distribution function of τ is continuous, then τ is an $\mathbb{H}$-totally inaccessible stopping time. (See Dellacherie and Meyer [44, Chapter IV, p.239 in the French version].)

The goal is to compute the $\mathbb{H}$-compensator of τ.

Remark 2.1.1 Dellacherie [39, 44] considers the σ algebra $\mathcal{H}_t^0$ generated by $\tau \wedge t$ (which contains the atom $\{\tau \geq t\}$) and the associated filtration $\mathbb{H}^0$. This filtration is not continuous on right: $\mathcal{H}_{t+}^0$ is obtained by splitting the atom $\{\tau \geq t\}$ into $\{\tau = t\}$ and $\{\tau > t\}$. Setting $\mathcal{H}_t^* = H_{t+}^0$, the random time τ is an $\mathbb{H}^*$ stopping time, but is not an $\mathbb{H}^0$ stopping time (hence $\mathbb{H}^* = \mathbb{H}$). It is proved that any $\mathbb{H}^0$-stopping time is predictable and that, if the law of τ is atomic and not degenerate, then τ is $\mathbb{H}$-accessible and not $\mathbb{H}$-predictable.

2.1.1 Key Lemma

Lemma 2.1.2 *If X is any integrable, $\mathcal{A}$-measurable r.v., one has*

$$\mathbb{E}(X|\mathcal{H}_s)\mathbb{1}_{\{s<\tau\}} = \mathbb{1}_{\{s<\tau\}}\frac{\mathbb{E}(X\mathbb{1}_{\{s<\tau\}})}{\mathbb{P}(s<\tau)}\,. \tag{2.1.1}$$

PROOF: The r.v. $\mathbb{E}(X|\mathcal{H}_s)$ is $\mathcal{H}_s$-measurable. Therefore, it can be written in the form $\mathbb{E}(X|\mathcal{H}_s) = g(\tau)\mathbb{1}_{\{s\geq\tau\}} + h(s)\mathbb{1}_{\{s<\tau\}}$ for some functions g,h. By multiplying both members by $\mathbb{1}_{\{s<\tau\}}$, and taking the expectation, we obtain, using the fact that $\{s<\tau\}\in\mathcal{H}_s$,

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{s<\tau\}}\mathbb{E}(X|\mathcal{H}_s)] &= \mathbb{E}[\mathbb{E}(\mathbb{1}_{\{s<\tau\}}X|\mathcal{H}_s)] = \mathbb{E}[\mathbb{1}_{\{s<\tau\}}X]\\ &= \mathbb{E}(h(s)\mathbb{1}_{\{s<\tau\}}) = h(s)\mathbb{P}(s<\tau)\,.\end{aligned}$$

Hence, if $\mathbb{P}(s<\tau)\neq 0$, $h(s) = \dfrac{\mathbb{E}(X\mathbb{1}_{\{s<\tau\}})}{\mathbb{P}(s<\tau)}$ gives the desired result. If, for some s, one has $\mathbb{P}(s<\tau)=0$, then $\{\tau>s\}$ is a negligeable set and $\mathbb{1}_{s<\tau}=0$ a.s. Then, in the right-hand side of (2.1.1), we set $\frac{0}{0}=0$. □

Exercise 2.1.3 Assume that Y is $\mathcal{H}_\infty$-measurable, so that $Y=h(\tau)$ for some Borel measurable function $h:\mathbb{R}_+\to\mathbb{R}$ and that $F(t)<1$ for $t>0$, F being continuous. Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = \mathbb{1}_{\{\tau\leq t\}}h(\tau) + \frac{1}{1-F(t)}\mathbb{1}_{\{t<\tau\}}\int_t^\infty h(u)\,dF(u). \tag{2.1.2}$$

Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = \mathbb{1}_{\{\tau\leq t\}}h(\tau) + \mathbb{1}_{\{t<\tau\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\,d\Gamma(u)\,.$$

Find a predictable process φ so that $dY_t = \varphi_t dM_t$. ◁

2.1.2 Some Martingales

In all this section, we assume that F is continuous. The general case can be found in [8].

Proposition 2.1.4 *Assuming that F is continuous and $F(t)<1,\forall t$, the process $(M_t, t\geq 0)$ defined as*

$$M_t = H_t - \int_0^{\tau\wedge t}\frac{dF(s)}{1-F(s)} = H_t - \int_0^t(1-H_{s-})\frac{dF(s)}{1-F(s)} = H_t + \int_0^t(1-H_{s-})\frac{dG(s)}{G(s)}$$

is an $\mathbb{H}$-martingale.

PROOF: Let $s<t$. Then:

$$\mathbb{E}(H_t-H_s|\mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\mathbb{E}(\mathbb{1}_{\{s<\tau\leq t\}}|\mathcal{H}_s) = \mathbb{1}_{\{s<\tau\}}\frac{F(t)-F(s)}{1-F(s)}\,, \tag{2.1.3}$$

which follows from (2.1.1) with $X=\mathbb{1}_{\{\tau\leq t\}}$.
On the other hand, the quantity

$$C := \mathbb{E}\left[\int_s^t(1-H_{u-})\frac{dF(u)}{1-F(u)}\,|\,\mathcal{H}_s\right]\,,$$

is equal to

$$\begin{aligned} C &= \int_s^t \frac{dF(u)}{1-F(u)} \mathbb{E}\left[\mathbb{1}_{\{\tau>u\}}|\mathcal{H}_s\right] \\ &= \mathbb{1}_{\{\tau>s\}} \int_s^t \frac{dF(u)}{1-F(u)} \left(1 - \frac{F(u)-F(s)}{1-F(s)}\right) = \mathbb{1}_{\{\tau>s\}} \int_s^t \frac{dF(u)}{1-F(s)} \\ &= \mathbb{1}_{\{\tau>s\}} \frac{F(t)-F(s)}{1-F(s)} \end{aligned}$$

which, from (2.1.3) proves the desired result. □

The (continuous increasing) function

$$\Gamma(t) := \int_0^t \frac{dF(s)}{1-F(s)} = -\ln(1-F(t)) = -\ln(G(t))$$

is called the **hazard function** of τ. Note, for future use, that $dF(t) = G(t)d\Gamma(t) = e^{-\Gamma(t)}d\Gamma(t)$. From Proposition 2.1.4, we obtain that the process $M_t := H_t - \Gamma(t \wedge \tau)$ is an $\mathbb{H}$ martingale, hence the Doob-Meyer decomposition of the submartingale H is $H_t = M_t + \Gamma(t \wedge \tau)$. The (predictable) process $A_t = \Gamma(t \wedge \tau)$ is called the **compensator** of H.
Moreover, if F is differentiable with derivative f, the process

$$M_t = H_t - \int_0^{\tau\wedge t} \gamma(s)ds = H_t - \int_0^t \gamma(s)(1-H_s)ds$$

is a martingale, where $\gamma(s) = \dfrac{f(s)}{1-F(s)}$ is a deterministic non-negative function, called **the intensity of** τ.

Proposition 2.1.5 *Assume that F is a continuous function. For any (bounded) Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, the process*

$$M_t^h = \mathbb{1}_{\{\tau\le t\}}h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u) \qquad (2.1.4)$$

is an $\mathbb{H}$-martingale. Moreover, $dM_t^h = h(t)dM_t$.

PROOF: On the one hand, for $s < t$,

$$\begin{aligned} \mathbb{E}\big(h(\tau)\mathbb{1}_{\{s<\tau\le t\}} \,|\, \mathcal{H}_s\big) &= \mathbb{1}_{\{s<\tau\}} \frac{1}{\mathbb{P}(s<\tau)} \mathbb{E}(h(\tau)\mathbb{1}_{s<\tau\le t}) = \mathbb{1}_{\{s<\tau\}} e^{\Gamma(s)} \int_s^t h(u)dF(u) \\ &= \mathbb{1}_{\{s<\tau\}} e^{\Gamma(s)} \int_s^t h(u)e^{-\Gamma(u)}\, d\Gamma(u). \end{aligned}$$

On the other hand, we get

$$J := \mathbb{E}\Big(\int_{s\wedge\tau}^{t\wedge\tau} h(u)\, d\Gamma(u)|\mathcal{H}_s\Big) = \mathbb{E}\big(\tilde{h}(\tau)\mathbb{1}_{\{s<\tau\le t\}} + \tilde{h}(t)\mathbb{1}_{\{\tau>t\}} \,|\, \mathcal{H}_s\big)$$

where, for fixed s, we set $\tilde{h}(t) = \int_s^t h(u)\, d\Gamma(u)$. Consequently,

$$J = \mathbb{1}_{\{s<\tau\}} e^{\Gamma(s)} \Big(\int_s^t \tilde{h}(u)e^{-\Gamma(u)}\, d\Gamma(u) + e^{-\Gamma(t)}\tilde{h}(t)\Big) =: \mathbb{1}_{\{s<\tau\}} e^{\Gamma(s)} \widetilde{J}.$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned}\widetilde{J} &= \int_s^t d\Gamma(u)e^{-\Gamma(u)} \int_s^u h(v)\, d\Gamma(v) + e^{-\Gamma(t)}\tilde{h}(t) \\ &= \int_s^t d\Gamma(u)h(u) \int_u^t e^{-\Gamma(v)}\, d\Gamma(v)\; + e^{-\Gamma(t)} \int_s^t h(u)\, d\Gamma(u) \\ &= \int_s^t h(u)e^{-\Gamma(u)}\, d\Gamma(u),\end{aligned}$$

as expected. Writing

$$M_t^h = \int_0^t h(u)dH_u - \int_0^t (1-H_u)h(u)\, d\Gamma(u)\,,$$

the differential form of M^h is obtained. □

Example 2.1.6 In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the first time when N jumps, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s)ds$ is a martingale (indeed, N can be viewed as a standard Poisson process $\widetilde{N}$ of intensity 1, changed of time; $N_t = \widetilde{N}_{\Lambda(t)}$ with $\Lambda_t = \int_0^t \lambda(s)ds$. (We shall come back to this change of time methodology latter). Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(s)ds$ is a martingale.

Exercise 2.1.7 Take the example of Exercise 2.1.3 and assume that Γ is continuous. Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{t<\tau\}} \int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\, d\Gamma(u)\,.$$

Find a predictable process φ so that $dY_t = \varphi_t dM_t$. ◁

Exercise 2.1.8 Let B be a Brownian motion and $\tau = \inf\{t\ B_t = a\}$. Find the $\mathbb{F}^B$ compensator of τ. Find the $\mathbb{F}^0$ compensator of τ, when $\mathbb{F}^0$ is the trivial filtration. ◁

Exercise 2.1.9 a) Prove that the process $L_t := 1\!\!1_{\{\tau>t\}} \exp\left(\int_0^t \gamma(s)ds\right)$ is an $\mathbb{H}$-martingale and

$$L_t = 1 - \int_{]0,t]} L_{u-} dM_u \tag{2.1.5}$$

In particular, for $t < T$,

$$\mathbb{E}(1\!\!1_{\{\tau>T\}}|\mathcal{H}_t) = 1\!\!1_{\{\tau>t\}} \exp\left(-\int_t^T \gamma(s)ds\right)\,.$$

b) Let $d\mathbb{Q}_{|\mathcal{H}_t} = L_t d\mathbb{P}_{|\mathcal{H}_t}$. Prove that $\mathbb{Q}(\tau \le t) = 0$. ◁

Exercise 2.1.10 a) Let F be continuous and $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Prove that the process

$$Y_t := \exp\left(1\!\!1_{\{\tau\le t\}}h(\tau)\right) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u) \tag{2.1.6}$$

is a $\mathbb{H}$-martingale. Find a predictable process φ such that

$$dY_t = \varphi_t dM_t$$

◁

Exercise 2.1.11 Assume that Γ is a continuous function. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Prove that the process

$$Y_t := (1 + \mathbb{1}_{\tau \le t} h(\tau)) \exp\Big(- \int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big) . \tag{2.1.7}$$

is an $\mathbb{H}$-martingale. Find a predictable process φ such that $dY_t = \varphi_t dM_t$. Give a condition on h so that Y is positive. In that case, find a predictable process ψ such that $dY_t = Y_{t-}\psi_t dM_t$. ◁

Exercise 2.1.12 In this exercise, F is only continuous on right, and $F(t-)$ is the left limit of F at point t. Prove that the process $(M_t, t \ge 0)$ defined as

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1-F(s-)} = H_t - \int_0^t (1-H_s)\frac{dF(s)}{1-F(s-)}$$

is an $\mathbb{H}$-martingale. ◁

2.2 Compensator of a Random Time with respect to a Reference Filtration

We denote (with an abuse of notation) by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ the enlarged filtration which is the smallest right-continuous filtration which contains $\mathbb{F}$, making τ a stopping time. More precisely

$$\mathcal{G}_t = \cap_{s>t} F_s \vee \mathcal{H}_s$$

It is straightforward to establish that any $\mathcal{G}_t$-measurable random variable is equal, on the set $\{\tau > t\}$, to an $\mathcal{F}_t$-measurable random variable. Indeed, $\mathcal{G}_t$-measurable random variables are generated by $x_t(g(\tau)\mathbb{1}_{\tau\le t} + h(t)\mathbb{1}_{t<\tau})$, where x_t is $\mathcal{F}_t$ measurable and g, h are Borel functions. In particular, if Y is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process $Y^{\mathbb{F}}$, called the predefault-value of Y, such that $\mathbb{1}_{\{t<\tau\}} Y_t = \mathbb{1}_{\{t<\tau\}} Y^{\mathbb{F}}_t$. Under the standing assumption that $G_t := P(\tau > t \,|\, \mathcal{F}_t) > 0$ for $t \in \mathbb{R}_+$, the uniqueness of pre-default value process follows from [41, p.186]. Moreover, if Y is $\mathbb{G}$-predictable its pre-default value $Y^{\mathbb{F}}$ coincide up to τ *included* (see [41, p.186]), namely,

$$\mathbb{1}_{\{t\le\tau\}} Y_t = \mathbb{1}_{\{t\le\tau\}} Y^{\mathbb{F}}_t .$$

If Y is $\mathbb{G}$-adapted, it is standard to check that $Y \ge 0$ implies $Y^{\mathbb{F}} \ge 0$.

2.2.1 Key Lemma

We denote by $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$ the conditional cumulative probability of τ given the information $\mathcal{F}_t$ and we set[1] $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = 1 - F_t$. We assume $G_t > 0$, for $t > 0$. See [8] for the case where G can vanish.

Lemma 2.2.1 ***Key Lemma 1.*** *Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then, for $t \le T$*

$$\mathbb{E}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t}\mathbb{E}(XG_T|\mathcal{F}_t) . \tag{2.2.1}$$

PROOF: Note that

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(X|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} x_t$$

[1] Latter on, we shall denote frequently by Z this quantity, as it is done in the literature on enlargement of filtration.

where x_t is $\mathcal{F}_t$-measurable, and taking conditional expectation w.r.t. $\mathcal{F}_t$ of both members, we deduce

$$x_t = \frac{\mathbb{E}(X 1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)} = 1\!\!1_{\{\tau>t\}} \frac{1}{G_t}\mathbb{E}(XG_T|\mathcal{F}_t)\,.$$

□

Lemma 2.2.2 ***Key lemma 2.*** *Let h be an $\mathbb{F}$-predictable process. Then, for $t<T$,*

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = h_\tau 1\!\!1_{\{\tau<t\}} + 1\!\!1_{\{\tau>t\}}\frac{1}{G_t}\mathbb{E}(\int_t^T h_u dF_u|\mathcal{F}_t) \tag{2.2.2}$$

PROOF: In a first step, the result is established for processes h of the form $h_t = 1\!\!1_{]u,v]}(t)K_u$ where $K_u \in \mathcal{F}_u$. In that case, for $t<u<v<T$, applying the key lemma

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) \quad = \quad \mathbb{E}(K_u 1\!\!1_{u<\tau<v}|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{1}{G_t}\mathbb{E}(K_u 1\!\!1_{u<\tau<v}|\mathcal{F}_t)$$

It remains to note that

$$\begin{aligned}
\mathbb{E}(K_u 1\!\!1_{u<\tau<v}|\mathcal{F}_t) &= \mathbb{E}(K_u 1\!\!1_{\tau<v}|\mathcal{F}_t) - \mathbb{E}(K_u 1\!\!1_{\tau<u}|\mathcal{F}_t) \\
&= \mathbb{E}(K_u(1-F_v)|\mathcal{F}_t) - \mathbb{E}(K_u(1-F_u)|\mathcal{F}_t) = \mathbb{E}(\int_t^T h_r dF_r|\mathcal{F}_t)
\end{aligned}$$

The other cases are done in the same way. The result follows by approximation. □

As we shall see, this elementary result will allow us to compute the value of credit derivatives.

Comment 2.2.3 It can be useful to understand the meaning of the lemma in the case where, as in the structural model, the default time is an $\mathbb{F}$-stopping time.
We are not interested in this lemma with $\mathbb{G}$-predictable processes, mainly because any $\mathbb{G}$-predictable process is equal, on $\{t\le\tau\}$ to an $\mathbb{F}$-predictable process.

2.2.2 Martingales

Proposition 2.2.4 *The process $(F_t, t\ge 0)$ is an $\mathbb{F}$-submartingale. The process G is an $\mathbb{F}$-supermartingale. Furthemore,*

$$\{\tau>t\}\subset\{G_t>0\} \tag{2.2.3}$$

PROOF: From definition, and from the increasing property of the process H, for $s<t$:

$$\mathbb{E}(F_t|\mathcal{F}_s) \quad = \quad \mathbb{E}\left(\mathbb{E}(H_t|\mathcal{F}_t)\,|\mathcal{F}_s\right) = \mathbb{E}\left(H_t|\mathcal{F}_s\right) \ge \mathbb{E}\left(H_s|\mathcal{F}_s\right) = F_s\,.$$

Let $A_t = \{G_t>0\}$. Then $\mathbb{P}(A_t^c\cap\{\tau>t\}) = \mathbb{E}(1\!\!1_{A_t^c}\mathbb{P}(\tau>t|\mathcal{F}_t)) = 0$. □

As a supermartingale, G admits a Doob-Meyer decomposition

$$G_t = \mu_t - A_t^p \tag{2.2.4}$$

where μ is a martingale and A^p is a predictable increasing process (we have used that G, being bounded is of class (D)).

Proposition 2.2.5 *(i) If $G>0$, the process $L_t = (1-H_t)/G_t$ is a $\mathbb{G}$ -martingale.*

(ii) If X is an $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale.

(iii) If the process G is decreasing and continuous, the process $M_t = H_t - \Gamma(t\wedge\tau)$ is a $\mathbb{G}$ -martingale where $\Gamma = -\ln G$.

PROOF: (i) From the key lemma, for $t > s$

$$\mathbb{E}(L_t|\mathcal{G}_s) = \mathbb{E}(1\!\!1_{\{\tau>t\}}\frac{1}{G_t}|\mathcal{G}_s) = 1\!\!1_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(1\!\!1_{\{\tau>t\}}\frac{1}{G_t}|\mathcal{F}_s) = 1\!\!1_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\frac{1}{G_t}G_t|\mathcal{F}_s) = 1\!\!1_{\{\tau>s\}}\frac{1}{G_s} = L_s$$

(ii) From the key lemma,

$$\begin{aligned}
\mathbb{E}(L_tX_t|\mathcal{G}_s) &= \mathbb{E}(1\!\!1_{\{\tau>t\}}L_tX_t|\mathcal{G}_s)\\
&= 1\!\!1_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(1\!\!1_{\{\tau>t\}}\frac{1}{G_t}X_t|\mathcal{F}_s)\\
&= 1\!\!1_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\mathbb{E}(1\!\!1_{\{\tau>t\}}|\mathcal{F}_t)\frac{1}{G_t}X_t|\mathcal{F}_s) = L_s\mathbb{E}(X_t|\mathcal{F}_s) = L_sX_s.
\end{aligned}$$

(iii) From integration by parts formula (H is a finite variation process, and Γ an increasing continuous process):

$$dL_t = (1-H_t)e^{\Gamma_t}d\Gamma_t - e^{\Gamma_t}dH_t$$

and the process $M_t = H_t - \Gamma(t\wedge\tau)$ can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]}(1-H_u)d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u}dL_u$$

and is a $\mathbb{G}$-local martingale since L is $\mathbb{G}$-martingale. (It can be noted that, if Γ is not increasing, the differential of e^Γ is more complicated.) □

Comment 2.2.6 (a) Assertion (ii) seems to be related with a change of probability. It is important to note that here, one changes the filtration, not the probability measure. Moreover, setting $d\mathbb{Q}^* = Ld\mathbb{P}$ does not define a probability $\mathbb{Q}$ **equivalent** to $\mathbb{P}$, since the positive martingale L vanishes. The probability $\mathbb{Q}^*$ would be only absolutely continuous w.r.t. $\mathbb{P}$. See Collin-Dufresne and Hugonnier [36].
(b) If G can vanish, the process L in Lemma 2.2.5 is a supermartingale. See [8] for a general study.

Proposition 2.2.7 *Let A^p be defined in (2.2.4). The process*

$$M_t = H_t - \int_0^{t\wedge\tau}\frac{dA^p_u}{G_{u-}} =: H_t - \Lambda_{t\wedge\tau}$$

is a $\mathbb{G}$-martingale.

PROOF: We give the proof in the case where G is continuous in two steps. In the proof $A = A^p$.
In a first step, we prove that, for $s < t$

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1-\mathbb{P}(t<\tau|\mathcal{G}_s) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(G_t|\mathcal{F}_s) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(\mu_t - A_t|\mathcal{F}_s)\\
&= 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}(\mu_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}(\mu_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s))\\
&= 1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)\,.
\end{aligned}$$

In a second step, we prove that, setting, for any v, $K_v = \int_0^v (1-H_s)\frac{dA_s}{G_s}$,

$$\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) = K_{s\wedge\tau} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed, from the key formula, for fixed t and $h_u = K_{t\wedge u}$

$$\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{t\wedge\tau}\mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^\infty K_{t\wedge u}dF_u|\mathcal{F}_s\right) \\
&= K_\tau\mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^t K_u dF_u + \int_t^\infty K_t dF_u|\mathcal{F}_s\right) \\
&= K_{s\wedge\tau}\mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^t K_u dF_u + K_t G_t|\mathcal{F}_s\right)
\end{aligned}$$

We now use IP formula, using the fact that K has finite variation and is continuous

$$d(K_t(1-F_t)) = -K_t dF_t + (1-F_t)dK_t = -K_t dF_t + dA_t$$

hence

$$\int_s^t K_u dF_u + K_t(1-F_t) = -K_t(1-F_t) + K_s(1-F_s) + A_t - A_s + K_t(1-F_t) = K_s(1-F_s) + A_t - A_s\,.$$

It follows that

$$\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{s\wedge\tau}\mathbb{1}_{\tau\leq s} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(K_s G_s + A_t - A_s|\mathcal{F}_s\right) \\
&= K_{s\wedge\tau} + \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right)\,.
\end{aligned}$$

Assuming that A is absolutely continuous w.r.t. the Lebesgue measure and denoting by a its derivative, we have proved the existence of a $\mathbb{F}$-adapted process λ, called the intensity rate such that the process

$$H_t - \int_0^{t\wedge\tau}\lambda_u du = H_t - \int_0^t (1-H_u)\lambda_u du \tag{2.2.5}$$

is a $\mathbb{G}$-martingale. More precisely, $\lambda_s = \frac{a_s}{1-F_s}$.
For the general case, see Bielecki and Rutkowski [27] or Elliott et al [53] □

Lemma 2.2.8 *If (2.2.5) holds, the process λ satisfies*

$$\lambda_t = \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{P}(t<\tau<t+h|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}\,.$$

Proof: The martingale property of M implies that

$$\mathbb{E}(\mathbb{1}_{t<\tau<t+h}|\mathcal{G}_t) = \int_t^{t+h}\mathbb{E}((1-H_s)\lambda_s|\mathcal{G}_t)ds$$

It follows that, on $\{t<\tau\}$

$$\lambda_t = \frac{1}{h}\lim\mathbb{P}(t<\tau<t+h|\mathcal{G}_t) = \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{P}(t<\tau<t+h|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}\,.$$

□

Comment 2.2.9 We assume G continuous and positive. We recall that the Doob-Meyer decomposition of G is denoted $G_t = \mu_t - A_t$. From $L_t = (1-H_t)(G_t)^{-1}$, one obtains

$$dL_t = -(1-H_{t-})\frac{1}{G_t^2}(d\mu_t - dA_t) + \frac{1}{G_t^3}d\langle\mu\rangle_t - \frac{1}{G_t}dH_t$$

it follows that

$$dL_t - \frac{1}{G_t}dM_t = -(1-H_t)\frac{1}{G_t^2}\left(d\mu_t - \frac{1}{G_t}d\langle\mu\rangle_t\right)$$

hence, due to the $\mathbb{G}$-martingale property of L, the quantity $(1-H_t)\frac{1}{G_t^2}\left(d\mu_t - \frac{1}{G_t}d\langle\mu\rangle_t\right)$ corresponds to a $\mathbb{G}$-local martingale.

Proposition 2.2.10 *Let H^p be the $\mathbb{G}$-predictable compensator of H, i.e., the $\mathbb{G}$-predictable increasing process such that $H - H^p$ is a $\mathbb{G}$-martingale. The random variable H^p_τ has a unit exponential law.*

PROOF: Let φ be a bounded Borel function, $\Phi(t) = \int_0^t \varphi(s)ds$ and

$$M_t^\varphi := \int_0^t \varphi(H_s^p)dM_s = \varphi(H_\tau^p)1\!\!1_{\tau\le t} - \int_0^t \varphi(H_s^p)dH_s^p = \varphi(H_\tau^p)1\!\!1_{\tau\le t} - \Phi(H_t^p)$$

Then, for $t=\infty$, using the fact that $H^p_\infty = H^p_\tau$, one has $\mathbb{E}(\varphi(H^p_\tau)) = \mathbb{E}(\Phi(H^p_\tau))$ and the result follows. □

2.2.3 Covariation process

We suppose A^p continuous, and write $M_t = H_t - \Lambda_{t\wedge\tau}$ the fundamental martingale M, where A^p is continuous. The covariation process of M is obviously H: indeed, M being a pure jump martingale, $M_t^2 - \sum_{s\le t}(\Delta M_s)^2$ is a martingale. It suffices to note that $(\Delta M_s)^2 = \Delta M_s = \Delta H_s$ so that $\sum_{s\le t}(\Delta M_s)^2 = H_t$. It follows that $M_t^2 - (H_t - \Lambda(t\wedge\tau)) - \Lambda(t\wedge\tau) = M_t^2 - M_t - \Gamma(t\wedge\tau)$ is a martingale, so that $M_t^2 - \Lambda(t\wedge\tau)$ is a martingale too, and the predictable covariation process is $\Lambda(t\wedge\tau)$.

2.3 Cox Processes and Extensions

In this section, we present a particular construction of random times. This construction is the basic one to define a default time in finance.In a credit risk setting, the random variable τ represents the time when a default occurs. In the literature, models for default times are often based on a threshold: the default occurs when some driving process X reaches a given barrier. Based on this observation, we consider the random time on $I\!R_+$ in a general threshold model. Let X be a stochastic process and Θ be a barrier which we shall make precise later. Define the random time as the first passage time

$$\tau := \inf\{t : X_t \ge \Theta\}.$$

In classical structural models, a reference filtration $\mathbb{F}$ is given, the process X is an $\mathbb{F}$-adapted process associated with the value of a firm and the barrier Θ is a constant. So, τ is an $\mathbb{F}$-stopping time. If τ is a predictable stopping time (e.g., if $\mathbb{F}$ is a Brownian filtration), the compensator of $H_t = 1\!\!1_{\tau\le t}$ is H_t. The goal is then to compute the conditional law of the default $P(\tau > \theta|\mathcal{F}_t^X)$, for $\theta > t$

In reduced form approach (say, if τ is not the first time where a process reaches a constant barrier), we shall deal with two kinds of information: some information denoted as $(\mathcal{F}_t, t\ge 0)$ and the information from the default time, i.e. the knowledge of the time where the default occurred in the past, it the default has appeared. More precisely, this information is modeled by the filtration $\mathbb{H}$ generated by the default process H (completed with negligeable sets).

At the intuitive level, $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). This filtration can also be a subfiltration of the prices. The case where $\mathbb{F}$ is the trivial filtration is exactly what we have studied in the toy model. Though in typical examples $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbb{F}$.

2.3.1 Construction of Cox Processes with a given stochastic intensity

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$. *A nonnegative $\mathbb{F}$-adapted process λ is given.* We assume that there exists, on the space $(\Omega, \mathcal{G}, \mathbb{P})$, a random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law: $\mathbb{P}(\Theta \geq t) = e^{-t}$. We define the default time τ as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, using the increasing property of Λ, one gets $\{\tau > s\} = \{\Lambda_s < \Theta\}$. We assume that $\Lambda_t < \infty, \forall t$, $\Lambda_\infty = \infty$, hence τ is a real-valued r.v.. One can also define τ as

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq -\ln U\}$$

where U has a uniform law and is independent of $\mathcal{F}_\infty$. Indeed, the r.v. $-\ln U$ has an exponential law of parameter 1, since $\{-\ln U > a\} = \{U < e^{-a}\}$.

We write as usual $H_t = 1\!\!1_{\{\tau \leq t\}}$ and $\mathcal{H}_t = \sigma(H_s \, : \, s \leq t)$. We introduce the smallest right-continuous filtration $\mathbb{G}$ which contains $\mathbb{F}$ and turns τ in a stopping time. (We denote by $\mathbb{F}$ the original **F**iltration and by $\mathbb{G}$ the enlar**G**ed one.) As already said, we shall write $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

It is easy to describe the events which belong to the σ-field $\mathcal{G}_t$ on the set $\{\tau > t\}$. Indeed, if $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies $1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, where y_t is an $\mathcal{F}_t$-measurable random variable.

Comments 2.3.1 (i) In order to construct the r.v. Θ, one needs to enlarge the probability space as follows. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be an auxiliary probability space with a r.v. Θ with exponential law. We introduce the product probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{Q}}) = (\Omega \times \hat{\Omega}, \mathcal{F}_\infty \otimes \hat{\mathcal{F}}, \mathbb{Q} \otimes \hat{\mathbb{P}})$.
(ii) Another construction for the default time τ is to choose $\tau = \inf \{t \geq 0 \, : \, \tilde{N}_{\Lambda_t} = 1\}$, where $\Lambda_t = \int_0^t \lambda_s \, ds$ and $\tilde{N}$ is a Poisson process with intensity 1, independent of the filtration $\mathbb{F}$. This second method is in fact equivalent to the first. Cox processes are used in a great number of studies (see, e.g., [101])

2.3.2 Conditional Expectations

Lemma 2.3.2 *The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$*

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \exp\big(-\Lambda_s\big)\,.$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s < \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \leq t$, we obtain

$$\mathbb{P}(\tau > s | \mathcal{F}_t) = \mathbb{P}\Big(\Lambda_s < \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\big(-\Lambda_s\big).$$

In particular, we have

$$\mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\tau \le t|\mathcal{F}_\infty), \tag{2.3.1}$$

and, for $t \ge s$, $\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$ is here an increasing process, as the right-hand side of (2.3.1) is. □

The conditional density of τ, and the law of τ can be easily computed. One has $\mathbb{P}(\tau > t) = \mathbb{E}(e^{-\Lambda_t})$, so that $\mathbb{P}(\tau \in d\theta) = \mathbb{E}(\lambda_\theta e^{-\Lambda_\theta})d\theta$ and, for $\theta \le t$

$$\mathbb{P}(\tau \in d\theta|\mathcal{F}_t) = \lambda_\theta e^{-\Lambda_\theta} d\theta$$

For $\theta > t$, one has $\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \mathbb{P}(\tau > \theta|\mathcal{F}_\theta|\mathcal{F}_t) = \mathbb{E}(e^{-\Lambda_\theta}|\mathcal{F}_t) = \mathbb{E}(\int_\theta^\infty \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t)$, hence

$$\mathbb{P}(\tau \in d\theta|\mathcal{F}_t) = \mathbb{E}(\lambda_\theta e^{-\Lambda_\theta}|\mathcal{F}_t)d\theta$$

Remark 2.3.3 If the process λ is not non-negative, we get,

$$\{\tau > s\} = \{\sup_{u \le s} \Lambda_u < \Theta\}\,,$$

hence for $s < t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u\le s} \Lambda_u)\,.$$

More generally, some authors define the default time as

$$\tau = \inf\{t \ge 0 \,:\, X_t \ge \Theta\}$$

where X is a given $\mathbb{F}$-semi-martingale. Then, for $s \le t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u\le s} X_u)\,.$$

Exercise 2.3.4 Prove that τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function. ◁

2.3.3 Immersion property

Lemma 2.3.5 *Let X be an $\mathcal{F}_\infty$-measurable integrable r.v.. Then*

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)\,. \tag{2.3.2}$$

Proof: To prove that $\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)$, it suffices to check that

$$\mathbb{E}(B_t h(\tau\wedge t)X) = \mathbb{E}(B_t h(\tau \wedge t)\mathbb{E}(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = 1\!\!1_{[0,a]}$. For $t \le a$, the equality is obvious. For $t > a$, we have from (2.3.1)

$$\begin{aligned}\mathbb{E}(B_t 1\!\!1_{\{\tau\le a\}}\mathbb{E}(X|\mathcal{F}_t)) &= \mathbb{E}(\mathbb{E}(B_t X|\mathcal{F}_t)\mathbb{E}(1\!\!1_{\{\tau\le a\}}|\mathcal{F}_t)) = \mathbb{E}(X B_t \mathbb{E}(1\!\!1_{\{\tau\le a\}}|\mathcal{F}_t))\\ &= \mathbb{E}(B_t X \mathbb{E}(1\!\!1_{\{\tau\le a\}}|\mathcal{F}_\infty)) = \mathbb{E}(B_t X 1\!\!1_{\{\tau\le a\}})\end{aligned}$$

as expected. □

Remark 2.3.6 Let us remark that (2.3.2) implies that every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. However, equality (2.3.2) does not apply to any $\mathcal{G}_\infty$-measurable random variable; in particular $\mathbb{P}(\tau \le t|\mathcal{G}_t) = 1\!\!1_{\{\tau\le t\}}$ is not equal to $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$.

This lemma implies that any (u.i.) $\mathbb{F}$-martingale is a $\mathbb{G}$ martingale. This property is known as the immersion property of $\mathbb{F}$ with respect to $\mathbb{G}$ and will be studied in the next chapter. Let us give another proof of this result.

Lemma 2.3.7 *In a Cox model, any $\mathbb{F}$-martingale is a $\mathbb{G}$ martingale*

PROOF: Since Θ is independent from $\mathbb{F}$, it is obvious that any $\mathbb{F}$-martingale M is an $\mathbb{F}^\tau = \mathbb{F} \vee \sigma(\Theta)$ martingale. Since $\mathbb{G} \subset \mathbb{F}^\tau$, it follows that M is a $\mathbb{G}$ martingale. $\square$

Exercise 2.3.8 Prove that $\mathbb{H}$ is, in general, not immersed in $\mathbb{G}$. Prove that, if λ is deterministic, $\mathbb{H}$ is immersed in $\mathbb{G}$. $\lhd$

2.3.4 Predictable Representation Theorem, Change of Probability

In this section, we assume the condition **(C)**, that is any $\mathbb{F}$-martingale is continuous. We study the form of a general u.i. $\mathbb{G}$ martingale. Restricting in the last step our attention to the case where $\mathbb{F}$ is a Brownian filtration, we shall establish a predictable representation theorem, similar to the one given in Kusuoka [100].

Predictable Representation Theorem

We start with u.i. $\mathbb{G}$-martingales of the form $Y_t = \mathbb{E}(Xf(\tau)|\mathcal{G}_t)$ where $X \in \mathcal{F}_\infty$ is integrable and f is a bounded Borel function. From the key lemma

$$Y_t = f(\tau)\mathbb{E}(X|\mathcal{G}_t)1\!\!1_{t>\tau} + 1\!\!1_{t\le\tau}e^{\Lambda_t}\mathbb{E}(X\int_t^\infty f(u)e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t) =: f(\tau)\mathbb{E}(X|\mathcal{G}_t)1\!\!1_{t>\tau} + 1\!\!1_{t\le\tau}Y_t^{\mathbb{F}}$$

From immersion, and hypothesis (C), $\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t) =: X_t$ is a continuous $\mathbb{F}$-martingale. We write

$$\mathbb{E}(X\int_t^\infty f(u)e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t) = X_t^f - X_t\int_0^t f(u)e^{-\Lambda_u}\lambda_u du$$

where $X_t^f := \mathbb{E}(X\int_0^\infty f(u)e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. Finally, introducing the $\mathbb{G}$ martingale $L_t = 1\!\!1_{t\le\tau}e^{\Lambda_t}$

$$Y_t = X_t\int_0^t f(u)dH_u + L_t\left(X_t^f - X_t\int_0^t f(u)e^{-\Lambda_u}\lambda_u du\right)$$

By integration by parts, using that the $\mathbb{F}$-martingales are orthogonal to L, and after easy simplifications, we get

$$dY_t = X_t f(t)(dH_t - (1-H_t)\lambda_t dt) + \psi_t dL_t + \varphi_t dX_t + L_{t-}dX_t^f$$

where $\psi_t = X_t^f - X_t\int_0^t f(u)e^{-\Lambda_u}\lambda_u du, \varphi_t = \int_{]0,t[} f(u)dH_u - L_{t-}\int_0^t f(u)\lambda_u e^{\Lambda_u}du$ are $\mathbb{G}$-predictable processes. Finally

$$dY_t = (X_t f(t) - \psi_t L_{t-})dM_t + \varphi_t dX_t + L_{t-}dX_t^f$$

In the case where $\mathbb{F}$ is a Brownian filtration, any continuous $\mathbb{F}$-martingale admits a representation w.r.t. the Brownian motion W. Being true for u.i. martingales, of the specific form, the result extend and we have obtained

Theorem 2.3.9 *In the Cox model, if $\mathbb{F}$ is a Brownian filtration generated by W, any $\mathbb{G}$ martingale admits a representation of the form*

$$Y_t = Y_0 + \int_0^t \psi_s dM_s + \int_0^t \varphi_s dW_s$$

where ψ and φ are predictable processes, and $\psi_s = Y_s - Y_s^{\mathbb{F}}$ where $Y^{\mathbb{F}}$ is the predefault value of Y.

Remark 2.3.10 This result will be extended in Theorem 4.2.14.

Change of Probability

We assume that we are under the conditions of the previous theorem, i.e. under condition (**C**), in a Cox model. We are interested with the impact of a change of probability. Due to Theorem 2.3.9, any equivalent probability measure $\mathbb{Q}$ can be written as $d\mathbb{Q}_{|\mathcal{G}_t} = L_t d\mathbb{P}|\mathcal{G}_t$; where L satisfies

$$dL_t = L_{t-}\left(\psi_t dW_t + \gamma_t dM_t\right)$$

where ψ and γ are predictable processes, with $\gamma > -1$ to preserve positivity of L. Indeed, the process L can be written as

$$L_t = \exp\big(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\big)\exp\big(-\int_0^t \gamma_s \lambda_s^{\mathbb{G}} ds\big)\left(1+\gamma_\tau\right)^{H_t}$$

where $\lambda_t^{\mathbb{G}} = \lambda_t(1-H_t)$. Under $\mathbb{Q}$, the processes $W^{\mathbb{Q}}$ and M^{Q} defined below, ae $\mathbb{Q}$ martingales:

$$W_t^{\mathbb{Q}} := W_t - \int_0^t \psi_s ds\,, \quad M_t^{\mathbb{Q}} = M_t - \int_0^t \lambda_s^{\mathbb{G}}\gamma_s ds$$

Note that the $\mathbb{F}$-intensity of τ under $\mathbb{Q}$ is $\lambda_t^{\mathbb{Q}} = \lambda_t(1+\gamma_t)$ (so that $H - \int_0^{\cdot}(1-H_s)\lambda_s(1+\gamma_s)ds$ is a $(\mathbb{Q},\mathbb{G})$ martingale).

In general, the immersion hypothesis between $\mathbb{F}$ and $\mathbb{G}$ is not satisfied under $\mathbb{Q}$ (see Coculescu et al. [34], Section 4.1.2 and Section 8.6 for a counterexample). However, if ψ is taken as $\mathbb{F}$-predictable, then, from Bayes formula, denoting$\ell_t = \mathbb{E}_{\mathbb{P}}(L_t|\mathcal{F}_t) = \exp\big(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\big)$, one has

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = \frac{1}{\ell_t}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\tau>t}L_t|\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\tau>t}\exp\big(-\int_0^t \gamma_s\lambda_s ds\big)|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda_s(1+\gamma_s)ds\right)$$

In can be noted that $\Theta \in \mathcal{G}_\tau$ (indeed $\Theta = \int_0^\tau \lambda_s ds$) and that, under a change of probability in the filtration $\mathbb{G}$, the independence of $\mathbb{F}$ and Θ can fail.

2.3.5 Extension to different barrier

One can define the time of default as

$$\tau = \inf\{t\,:\,\Lambda_t \geq \Sigma\}$$

where Σ a non-negative r.v. independent of $\mathcal{F}_\infty$. This model reduces to the previous one: if Φ is the cumulative function of Σ, the r.v. $\Phi(\Sigma)$ has a uniform distribution and

$$\tau = \inf\{t\,:\,\Phi(\Lambda_t) \geq \Phi(\Sigma)\} = \inf\{t\,:\,\Psi^{-1}[\Phi(\Lambda_t)] \geq \Theta\}$$

where Ψ is the cumulative function of the exponential law. Then,

$$F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\Lambda_t \geq \Sigma|\mathcal{F}_t) = 1 - \exp\left(-\Psi^{-1}(\Phi(\Lambda_t))\right)\,.$$

2.3.6 Dynamics of prices in a default setting

We assume here that $\mathbb{F}$-martingales are continuous.

Defaultable Zero-Coupon Bond

A defaultable Zero-coupon Bond of maturity T pays one monetary unit at time T, if the default has not occurred before T. Let $\mathbb{Q}$ be a risk-neutral probability and $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T given by

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(- \int_t^T r_s\, ds\big) \,\Big|\, \mathcal{F}_t\Big).$$

The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(1\!\!1_{\{T<\tau\}} \exp\big(- \int_t^T r_s\, ds\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= 1\!\!1_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(- \int_t^T [r_s + \lambda_s]\, ds\big) \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

Here, we are working in a Cox model under $\mathbb{Q}$, i.e. $\tau = \inf\{t \int_0^t \lambda_s ds \geq \Theta\}$ where Θ is independent of $\mathbb{F}$ under $\mathbb{Q}$. In particular, $\mathbb{Q}(\tau > t|\mathcal{F}_t) = \exp - \int_0^t \lambda_s ds)$. Then, in the case $r = 0$,

$$D(t,T) = 1\!\!1_{t<\tau} e^{\Lambda_t} \mathbb{Q}(\tau > T|\mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T}|\mathcal{F}_t)$. Then,

$$dD(t,T) = m_t dL_t + L_{t-} dm_t = -m_t L_{t-} dM_t + L_{t-} dm_t = -D(t-,T) dM_t + L_{t-} dm_t$$

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Hence $D(t,T) = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = -D(t-,T) dM_t\,.$$

Remark 2.3.11 If $\mathbb{P}$ is a probability such that Θ is independent of $\mathcal{F}_\infty$ and $\mathbb{Q}$ a probability equivalent to $\mathbb{P}$, it is not true, in general that Θ is independent of $\mathcal{F}_\infty$ and has an exponential law under $\mathbb{Q}$. Changes of probabilities that preserve the independence of Θ and $\mathcal{F}_\infty$ change the law of Θ, hence the intensity.

Exercise 2.3.12 Write the risk-neutral dynamics of D for a general interest rate r. $\lhd$

Recovery with Payment at maturity

We assume here that $r = 0$. We consider a contract which pays K_τ at date T, if $\tau \leq T$ and no payment in the case $\tau > T$, where K is a given $\mathbb{F}$-predictable process.

An immediate application of the key lemma shows that the price at time t of this contract is

$$\begin{aligned} S_t &= E(K_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} E(K_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) \\ &= K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u dF_u|\mathcal{F}_t) \end{aligned}$$

where $F_u = P(\tau \leq u|\mathcal{F}_u) = 1 - e^{-\Lambda_u}$, or

$$S_t = K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u e^{-\Lambda_u} \lambda_u du|\mathcal{F}_t)$$

or

$$S_t = \int_0^t K_u dH_u + L_t \left(- \int_0^t K_u e^{-\Lambda_u} \lambda_u du + m_t^K \right)$$

where $m_t^K = E(\int_0^T K_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$. From $dL_t = -L_{t-}dM_t$ and

$$d(Lm^K)_t = L_{t-}dm_t^K + m_{t-}^K dL_t + d[m^K, L]_t = L_{t-}dm_t^K + m_{t-}^K dL_t$$

we deduce that

$$dS_t = K_t(dH_t - \lambda_t(1-H_t)dt) - S_{t-}dM_t + L_t dm_t^K = (K_t - S_{t-})dM_t + L_t dm_t^K$$

Note that, since m^K is continuous, its covariation process with L is null and that one can write $L_t dm_t^K$ instead of $L_{t-}dm_t^K$. Note also that, from the definition, the process S is a $\mathbb{G}$-martingale. This can be checked looking at the dynamics, since m^K is a $\mathbb{F}$, hence a $\mathbb{G}$, martingale.(WHY?)

Exercise 2.3.13 Write the risk-neutral dynamics of the price of the recovery for a general interest rate r. $\lhd$

Recovery with Payment at Default Time

Let K be a given $\mathbb{F}$-predictable process. The payment K_τ is done at time τ. Then, in the case $r = 0$,

$$S_t = 1\!\!1_{t<\tau} E(K_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u dF_u|\mathcal{F}_t)\,.$$

The dynamics of S is

$$dS_t = -S_{t-}dM_t + L_t(dm_t^K - K_t e^{-\Lambda_t}\lambda_t)dt = -S_{t-}dM_t + (1-H_t)(e^{\Lambda_t}dm_t^K - K_t\lambda_t)dt$$

and the process $S_t + K_\tau 1\!\!1_{\{\tau<t\}} = S_t + \int_0^t K_s dH_s = E(K_\tau|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale, as well as the process $S_t + \int_0^{t\wedge\tau} K_s\lambda_s ds$. The quantity $K_t\lambda_t$ which appears in the dynamics of S can be interpreted as a dividend K_t paid at rate λ_t (or with probability $\lambda_t dt = P(t<\tau<t+dt|\mathcal{F}_t)/P(t<\tau|\mathcal{F}_t)$)

Price and Hedging a Defaultable Call

We assume that

- the savings account $Y_t^0 = 1$
- a risky asset with risk-neutral dynamics

  $$dY_t = Y_t\sigma dW_t$$

  where W is a Brownian motion and σ is a constant
- a DZC of maturity T with price $D(t,T)$

are traded. The reference filtration is that of the BM W. The price of a defaultable call with payoff $1\!\!1_{T<\tau}(Y_T - K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(1\!\!1_{T<\tau}(Y_T-K)^+|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T}(Y_T-K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T-K)^+|\mathcal{F}_t)$. Hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-}dM_t$$

- In the particular case where λ is deterministic,

$$m_t^Y = e^{-\Lambda_T}E((Y_T-K)^+|\mathcal{F}_t) = e^{-\Lambda_T}C_t^Y$$

where C^Y is the price of a call in the Black Scholes model. This quantity is $C^Y_t = C^Y(t, Y_t)$ and satisfies $dC^Y_t = \Delta_t dY_t$ where $\Delta_t$is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$)

$$C_t = 1\!\!1_{t<\tau} e^{\Lambda_t} e^{-\Lambda_T} C^Y(t, Y_t) = L_t e^{-\Lambda_T} C^Y(t, Y_t) = D(t, T) C^Y(t, Y_t)$$

From

$$C_t = D(t, T) C^Y(t, Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC^Y + C^Y dL_t) = e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \\ &= e^{-\Lambda_T}(L_t \Delta_t dY_t - C^Y L_t dM_t) \end{aligned}$$

Therefore, using that $dD(t, T) = m_t dM_t = -e^{-\Lambda_T} L_t dM_t$ we get

$$dC_t = e^{-\Lambda_T} L_t \Delta_t dY_t - C^Y dD(t, T) = e^{-\Lambda_T} L_t \Delta_t dY_t + \frac{C_t}{D(t, T)} dD(t, T)$$

hence, an hedging strategy consists of holding in particular $\frac{C_t}{D(t,T)}$ DZCs.

- In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t, T)} dD(t, T) + L_t \frac{m^Y_t}{m_t} dm_t + L_t dm^Y_t = \frac{C_{t-}}{D(t, T)} dD(t, T) + \vartheta_t dY_t$$

An hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

Credit Default Swap

Definition 2.3.14 *A T-maturity credit default swap (CDS) with a constant rate κ and recovery at default is a contract. The seller agrees to pay the recovery at default time, the buyer pays (in continuous time) the premium κ till maturity or to default time, whichever occurs the first. The $\mathbb{F}$-predictable process $\delta : [0, T] \to \mathbb{R}$ represents the default protection, and the constant κ is the fixed CDS rate (also termed the spread or premium of the CDS).*

Let $B_t = \exp \int_0^t r_s ds$. The cumulative ex-dividend price of a CDS equals, for any $t \in [0, T]$, to the expectation of the remaining discounted future payoffs

$$S_t = B_t \mathbb{E}_{\mathbb{Q}}((B_\tau)^{-1} \delta_\tau 1\!\!1_{t<\tau\le T} - \int_t^{T\wedge\tau} \kappa B_s^{-1} ds | \mathcal{G}_t)$$

The cumulative price is

$$S_t = B_t \mathbb{E}_{\mathbb{Q}}((B_\tau)^{-1} \delta_\tau 1\!\!1_{\tau\le T} - \int_0^{T\wedge\tau} \kappa B_s^{-1} ds | \mathcal{G}_t)$$

We denote by D the dividend process associated with the CDS:

$$D_t = Z_t 1\!\!1_{\tau\le t} - \kappa(t \wedge \tau)$$

An immediate application of the key lemma gives the following result

Proposition 2.3.15 *The ex-dividend price of a CDS equals, for any $t \in [0, T]$,*

$$S_t(\kappa) = 1\!\!1_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_t^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right), \tag{2.3.3}$$

and thus the cumulative price of a CDS equals, for any $t \in [0, T]$,

$$S^{\text{cum}}_t(\kappa) = 1\!\!1_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_t^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right) + B_t \int_{]0,t]} B_u^{-1} \, dD_u. \tag{2.3.4}$$

An easy computation yields to

Corollary 2.3.16 *The dynamics of the ex-dividend price $S(\kappa)$ on $[0,T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\,dM_t + (1-H_t)\big(r_tS_t + \kappa - \lambda_t\delta_t\big)\,dt + (1-H_t)G_t^{-1}B_t\,dn_t,$$

where the $\mathbb{F}$-martingale n is given by the formula

$$n_t = \mathbb{E}_{\mathbb{Q}}\left(\int_0^T B_u^{-1}G_u\delta_u\lambda_u\,du - \kappa\int_0^T B_u^{-1}G_u\,du\,\Big|\,\mathcal{F}_t\right). \tag{2.3.5}$$

The dynamics of the cumulative price $S^{\mathrm{cum}}(\kappa)$ on $[0,T]$ are

$$dS_t^{\mathrm{cum}}(\kappa) = r_tS_t^{\mathrm{cum}}(\kappa)\,dt + \big(\delta_t - S_{t-}(\kappa)\big)\,dM_t + (1-H_t)G_t^{-1}B_t\,dn_t$$

2.3.7 Generalisation

We start with the filtered space $(\Omega,\mathcal{G},\mathbb{F},\mathbb{P})$ and the random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law. We define the default time τ as the first time when the increasing process Γ is above the random level Θ, i.e.,

$$\tau = \inf\{t \geq 0 \,:\, \Gamma_t \geq \Theta\}.$$

We do not assume any more that Γ is absolutely continuous, and we are even interested with the case where Γ fails to be continuous.

The same proof as before yields to

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t}$$

However, since Γ fails to be predictable, the compensator of H is no more Γ.

Let us study the following example. Let X be a compound Poisson process, with positive jumps, i.e.,

$$X_t = \sum_{n=1}^{N_t} Y_n$$

where N is a Poisson process and Y_n positive random variable, i.i.d. and independent from N.

Let $\psi(u) = \int_0^\infty (1-e^{uy})F(dy)$ where F is the cumulative distribution function of Y_1. Then, $e^{uX_t + t\lambda\psi(u)}$ is a martingale. Then, from $G_t = e^{-X_t} = e^{-X_t+t\lambda\psi(-1)}e^{-t\lambda\psi(-1)} = n_te^{-t\lambda\psi(-1)}$ where n is a martingale one deduce, by integration by parts the Doob-Meyer decomposition that

$$dG_t = e^{-t\lambda\psi(-1)}dN_t - e^{-t\lambda\psi(-1)}n_t\lambda\psi(-1)dt$$

and it follows that

$$1\!\!1_{\tau\leq t} - (t\wedge\tau)\lambda\psi(-1)$$

is a martingale.

One can also compute directly the Doob-Meyer decomposition of supermartingale G from Itô's formula. Let μ the jump measure of X

$$e^{-X_t} = 1 + \int_0^t\int (e^{-(X_{u-}+y)} - e^{-X_{u-}})\;(\mu(du,ds) - du\lambda F(dy)) + \int_0^t\int (e^{-(X_u+y)} - e^{-X_u})du\lambda F(dy)$$

where the quantity $\int_0^t\int (e^{-(X_{u-}+y)} - e^{-X_{u-}})\,((\mu(du,ds) - du\lambda F(dy)))$ represents a martingale. Hence the form of the compensator.

Chapter 3

Two Defaults

3.1 Two defaults, trivial reference filtration

As usual, a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is given. Let us first study the case with two random times τ_1, τ_2. We denote by $\tau_{(1)} = \inf(\tau_1, \tau_2)$ and $\tau_{(2)} = \sup(\tau_1, \tau_2)$, and we assume, for simplicity, that $\mathbb{P}(\tau_1 = \tau_2) = 0$. We denote by $(H^i_t, t \geq 0)$ the default process associated with $\tau_i, (i = 1, 2)$, and by $H_t = H^1_t + H^2_t$ the process associated with the two defaults. As before, $\mathbb{H}^i$ is the filtration generated by the process H^i and $\mathbb{H}$ is the filtration generated by the process H. The filtration $\mathbb{G}$ is generated by two processes H^i. The σ-algebra $\mathcal{G}_t = \mathcal{H}^1_t \vee \mathcal{H}^2_t$ is equal to $\sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t)$. It is useful to note that $\mathcal{G}_t$ is strictly greater than $\mathcal{H}_t$. Exemple: assume that τ_1 and τ_2 are independent and identically distributed. Then, obviously, for $u < t$

$$P(\tau_1 < \tau_2 | \tau_{(1)} = u, \tau_{(2)} = t) = 1/2\,,$$

hence $\sigma(\tau_1, \tau_2) \neq \sigma(\tau_{(1)}, \tau_{(2)})$.

3.1.1 Computation of joint laws

A $\mathcal{H}^1_t \vee \mathcal{H}^2_t$-measurable random variable is equal to

- a constant on the set $t < \tau_{(1)}$,

- a $\sigma(\tau_{(1)})$-measurable random variable on the set $\tau_{(1)} \leq t < \tau_{(2)}$, i.e., a $\sigma(\tau_1)$-measurable random variable on the set $\tau_1 \leq t < \tau_2$, and a $\sigma(\tau_2)$-measurable random variable on the set $\tau_2 \leq t < \tau_1$

- a $\sigma(\tau_1, \tau_2)$-measurable random variable on the set $\tau_2 \leq t$.

We denote by G the survival probability of the pair (τ_1, τ_2), i.e.,

$$G(t, s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)\,.$$

We shall also use the notation

$$g(s) = \frac{d}{ds} G(s, s) = \partial_1 G(s, s) + \partial_2 G(s, s)$$

where $\partial_1 G$ is the partial derivative of G with respect to the first variable (resp. $\partial_2 G$ is the partial derivative of G with respect to the second variable).

• We present in a first step some computations of conditional laws.

$$\begin{aligned}
\mathbb{P}(\tau_{(1)} > s) &= \mathbb{P}(\tau_1 > s, \tau_2 > s) = G(s, s) \\
\mathbb{P}(\tau_{(2)} > t | \tau_{(1)} = s) &= \frac{1}{g(s)} \left(\partial_1 G(s, t) + \partial_2 G(t, s) \right), \text{ for } t > s \\
&= 1, \text{ for } s \geq t
\end{aligned}$$

• We also compute conditional expectation in the filtration $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$: For $t < T$

$$
\begin{aligned}
\mathbb{P}(T < \tau_{(1)} | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_{(1)}} \frac{\mathbb{P}(T < \tau_{(1)})}{\mathbb{P}(t < \tau_{(1)})} = \mathbb{1}_{t<\tau_{(1)}} \frac{G(T,T)}{G(t,t)} \\
\mathbb{P}(T < \tau_1 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_1} \frac{\mathbb{P}(T < \tau_1 | \mathcal{H}_t^2)}{\mathbb{P}(t < \tau_1 | \mathcal{H}_t^2)} + \mathbb{1}_{\tau_1 < t} \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{\mathbb{P}(T < \tau_1, t < \tau_2)}{\mathbb{P}(t < \tau_1, t < \tau_2)} + \mathbb{1}_{\tau_2 \le t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + \mathbb{1}_{\tau_1 < t} \\
&= \mathbb{1}_{t<\tau_1} \left(\mathbb{1}_{t<\tau_2} \frac{G(T,t)}{G(t,t)} + \mathbb{1}_{\tau_2 < t} \frac{\mathbb{P}(T < \tau_1 | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} \right) + \mathbb{1}_{\tau_1 < t} \\
\mathbb{P}(\tau_{(2)} \le T | \mathcal{H}_t^1 \vee \mathcal{H}_t^2) &= \mathbb{1}_{t<\tau_{(1)}} \frac{\mathbb{P}(t \le \tau_{(1)} < \tau_{(2)} < T)}{\mathbb{P}(t < \tau_{(1)})} + \mathbb{1}_{\tau_1 \le t < \tau_2} \frac{\mathbb{P}(t < \tau_2 < T | \tau_1)}{\mathbb{P}(t < \tau_2 | \tau_1)} \\
&\quad + \mathbb{1}_{\tau_2 \le t < \tau_1} \frac{\mathbb{P}(t < \tau_1 < T | \tau_2)}{\mathbb{P}(t < \tau_1 | \tau_2)} + \mathbb{1}_{\tau_{(2)} < t} \, .
\end{aligned}
$$

• The computation of $\mathbb{P}(T < \tau_1 | \tau_2)$ can be done as follows: the function h such that $\mathbb{P}(T < \tau_1 | \tau_2) = h(\tau_2)$ satisfies

$$\mathbb{E}(h(\tau_2)\varphi(\tau_2)) = \mathbb{E}(\varphi(\tau_2)\mathbb{1}_{T<\tau_1})$$

for any Borel (bounded) function φ. This implies that (assuming that the pair (τ_1, τ_2) has a density f)

$$\int_0^\infty dv h(v)\varphi(v) \int_0^\infty du f(u,v) = \int_0^\infty dv \varphi(v) \int_T^\infty du f(u,v)$$

or

$$\int_0^\infty dv h(v)\varphi(v)\partial_2 G(0,v) = \int_0^\infty dv \varphi(v)\partial_2 G(T,v)$$

hence, $h(v) = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$.
We can also write

$$\mathbb{P}(T < \tau_1 | \tau_2 = v) = \frac{\mathbb{P}(T < \tau_1, \tau_2 \in dv)}{\mathbb{P}(\tau_2 \in dv)} = -\frac{1}{\mathbb{P}(\tau_2 \in dv)} \frac{d}{dv} \mathbb{P}(\tau_1 > T, \tau_2 > v)\, dv = \frac{\partial_2 G(T,v)}{\partial_2 G(0,v)}$$

hence, on the set $\tau_2 < T$,

$$\mathbb{P}(T < \tau_1 | \tau_2) = h(\tau_2) = \frac{\partial_2 G(T,\tau_2)}{\partial_2 G(0,\tau_2)}$$

• In the same way, for $T > t$

$$\mathbb{P}(\tau_1 \le T < \tau_2 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}} = \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} \Psi(\tau_1)$$

where Ψ satisfies

$$\mathbb{E}(\varphi(\tau_1)\mathbb{1}_{\tau_1 \le t < T < \tau_2}) = \mathbb{E}(\varphi(\tau_1)\Psi(\tau_1)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}})$$

for any Borel (bounded) function φ. In other terms

$$\int_0^t du \varphi(u) \int_T^\infty dv f(u,v) = \int_0^t du \varphi(u)\Psi(u) \int_t^\infty dv f(u,v)$$

or

$$\int_0^t du \varphi(u)\partial_1 G(u,T) = \int_0^t du \varphi(u)\Psi(u)\partial_1 G(u,t)\, .$$

This implies that

$$\Psi(u) = \frac{\partial_1 G(u,T)}{\partial_1 G(u,t)}$$

$$\mathbb{P}(\tau_1 \le T < \tau_2 | \mathcal{H}_t^1 \vee \mathcal{H}_t^2)\mathbb{1}_{\{\tau_1 \le t < \tau_2\}} = \mathbb{1}_{\{\tau_1 \le t < \tau_2\}} \frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)}\, .$$

3.1.2 Value of credit derivatives

We assume in this section that the default-free interest rate is null $r = 0$.

We introduce two kinds of credit derivatives

A defaultable zero-coupon related to the default times τ^i delivers 1 monetary unit if (and only if) τ_i is greater that T: $D^i(t,T) = \mathbb{E}(1\!\!1_{\{T<\tau_i\}}|\mathcal{H}^1_t \vee \mathcal{H}^2_t)$

A contract which pays (at time T) R^1 is one default occurs before T and R_2 if the two defaults occur before T: $CD_t = \mathbb{E}(R_1 1\!\!1_{\{0<\tau_{(1)}\le T\}} + R_2 1\!\!1_{\{0<\tau_{(2)}\le T\}}|\mathcal{H}^1_t \vee \mathcal{H}^2_t)$

We obtain

$$\begin{aligned}
D^1(t,T) &= 1\!\!1_{\{\tau_1>t\}}\left(1\!\!1_{\{\tau_2\le t\}}\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)} + 1\!\!1_{\{\tau_2>t\}}\frac{G(T,t)}{G(t,t)}\right)\\
D^2(t,T) &= 1\!\!1_{\{\tau_2>t\}}\left(1\!\!1_{\{\tau_1\le t\}}\frac{\partial_1 G(\tau_1,T)}{\partial_2 G(\tau_1,t)} + 1\!\!1_{\{\tau_1>t\}}\frac{G(t,T)}{G(t,t)}\right)\\
CD_t &= R_1 1\!\!1_{\{\tau_{(1)}>t\}}\left(\frac{G(t,t)-G(T,T)}{G(t,t)}\right) + R_2 1\!\!1_{\{\tau_{(2)}\le t\}} + R_1 1\!\!1_{\{\tau_{(1)}\le t\}}\\
&\quad + R_2 1\!\!1_{\{\tau_{(2)}>t\}}\left\{I_t(0,1)\left(1-\frac{\partial_2 G(T,\tau_2)}{\partial_2 G(t,\tau_2)}\right) + I_t(1,0)\left(1-\frac{\partial_1 G(\tau_1,T)}{\partial_1 G(\tau_1,t)}\right)\right.\\
&\quad \left. + I_t(0,0)\left(1-\frac{G(t,T)+G(T,t)-G(T,T)}{G(t,t)}\right)\right\}
\end{aligned}$$

where we have denoted

$$\begin{aligned}
I_t(1,1) &= 1\!\!1_{\{\tau_1\le t,\tau_2\le t\}}\,, \qquad & I_t(0,0) &= 1\!\!1_{\{\tau_1>t,\tau_2>t\}}\\
I_t(1,0) &= 1\!\!1_{\{\tau_1\le t,\tau_2>t\}}\,, \qquad & I_t(0,1) &= 1\!\!1_{\{\tau_1>t,\tau_2\le t\}}
\end{aligned}$$

More generally, some easy computation leads to

$$\mathbb{E}(h(\tau_1,\tau_2)|\mathcal{H}_t) = I_t(1,1)h(\tau_1,\tau_2) + I_t(1,0)\Psi_{1,0}(\tau_1) + I_t(0,1)\Psi_{0,1}(\tau_2) + I_t(0,0)\Psi_{0,0}$$

where

$$\begin{aligned}
\Psi_{1,0}(u) &= -\frac{1}{\partial_1 G(u,t)}\int_t^\infty h(u,v)\partial_1 G(u,dv)\\
\Psi_{0,1}(v) &= -\frac{1}{\partial_2 G(t,v)}\int_t^\infty h(u,v)\partial_2 G(du,v)\\
\Psi_{0,0} &= \frac{1}{G(t,t)}\int_t^\infty\int_t^\infty h(u,v)G(du,dv)
\end{aligned}$$

The next result deals with the valuation of a first-to-default claim in a bivariate set-up. Let us stress that the concept of the (tentative) price will be later supported by strict replication arguments. In this section, by a *pre-default price* associated with a $\mathbb{G}$-adapted price process π, we mean here the (deterministic) function $\widetilde{\pi}$ such that $\pi_t 1\!\!1_{\{\tau_{(1)}>t\}} = \widetilde{\pi}(t)1\!\!1_{\{\tau_{(1)}>t\}}$ for every $t\in[0,T]$. In other words, the pre-default price $\widetilde{\pi}$ and the price π coincide prior to the first default only.

Definition 3.1.1 *Let $Z_i, i = 1,2$ be two functions, and X a constant. A FtD claim pays $Z_1(\tau_1)$ at time τ_1 if $\tau_1 < T, \tau_1 < \tau_2$, pays $Z_2(\tau_2)$ at time τ_2 if $\tau_2 < T, \tau_2 < \tau_1$, and X at maturity if $\tau_1 \wedge \tau_2 > T$*

Proposition 3.1.2 *The pre-default price of a FtD claim $(X, 0, Z, \tau_{(1)})$, where $Z = (Z_1, Z_2)$ and $X = c(T)$, equals*

$$\frac{1}{G(t,t)}\left(-\int_t^T Z_1(u)\,G(du,u) - \int_t^T Z_2(v)G(v,dv) + XG(T,T)\right).$$

PROOF: The price can be expressed as

$$\mathbb{E}_{\mathbb{Q}}(Z_1(\tau_1)1\!\!1_{\{\tau_1\le T,\tau_2>\tau_1\}}|\mathcal{G}_t)+\mathbb{E}_{\mathbb{Q}}(Z_2(\tau_2)1\!\!1_{\{\tau_2\le T,\tau_1>\tau_2\}}|\mathcal{G}_t)+\mathbb{E}_{\mathbb{Q}}(c(T)1\!\!1_{\{\tau_{(1)}>T\}}|\mathcal{G}_t).$$

The pricing formula now follows by evaluating the conditional expectation, using the joint distribution of default times under the martingale measure $\mathbb{Q}$. □

Comment 3.1.3 Same computations appear in Kurtz and Riboulet [99]

3.1.3 Martingales

We present the computation of the martingales associated with the times τ_i in different filtrations. In particular, we shall obtain the computation of the intensities in various filtrations.

We have established that, if $\mathbb{F}$ is a given reference filtration and if $G_t=\mathbb{P}(\tau>t|\mathcal{F}_t)$ is the Azéma supermartingale admitting a Doob-Meyer decomposition $G_t=Z_t-\int_0^t a_s ds$, then the process

$$H_t-\int_0^{t\wedge\tau}\frac{a_s}{G_{s-}}ds$$

is a $\mathbb{G}$-martingale, where $\mathbb{G}=\mathbb{F}\vee\mathbb{H}$ and $\mathcal{H}_t=\sigma(t\wedge\tau)$.

• **Filtration $\mathbb{H}^i$.** We study the decomposition of the semi-martingales H^i in the filtration $\mathbb{H}^i$. We set $F_i(s)=\mathbb{P}(\tau_i\le s)=\int_0^s f_i(u)du$. From our general result, recalled above, applied to the case where $\mathbb{F}$ is the trivial filtration, we obtain that for any $i=1,2$, the process

$$M_t^i=H_t^i-\int_0^{t\wedge\tau_i}\frac{f_i(s)}{1-F_i(s)}ds \tag{3.1.1}$$

is a $\mathbb{H}^i$-martingale.

• **Filtration $\mathbb{G}$.** We apply the general result to the case $\mathbb{F}=\mathbb{H}^2$ and $\mathbb{H}=\mathbb{H}^1$. Let

$$G_t^{1|2}=\mathbb{P}(\tau_1>t|\mathcal{H}_t^2)$$

be the Azéma supermartingale of τ_1 in the filtration $\mathbb{H}^2$, with Doob-Meyer decomposition $G_t^{1|2}=Z_t^{1|2}-\int_0^t a_s^{1|2}ds$ where $Z^{1|2}$ is a $\mathbb{H}^2$-martingale. Then, the process

$$H_t^1-\int_0^{t\wedge\tau_1}\frac{a_s^{1|2}}{G_{s-}^{1|2}}ds$$

is a $\mathbb{G}$-martingale. The process $A_t^{1|2}=\int_0^{t\wedge\tau_1}\frac{a_s^{1|2}}{G_{s-}^{1|2}}ds$ is the $\mathbb{G}$-adapted compensator of H^1. The same methodology can be applied for the compensator of H^2.

Proposition 3.1.4 *The process $M^{1,\mathbb{G}}$ defined as*

$$M_t^{1,\mathbb{G}} := H_t^1+\int_0^{t\wedge\tau_1\wedge\tau_2}\frac{\partial_1 G(s,s)}{G(s,s)}ds+\int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1}\frac{f(s,\tau_2)}{\partial_2 G(s,\tau_2)}ds$$

is a $\mathbb{G}$-martingale.

PROOF: Some easy computation enables us to write

$$\begin{aligned}G_t^{1|2} &= \mathbb{P}(\tau_1>t|\mathcal{H}_t^2)=H_t^2\mathbb{P}(\tau_1>t|\tau_2)+(1-H_t^2)\frac{\mathbb{P}(\tau_1>t,\tau_2>t)}{\mathbb{P}(\tau_2>t)}\\ &= H_t^2h^1(t,\tau_2)+(1-H_t^2)\psi(t)\end{aligned} \tag{3.1.2}$$

where

$$h^1(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)}\,; \quad \psi(t) = G(t,t)/G(0,t).$$

Function $t \to \psi(t)$ and process $t \to h(t,\tau_2)$ are continuous and of finite variation, hence integration by parts rule leads to

$$\begin{aligned}
dG_t^{1|2} &= h(t,\tau_2)dH_t^2 + H_t^2\partial_1 h(t,\tau_2)dt + (1-H_t^2)\psi'(t)dt - \psi(t)dH_t^2 \\
&= \left(h(t,\tau_2) - \psi(t)\right)dH_t^2 + \left(H_t^2\partial_1 h(t,\tau_2) + (1-H_t^2)\psi'(t)\right)dt \\
&= \left(\frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)} - \frac{G(t,t)}{G(0,t)}\right)dH_t^2 + \left(H_t^2\partial_1 h(t,\tau_2) + (1-H_t^2)\psi'(t)\right)dt
\end{aligned}$$

From the computation of the Stieljes integral, we can write

$$\begin{aligned}
\int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,\tau_2)}{\partial_2 G(0,\tau_2)}\right)dH_t^2 &= \left(\frac{G(\tau_2,\tau_2)}{G(0,\tau_2)} - \frac{\partial_2 G(\tau_2,\tau_2)}{\partial_2 G(0,\tau_2)}\right)1_{\{\tau_2 \le T\}} \\
&= \int_0^T \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right)dH_t^2
\end{aligned}$$

and substitute it in the expression of $dG^{1|2}$:

$$dG_t^{1|2} = \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right)dH_t^2 + \left(H_t^2\partial_1 h(t,\tau_2) + (1-H_t^2)\psi'(t)\right)dt$$

We now use that

$$dH_t^2 = dM_t^2 - \left(1 - H_t^2\right)\frac{\partial_2 G(0,t)}{G(0,t)}dt$$

where M^2 is a $\mathbb{H}^2$-martingale, and we get the $\mathbb{H}^2-$ Doob-Meyer decomposition of $G^{1|2}$:

$$\begin{aligned}
dG_t^{1|2} &= \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right)dM_t^2 - \left(1-H_t^2\right)\left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right)\frac{\partial_2 G(0,t)}{G(0,t)}dt \\
&\quad + \Big(H_t^2\partial_1 h^{(1)}(t,\tau_2) + (1-H_t^2)\psi'(t)\Big)dt
\end{aligned}$$

and from

$$\psi'(t) = \left(\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}\right)\frac{\partial_2 G(0,t)}{G(0,t)} + \frac{\partial_1 G(t,t)}{G(0,t)}$$

we conclude

$$dG_t^{1|2} = \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}\right)dM_t^2 + \left(H_t^2\partial_1 h^{(1)}(t,\tau_2) + (1-H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)}\right)dt$$

From (3.1.2), the process $G^{1|2}$ has a single jump of size $\frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} - \frac{G(t,t)}{G(0,t)}$. From (3.1.2),

$$G_t^{1|2} = \frac{G(t,t)}{G(0,t)} = \psi(t)$$

on the set $\tau_2 > t$, and its bounded variation part is $\psi'(t)$. The hazard process has a non null martingale part, except if $\frac{G(t,t)}{G(0,t)} = \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)}$ (this is the case if the default are independent). Hence, (H) hypothesis is not satisfied in a general setting between $\mathbb{H}^i$ and $\mathbb{G}$.

Remark 3.1.5 Note that

$$\begin{aligned} H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{1|2}}{G_s^{1|2}} ds &= H_t^1 - \int_0^{t\wedge\tau_1} \frac{H_s^2 \partial_1 h^1(s,\tau_2) - (1-H_s^2)\partial_1 G(s,s)/G(0,s)}{H_s^2 h^1(s,\tau_2) + (1-H_s^2)\psi(s)} ds \\ &= H_t^1 - \int_0^{t\wedge\tau_1} H_s^2 \frac{\partial_1 h^1(s,\tau_2)}{h^1(s,\tau_2)} - (1-H_s^2)\frac{\partial_1 G(s,s)/G(0,s)}{\psi(s)} ds \\ &= H_t^1 - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{\partial_1 h^1(s,\tau_2)}{h^1(s,\tau_2)} ds - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds \\ &= H_t^1 - \ln \frac{h^1(t\wedge\tau_1\wedge\tau_2,\tau_2)}{h^1(t\wedge\tau_1,\tau_2)} - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds \end{aligned}$$

It follows that the intensity of τ_1 in the $\mathbb{G}$-filtration is $\frac{\partial_1 G(s,s)}{G(s,s)}$ on the set $\{t < \tau_2 \wedge \tau_1\}$ and $\frac{\partial_1 h^1(s,\tau_2)}{h^1(s,\tau_2)}$ on the set $\{\tau_2 < t < \tau_1\}$. It can be proved that the intensity of $\tau_1 \wedge \tau_2$ is

$$\frac{\partial_1 G(s,s)}{G(s,s)} + \frac{\partial_2 G(s,s)}{G(s,s)} = \frac{g(t)}{G(t,t)}$$

where $g(t) = \frac{d}{dt} G(t,t)$

• **Filtration** $\mathbb{H}$ We reproduce now the result of Chou and Meyer [33], in order to obtain the martingales in the filtration $\mathbb{H}$, in case of two default times. Here, we denote by $\mathbb{H}$ the filtration generated by the process $H_t = H_t^1 + H_t^2$. This filtration is smaller than the filtration $\mathbb{G}$. We denote by $\tau_{(1)} = \tau_1 \wedge \tau_2$ the infimum of the two default times and by $\tau_{(2)} = \tau_1 \vee \tau_2$ the supremum. The filtration $\mathbb{H}$ is the filtration generated by $\sigma(\tau_{(1)} \wedge t) \vee \sigma - t_2 \wedge t)$, up to completion with negligeable sets.
Let us denote by $G_1(t)$ the survival distribution function of $\tau_{(1)}$, i.e., $G_1(t) = \mathbb{P}(\tau_1 > t, \tau_2 > t) = G(t,t)$ and by $G_2(t;u)$ the survival conditional distribution function of $\tau_{(2)}$ with respect to $\tau_{(1)}$, i.e., for $t > u$,

$$G_2(u;t) = \mathbb{P}(\tau_{(2)} > t | \tau_{(1)} = u) = \frac{1}{g(u)} \left(\partial_1 G(u,t) + \partial_2 G(t,u)\right),$$

where $g(t) = \frac{d}{dt} G(t,t) = \frac{1}{dt}\mathbb{P}(\tau_{(1)} \in dt)$. We shall also note

$$K(u;t) = \mathbb{P}(\tau_{(2)} - \tau_{(1)} > t | \tau_{(1)} = u) = G_2(u; t+u)$$

The process $M_t := H_t - \Lambda_t$ is a $\mathbb{H}$-martingale, where

$$\Lambda_t = \Lambda_1(t) \mathbb{1}_{t<\tau_{(1)}} + \left[\Lambda_1(\tau_{(1)}) + \Lambda_2(\tau_{(1)}, t - \tau_{(1)})\right] \mathbb{1}_{\tau_{(1)} \le t < \tau_{(2)}}$$

with

$$\Lambda_1(t) = -\int_0^t \frac{dG_1(s)}{G_1(s)} = \int_0^t \frac{g(s)}{G(s,s)} ds = -\ln \frac{G(t,t)}{G(0,0)} = -\ln G(t,t)$$

and

$$\Lambda_2(s;t) = -\int_0^t \frac{d_u K(s;u)}{K(s,u)} = -\ln \frac{K(s;t)}{K(s;0)}$$

hence

$$\begin{aligned} \Lambda_2(\tau_{(1)}, t - \tau_{(1)}) &= -\ln \frac{K(\tau_{(1)}; t - \tau_{(1)})}{K(\tau_{(1)}; 0)} = -\ln \frac{G_2(\tau_{(1)}; t)}{G_2(\tau_{(1)}; \tau_{(1)})} \\ &= -\ln \frac{\partial_1 G(\tau_{(1)}, t) + \partial_2 G(t, \tau_{(1)})}{\partial_1 G(\tau_{(1)}, \tau_{(1)}) + \partial_2 G(\tau_{(1)}, \tau_{(1)})} \end{aligned}$$

It is proved in Chou-Meyer [33] that any $\mathbb{H}$-martingale is a stochastic integral with respect to M. This result admits an immediate extension to the case of n successive defaults.
This representation theorem has an interesting consequence: a single asset is enough to get a complete market. This asset has final payoff $H_T - \Lambda_T$. It corresponds to a swap with cumulative premium leg Λ_t

3.1.4 Application of Norros lemma for two defaults

Norros's lemma

Proposition 3.1.6 *Let $\tau_i, i = 1, \cdots, n$ be n finite-valued random times and $\mathcal{G}_t = \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n$. Assume that*

(i) $P(\tau_i = \tau_j) = 0, \forall i \neq j$

(ii) there exists continuous processes Λ^i such that $M_t^i = H_t^i - \Lambda_{t\wedge\tau_i}^i$ are $\mathbb{G}$-martingales

then, the r.v's $\Lambda_{\tau_i}^i$ are independent with exponential law.

Proof. For any $\mu_i > -1$ the processes $L_t^i = (1+\mu_i)^{H_t^i} e^{-\mu_i \Lambda_{t\wedge\tau_i}^i}$, solution of

$$dL_t^i = L_{t-}^i \mu_i dM_t^i$$

are uniformly integrable martingales. Moreover, these martingales have no common jumps, and are orthogonal. Hence $E(\prod_i (1+\mu_i) e^{-\mu_i \Lambda_{\tau_i}^i}) = 1$, which implies

$$E(\prod_i e^{-\mu_i \Lambda_{\tau_i}^i}) = \prod_i (1+\mu_i)^{-1}$$

hence the independence property. □

Application

In case of two defaults, this implies that U_1 and U_2 are independent, where

$$U_i = \int_0^{\tau_i} \frac{a_i(s)}{G_i^*(s)} ds$$

and (with $h^{(1)}(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)}$, $h^{(2)}(u,t) = \frac{\partial_1 G(u,t)}{\partial_1 G(u,0)}$)

$$a_1(t) = -(1-H_t^2)\frac{\partial_1 G(t,t)}{G(0,t)} + H_t^2 \partial_1 h^{(1)}(t,\tau_2), \quad G_1^*(t) = H_t^2 h^{(1)}(t,\tau_2) + (1-H_t^2)\frac{G(t,t)}{G(0,t)},$$

$$a_2(t) = -(1-H_t^1)\frac{\partial_2 G(t,t)}{G(t,0)} + H_t^1 \partial_2 h^{(2)}(\tau_1,t), \quad G_2^*(t) = H_t^1 h^{(2)}(\tau_1,t) + (1-H_t^1)\frac{G(t,t)}{G(t,0)}$$

are independent. In a more explicit form,

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \ln \frac{h^{(1)}(\tau_1,\tau_2)}{h^{(1)}(\tau_1\wedge\tau_2,\tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \ln \frac{\partial_2 G(\tau_1,\tau_2)}{\partial_2 G(\tau_1\wedge\tau_2,\tau_2)}$$

is independent from

$$\int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)} ds + \ln \frac{h^{(2)}(\tau_1,\tau_2)}{h^{(2)}(\tau_1,\tau_1\wedge\tau_2)} = \int_0^{\tau_1\wedge\tau_2} \frac{\partial_2 G(s,s)}{G(s,s)} ds + \ln \frac{\partial_1 G(\tau_1,\tau_2)}{\partial_1 G(\tau_1,\tau_1\wedge\tau_2)}$$

Example of Poisson process

In the case where τ_1 and τ_2 are the two first jumps of a Poisson process, we have

$$G(t,s) = \begin{cases} e^{-\lambda t} \text{ for } s < t \\ e^{-\lambda s}(1+\lambda(s-t)) \text{ for } s > t \end{cases}$$

with partial derivatives

$$\partial_1 G(t,s) = \begin{cases} -\lambda e^{-\lambda t} \text{ for } t > s \\ -\lambda e^{-\lambda s} \text{ for } s > t \end{cases}, \quad \partial_2 G(t,s) = \begin{cases} 0 \text{ for } t > s \\ -\lambda^2 e^{-\lambda s}(s-t) \text{ for } s > t \end{cases}$$

and

$$h(t,s) = \begin{cases} 1 \text{ for } t > s \\ \frac{t}{s} \text{ for } s > t \end{cases}, \quad \partial_1 h(t,s) = \begin{cases} 0 \text{ for } t > s \\ \frac{1}{s} \text{ for } s > t \end{cases}$$

$$k(t,s) = \begin{cases} 0 \text{ for } t > s \\ 1 - e^{-\lambda(s-t)} \text{ for } s > t \end{cases}, \quad \partial_2 k(t,s) = \begin{cases} 0 \text{ for } t > s \\ \lambda e^{-\lambda(s-t)} \text{ for } s > t \end{cases}$$

Then, one obtains $U_1 = \tau_1$ et $U_2 = \tau_2 - \tau_1$

3.1.5 Dynamic of CDSs

Let us now examine the valuation of a single-name CDS written on the default τ_1. Our aim is to show that the dynamics of this CDS will be affected by the information on τ_2: when τ_2 occurs, the intensity of τ_1 changes, and this will change the parameters of the price dynamics. We reproduce some result appearing in Bielecki et al. [?]

We consider a CDS

- with a constant spread κ
- which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 < T$, where δ is a deterministic function.

The value of the CDS takes the form

$$V_t(\kappa) = \widetilde{V}_t(\kappa) 1\!\!1_{t<\tau_2\wedge\tau_1} + \widehat{V}_t(\kappa) 1\!\!1_{\tau_1\wedge\tau_2\le t<\tau_1}\,.$$

First, we restrict our attention to the case $t < \tau_2 \wedge \tau_1$.

Proposition 3.1.7 *On the set $t < \tau_2 \wedge \tau_1$, the value of the CDS is*

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)}\left(-\int_t^T \delta(u)\partial_1 G(u,t)\,du - \kappa \int_t^T G(u,t)\,du \right).$$

Proof: The value $V(\kappa)$ of this CDS, computed in the filtration $\mathbb{H}$, i.e., taking care on the information on the second default contained in that filtration, is

$$V_t(\kappa) = 1\!\!1_{t<\tau_1}\mathbb{E}\left(\delta(\tau_1)1\!\!1_{\tau_1\le T} - \kappa((T\wedge\tau_1)-t)|\mathcal{H}_t\right)$$

Let us denote by $\tau = \tau_1 \wedge \tau_2$ the first default time. Then, $1\!\!1_{\{t<\tau\}}V_t(\kappa) = 1\!\!1_{\{t<\tau\}}\widetilde{V}_t(\kappa)$, where

$$\begin{aligned}
\widetilde{V}_t(\kappa) &= \frac{1}{\mathbb{Q}(\tau>t)}\mathbb{E}\left(\delta(\tau_1)1\!\!1_{\tau_1\le T}1\!\!1_{t<\tau} - \kappa((T\wedge\tau_1)-t)1\!\!1_{t<\tau}\right) \\
&= \frac{1}{G(t,t)}\mathbb{E}\left(\delta(\tau_1)1\!\!1_{\tau_1\le T}1\!\!1_{t<\tau} - \kappa((T\wedge\tau_1)-t)1\!\!1_{t<\tau}\right) \\
&= \frac{1}{G(t,t)}\left(\int_t^T \delta(u)\mathbb{Q}(\tau_1\in du, \tau_2>t)\right. \\
&\quad \left. -\kappa\int_t^T (u-t)\mathbb{Q}(\tau_1\in du,\tau_2>t) - (T-t)\kappa\int_T^\infty \mathbb{Q}(\tau_1\in du,\tau_2>t)\right)
\end{aligned}$$

In other terms, using integration by parts formula

$$\widetilde{V}_t(\kappa) = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u)\partial_1 G(u,t)\,du - \kappa \int_t^T G(u,t)\,du \right)$$

□

Proposition 3.1.8 *On the event $\{\tau_2 \le t < \tau_1\}$, the CDS price equals*

$$\begin{aligned} V_t(\kappa) &= \hat{V}_t = 1\!\!1_{t<\tau_1}\mathbb{E}\left(\delta(\tau_1)1\!\!1_{\tau_1 \le T} - \kappa((T\wedge\tau_1) - t)|\sigma(\tau_2)\right) \\ &= \frac{1}{\partial_2 G(t,\tau_2)} \left(- \int_t^T \delta(u) f(u,\tau_2)\,du - \kappa \int_t^T \partial_2 G(u,\tau_2)\,du \right) := V_t^{1|2}(\tau_2) \end{aligned}$$

where

$$V_t^{1|2}(s) = \frac{1}{\partial_2 G(t,s)} \left(- \int_t^T \delta(u) f(u,s)\,du - \kappa \int_t^T \partial_2 G(u,s)\,du \right).$$

In the financial interpretation, $V_t^{1|2}(s)$ is the market price at time t of a CDS on the first credit name, under the assumption that the default τ_2 occurs at time s and the first name has not yet defaulted (recall that simultaneous defaults are excluded, since we have assumed that G is differentiable).
The price of a CDS is $V_t = \widetilde{V}_t 1\!\!1_{t<\tau_2\wedge\tau_1} + \widehat{V}_t 1\!\!1_{\tau_2\wedge\tau_1 \le t < \tau_1}$. Differentiating the deterministic function which gives the value of the CDS, we obtain

$$d\widetilde{V}_t(\kappa) = \left(\big(\widetilde{\lambda}_1(t) + \widetilde{\lambda}_2(t)\big)\widetilde{V}_t(\kappa) + \kappa - \widetilde{\lambda}_1(t)\delta(t) - \widetilde{\lambda}_2(t) V_t^{1|2}(t) \right) dt,$$

where for $i = 1,2$ the function $\widetilde{\lambda}_i(t)$ is the (deterministic) *pre-default intensity* of τ_i given in (**??**) and

$$d\widehat{V}_t(\kappa) = \left(\widetilde{\lambda}_t^{1|2}(\tau_2) \left(\widehat{V}_t(\kappa) - \delta(t) \right) + \kappa \right) dt$$

where $\widetilde{\lambda}_t^{1|2}(u)$ is given in (**??**).

Proposition 3.1.9 *The price of a CDS follows*

$$\begin{aligned} dV_t &= (1-H_t^1)(1-H_t^2)(\kappa - \delta(t)\widetilde{\lambda}^1(t))dt + (1-H_t^1)H_t^2(\kappa - \delta(t)\widetilde{\lambda}_t^{1|2})dt \\ &\quad -V_{t-}dM_t^1 + (1-H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \end{aligned} \tag{3.1.3}$$

Proof: Differentiating $V_t = \widetilde{V}_t(1-H_t^1)(1-H_t^2) + \hat{V}_t(1-H_t^1)H_t^2$ one obtains

$$\begin{aligned} dV_t &= (1-H_t^1)(1-H_t^2)d\widetilde{V}_t + (1-H_t^1)H_t^2 d\widehat{V}_t - V_{t-}dH_t^1 \\ &\quad +(1-H_t^1)(V_t^{1|2}(t) - \widetilde{V}_t)dH_t^2 \end{aligned}$$

which leads to the result after light computations □

Comment 3.1.10 As for a single name CDS, the quantity $-\delta(t)\widetilde{\lambda}^1(t)$ corresponds to the dividend δ to be paid at time t with probability λ^1 on the set $t < \tau_1 \wedge \tau_2$ and $\delta(t)\widetilde{\lambda}_t^{1|2}$ corresponds to the dividend δ to be paid at time t with probability $\lambda_t^{1|2}$ on the set $\tau_2 < t < \tau_1$. The quantity $V_t^{1|2}(t) - \widetilde{V}_t$ represents the jump in the value of the CDS, when default τ_2 occurs at time t.

The cumulative price of the CDS is

$$V_t^{\text{cum}} = \mathbb{E}(\delta(\tau)\mathbb{1}_{\tau\le T} - \kappa(T\wedge\tau)|\mathcal{H}_t)$$

It follows that

$$dV_t^{\text{cum}} = dV_t - \delta(t)dH_t^1 - \kappa(1-H_t^1)dt$$

hence, since the cumulative price is a martingale

$$dV_t = dm_t + \delta(t)\lambda_t^1 dt - \kappa(1-H_t^1)dt$$

$$\begin{aligned}
dV_t^{\text{cum}} &= (1-H_t^1)(1-H_t^2)(\kappa-\delta(t)\widetilde{\lambda}^1(t))dt + (1-H_t^1)H_t^2(\kappa-\delta(t)\widetilde{\lambda}_t^{1|2})dt \\
&\quad -V_{t-}dM_t^1 + (1-H_t^1)(V_t^{1|2}(t)-V_{t-})dM_t^2 - \delta(t)dH_t^1 - \kappa(1-H_t^1)dt \\
&= (\delta(t)-V_{t-})dM_t^1 + (1-H_t^1)(V_t^{1|2}(t)-V_{t-})dM_t^2
\end{aligned}$$

which is an easy way to obtain the drift term in (3.1.3).

3.1.6 CDSs as hedging assets

Assume now that a CDS written on τ_2 is also traded in the market. We denote by $V^i, i=1,2$ the prices of the two CDSs. Since the CDS are paying dividends, a self financing strategy consisting in ϑ^i shares of CDS's has value $X_t = \vartheta_t^1 V_t^1 + \vartheta_t^2 V_t^2$ and dynamics

$$\begin{aligned}
dX_t &= \vartheta_t^1 dV_t^{1,cum} + \vartheta_t^2 dV_t^{2,cum} \\
&= \vartheta_t^1\left((\delta^1(t)-V_{t-}^1)dM_t^1 + (1-H_t^1)(V_t^{1|2}(t)-\widetilde{V}_t^1)dM_t^2\right) \\
&\quad +\vartheta_t^2\left((\delta^2(t)-V_{t-}^2)dM_t^2 + (1-H_t^2)(V_t^{2|1}(t)-\widetilde{V}_t^2)dM_t^1\right) \\
&= \left(\vartheta_t^1(\delta^1(t)-V_{t-}^1) + \vartheta_t^2(1-H_t^2)(V_t^{2|1}(t)-\widetilde{V}_t^2\right) dM_t^1 \\
&\quad + \left(\vartheta_t^1(1-H_t^1)(V_t^{1|2}(t)-\widetilde{V}_t^1) + \vartheta_t^2(\delta^2(t)-V_{t-}^2)\right) dM_t^2
\end{aligned}$$

Let $A\in\mathcal{H}_T$ be a terminal payoff with price (we use the PRT to prove the existence of the coefficients π)

$$A_t = \mathbb{E}(A) + \int_0^t \pi_t^i dM_t^i\,.$$

. In order to hedge that claim, it remains to solve the linear system

$$\begin{aligned}
\vartheta_t^1(\delta^1(t)-V_{t-}^1) + \vartheta_t^2(1-H_t^2)(V_t^{2|1}(t)-\widetilde{V}_t^2) &= \pi_t^1 \\
\vartheta_t^1(1-H_t^1)(V_t^{1|2}(t)-\widetilde{V}_t^1) + \vartheta_t^2(\delta^2(t)-V_{t-}^2) &= \pi_t^2
\end{aligned}$$

Hence, on the set $t<\tau_1\wedge\tau_2$, noting that $V_t^i = \widehat{V}_t^i$ on that set,

$$\begin{aligned}
\vartheta_t^1 &= \frac{\pi_t^1(\delta^2(t)-V_t^2) - \pi_t^2(V_t^{2|1}-V_t^2)}{(\delta^1(t)-V_t^1)(\delta^2(t)-V_t^2) - (V_t^{1|2}-V_t^1)(V_t^{2|1}-V_t^2)} \\
\vartheta_t^2 &= \frac{\pi^2(\delta^1(t)-V_t^1) - \pi_t^1(V_t^{1|2}-V_t^1)}{(\delta^1(t)-V_t^1)(\delta^2(t)-V_t^2) - (V_t^{1|2}-V_t^1)(V_t^{2|1}-V_t^2)}
\end{aligned}$$

on the set $\tau_1<t<\tau_2$

$$\vartheta_t^1 = \frac{\pi_t^1(\delta^2(t)-V_t^2) - \pi_t^2(V_t^{2|1}-V_t^2)}{(\delta^1(t)-V_t^1)(\delta^2(t)-V_t^2)}, \quad \vartheta_t^2 = \frac{\pi_t^2}{\delta^2(t)-V_t^2}$$

on the set $\tau_2 < t < \tau_1$

$$\vartheta_t^1 = -\frac{\pi_t^1}{\delta^1(t) - V_t^1}, \quad \vartheta_t^2 = \frac{\pi_t^2(\delta^1(t) - V_t^1) - \pi_t^1(V_t^{1|2} - V_t^2)}{(\delta^1(t) - V_t^1)(\delta^2(t) - V_t^2)}$$

on the set $\tau_1 \vee \tau_2 < t$

$$\vartheta_t^1 = \frac{\pi_t^1}{\delta^1(t) - V_t^1}, \quad \vartheta_t^2 = \frac{\pi_t^2}{\delta^2(t) - V_t^2}\,.$$

As we saw above, for the case $A = h(\tau_1, \tau_2))$, one has a closed form for the coefficients π.

$$\begin{aligned} \pi_t^1 &= \big(h(t,\tau_2) - \psi_{0,1}(t,\tau_2)\big)H_t^2 + \big(\psi_{1,0}(t,t) - \psi_{0,0}(t)\big)(1 - H_t^2) \\ \pi_t^2 &= \big(h(\tau_1,t) - \psi_{1,0}(\tau_1,t)\big)H_t^1 + \big(\psi_{0,1}(t,t) - \psi_{0,0}(t)\big)(1 - H_t^1) \end{aligned}$$

3.2 Cox process modelling

We are now studying a financial market with null interest rate, and we work under the probability chosen by the market.

3.2.1 Independent Barriers

We now assume that n non negative processes $\lambda_i, i = 1, \dots, n$, $\mathbb{F}$-adapted are given and we denote $\Lambda_{i,t} = \int_0^t \lambda_{i,s} ds$. We assume the existence of n r.v. $U_i, i = 1, \cdots, n$ with uniform law, independent and independent of $\mathcal{F}_\infty$ and we define

$$\tau_i = \inf\{t \,:\, U_i \geq \exp(-\Lambda_{i,t})\}\,.$$

We introduce the following different filtrations
• $\mathbb{H}_i$ generated by $H_{i,t} = 1\!\!1_{\tau_i \leq t}$
• the filtration $\mathbb{G}$ defined as

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_{1,t} \vee \cdots \vee \mathcal{H}_{i,t} \vee \cdots \mathcal{H}_{n,t}$$

• the filtration $\mathbb{G}_i$ as $\mathcal{G}_{i,t} = \mathcal{F}_t \vee \mathcal{H}_{i,t}$
• $\mathbb{H}_{(-i)}$ the filtration

$$\mathcal{H}_{(-i),t} = \mathcal{H}_{1,t} \vee \cdots \vee \mathcal{H}_{i-1,t} \vee \mathcal{H}_{i+1,t} \cdots \mathcal{H}_{n,t}$$

Note the obvious inclusions

$$\mathbb{F} \subset \mathbb{G}_i \subset \mathbb{G}, \quad \mathbb{H}_{(-i)} \subset \mathbb{G} = \mathbb{G}_i \vee \mathbb{H}_{(-i)}$$

We note $\ell_i(t,T)$ the loss process

$$\ell_i(t,T) = \mathbb{E}(1\!\!1_{\tau_i \leq T}|\mathcal{G}_t) = \mathbb{P}(\tau_i \leq T|\mathcal{G}_t) = \mathbb{E}(H_{i,T}|\mathcal{G}_t)$$

and $\widetilde{D}_i(t,T) = \mathbb{E}(\exp(\Lambda_{i,t}\Lambda_{i,T})|\mathcal{F}_t)$ the predefault price if a DZC.

Lemma 3.2.1 *The following equalities holds*

$$\begin{aligned} \mathbb{P}(\tau_i \geq t_i, \forall i) &= \mathbb{E}(\exp - \sum_i \Lambda_{t_i,i}) && (3.2.1) \\ \mathbb{P}(\tau_i \geq t_i, \forall i|\mathcal{F}_t) &= \exp - \sum_i \Lambda_{t_i,i}, \ \forall t_i \leq t, && (3.2.2) \\ \mathbb{P}(\tau_i \geq t_i, \forall i|\mathcal{F}_t) &= \prod_i \mathbb{P}(\tau_i \geq t|\mathcal{F}_t), \ \forall t_i \leq t, \ \forall i && (3.2.3) \\ \mathbb{P}(\tau_i \geq t_i, \forall i|\mathcal{F}_t) &= \mathbb{E}(\exp - \sum_i \Lambda_{t_i,i}|\mathcal{F}_t), \ \forall t_i, && (3.2.4) \\ \mathbb{P}(\tau_i \geq t_i, \ \forall i|\mathcal{G}_t) &= \frac{\mathbb{P}(\tau_i \geq t_i, \ \forall i|\mathcal{F}_t)}{\mathbb{P}(\tau_i \geq t, \ \forall i|\mathcal{F}_t)} \quad \textit{on the set } \tau_i \geq t_i, \forall i && (3.2.5) \end{aligned}$$

PROOF: From the definition

$$\mathbb{P}(\tau_i \geq t_i, \forall i) = \mathbb{P}(\exp -\Lambda_{t_i,i} \geq U_i, \forall i) = \mathbb{E}(\exp -\sum_i \Lambda_{t_i,i})$$

where we have used that $\mathbb{P}(u_i \geq U_i) = u_i$ and $\mathbb{E}(\Psi(X,Y)) = \mathbb{E}(\psi(X))$ with $\psi(x) = \mathbb{E}(\Psi(x,Y))$ for independent r.v. X and Y.

In the same way,

$$\begin{aligned} \mathbb{P}(\tau_i \geq t_i, \forall i|\mathcal{F}_t) &= \mathbb{P}(\exp -\Lambda_{t_i,i} \geq U_i, \forall i|\mathcal{F}_t) \\ &= \exp -\sum_i \Lambda_{t_i,i} \end{aligned}$$

where we have used that $\mathbb{E}(\Psi(X,Y)|X)) = \psi(X)$ with $\psi(x) = \mathbb{E}(\Psi(x,Y)$ for independent r.v's X and Y, and that the $\Lambda_{t_i,i}$ are $\mathcal{F}_t$-measurable for $t_i \leq t$.

Lemma 3.2.2 *(a) Any bounded $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.*
(b) Any bounded $\mathbb{G}_i$-martingale is a $\mathbb{G}$-martingale

PROOF: (a) Using the caracterisation of conditional expectation, one has to check that

$$\mathbb{E}(\eta|\mathcal{F}_t) = \mathbb{E}(\eta|\mathcal{F}_\infty)$$

for any $\mathcal{G}_t$-measurable r.v. It suffices to prove the equality for

$$\eta = F_t h_1(t \wedge \tau_1) \cdots h_n(t \wedge \tau_n)$$

where $F_t \in \mathcal{F}_t$ and $h_i, i = 1, \cdots, n$ are bounded measurable functions. We can reduce attention to functions of the from $h_i(s) = 1\!\!1_{[0,a_i]}(s)$. If $a_i > t$, $h_i(t \wedge \tau_i) = 1$, so we can pay attention to the case where all the a_i's are smaller than t. The equality is now equivalent to

$$\mathbb{E}(\tau_i \leq a_i, \forall i|\mathcal{F}_t) = \mathbb{E}(\tau_i \leq a_i, \forall i|\mathcal{F}_\infty)$$

By definition

$$\mathbb{E}(\tau_i \leq a_i, \ \forall i|\mathcal{F}_t) = \mathbb{E}(\exp -\Lambda_{i,a_i} < U_i, \ \forall i|\mathcal{F}_t) = \Psi(\Lambda_{i,t}; i = 1, \cdots, n)$$

with $\Psi(u_i; i = 1, \cdots, n) = \prod(1 - u_i)$. The same computation leads to

$$\mathbb{E}(\tau_i \leq a_i, \forall i|\mathcal{F}_\infty) = \Psi(\Lambda_{i,a_i}, i = 1, \cdots, n)$$

(b) Using the same methodology, we are reduced to prove that for any bounded $\mathcal{G}_t$-measurable r.v. η,

$$\mathbb{E}(\eta|\mathcal{G}_{i,t}) = \mathbb{E}(\eta|\mathcal{G}_{(i,\infty)})$$

or even only that

$$\mathbb{E}(\eta_1\eta_2|\mathcal{G}_{i,t}) = \mathbb{E}(\eta_1\eta_2|\mathcal{G}_{(i,\infty)})$$

for $\eta_1 \in \mathcal{G}_{i,t}t$ and $\eta_2 \in \mathcal{H}_{(-i),t}$, that is

$$\mathbb{E}(\eta_2|\mathcal{G}_{i,t}) = \mathbb{E}(\eta_2|\mathcal{G}_{(i,\infty)})$$

To simplify, we assume that $i = 1$. Using the same elementary functions h as above, we have to prove that

$$\mathbb{E}(h_2(\tau_2 \wedge t) \cdots h_n(\tau_n \wedge a_n)|\mathcal{G}_{1,t}) = \mathbb{E}(h_2(\tau_2 \wedge t) \cdots h_n(\tau_n \wedge a_n)|\mathcal{G}_{(1,\infty)})$$

where $a_i < t$, that is

$$\mathbb{E}(1\!\!1_{\tau_2 \leq a_2} \cdots 1\!\!1_{\tau_n \leq a_n}|\mathcal{G}_{1,t}) = \mathbb{E}(1\!\!1_{\tau_2 \leq a_2} \cdots 1\!\!1_{\tau_n \leq a_n}|\mathcal{G}_{1,\infty})$$

Note that the vector $(U_2, \cdots, U_n)$ is independent from

$$\mathcal{G}_{1,\infty} = \mathcal{F}_\infty \vee \sigma(\tau_2) \vee \cdots \sigma(\tau_n) = \mathcal{F}_\infty \vee \sigma(U_2) \vee \cdots \sigma(U_n)$$

It follows that

$$\mathbb{E}(1\!\!1_{\tau_2 \le a_2} \cdots 1\!\!1_{\tau_n \le a_n} | \mathcal{G}_{1,\infty}) = \mathbb{E}(1\!\!1_{\exp -\Lambda_{2,a_2} \le U_2} \cdots 1\!\!1_{\exp -\Lambda_{n,a_n} \le U_n} | \mathcal{G}_{1,\infty}) = \prod_{i=2}^{n} (1 - \exp(-\Lambda_{i,a_i}))$$

Lemma 3.2.3 *The processes $M_{i,t} := H_{i,t} - \int_0^t (1 - H_{i,s}) \Lambda_{i,s} ds$ are $\mathbb{G}_i$-martingales and $\mathbb{G}$-martingales*

PROOF: We have shown that $M_{i,t} := H_{i,t} - \int_0^t (1 - H_{i,s}) \Lambda_{i,s} ds$ are $\mathbb{G}_i$-martingales. Now, from the lemma, $\mathbb{G}_i$ martingales are $\mathbb{G}$ martingales as well.

Lemma 3.2.4 *The processes $\ell_i(t, T)$ are $\mathbb{G}$-martingales and*

$$\ell_{i,t} = (1 - H_{i,t})(1 - \mathbb{E}(\exp(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t) + H_{i,t}$$

From the definition, the processes $\ell_i(t, T)$ are $\mathbb{G}$-martingales. From Lemma

$$\mathbb{P}(\tau_i \ge T, \ | \mathcal{G}_t) = 1\!\!1_{t < \tau_i} \frac{\mathbb{P}(\tau_i \ge T | \mathcal{F}_t)}{\mathbb{P}(\tau_i \ge t, \ | \mathcal{F}_t)} = (1 - H_{i,t}) \mathbb{E}(\exp -(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t)$$

hence $\ell_i(t, T) = H_{i,t} + (1 - H_{i,t}) \mathbb{E}(1 - \exp -(\Lambda_{i,t} - \Lambda_{i,T}) | \mathcal{F}_t)$

Chapter 4

Generalities and Immersion Property

From the end of the seventies, Jacod, Jeulin and Yor started a systematic study of the problem of enlargement of filtrations: namely, if $\mathbb{F}$ and $\mathbb{G}$ are two filtrations satisfying $\mathbb{F} \subset \mathbb{G}$, which $\mathbb{F}$-martingales M remain $\mathbb{G}$-semi-martingales and if it is the case, what is the semi-martingale decomposition of M in $\mathbb{G}$?

In the literature, there are mainly two kinds of enlargement of filtration:

• Initial enlargement of filtrations: in that case, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ where L is a r.v. (or, more generally $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}$ where $\widetilde{\mathcal{F}}$ is a σ-algebra, up to right-continuous regularization)

• Progressive enlargement of filtrations, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ with $\mathbb{H}$ the natural filtration of $H_t = 1\!\!1_{\{\tau \le t\}}$ where τ is a random time (or, more generally $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$ where $\widetilde{\mathbb{F}}$ is another filtration). In fact, very few studies are done in the case $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$. One exception is for $\widetilde{\mathcal{F}}_t = \sigma(J_t)$ where $J_t = \inf_{s \ge t} X_s$ when X is a three dimensional Bessel process (see [82]). See also the recent work of Kchia et al. [91].

Up to now, three lecture notes volumes have been dedicated to this question: Jeulin [82], Jeulin & Yor [86] and Mansuy & Yor [108]. There are also related chapters in the books of Protter [118] Dellacherie, Maisonneuve & Meyer [41], Jacod [75], Jeanblanc et al. [81] and Yor [135]. Chapter 20 in [41] (in French) contains a very general presentation of enlargement of filtration, based on fundamental results of the general theory of stochastic processes, developed in the previous chapters and books by the same authors. Chapter 12 in Yor [135] and the book Mansuy & Yor [108] are concerned with the case where all martingales in the reference filtration $\mathbb{F}$ are continuous. A survey, as well as many exercises, can be found in Mallein & Yor [106, Chapter 10]. More recently, Protter [118] and Jeanblanc, Yor & Chesney [81] have devoted a chapter of their books to the subject. The lecture by Song [123] is very complete and contains a deep general study. The reader can also find a summary and many examples of some classical problems in the lecture by Ouwehand [116].

Some first and important papers are Brémaud and Yor [30] (devoted to immersion case), Barlow [18] (for a specific study of honest times), Jacod [75, 76] and Jeulin & Yor [83]. A non-exhaustive list of references contains the papers of Ankirchner et al. [13], Nikeghbali [113] and Yoeurp [132].

Several thesis are devoted to this problem: Aksamit [2] Amendinger [9], Ankirchner [12], Bedini [20], Falala [28], Kchia [92], Kreher [97], Li [102], Song [122] and Wu [130].

Enlargement of filtration results are extensively used in finance to study two specific problems occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration, and change of prices dynamics, when an $\mathbb{F}$-martingale is no longer a $\mathbb{G}$-martingale. They are also a main stone for study of default risk.

An incomplete list of authors concerned with enlargement of filtration in finance for insider trading is: Ankirchner [13, 12], Amendinger [9, 10], Amendinger et al. [11], Baudoin [19], Corcuera et al. [37], Eyraud-Loisel [55], Florens & Fougère [59], Gasbarra et al. [65], Grorud & Pontier

[66], Hillairet [68], Hillairet and Jiao [70, 69], Imkeller [72], Karatzas & Pikovsky [90], Wu [130], Kohatsu-Higa & Øksendal [96], Zwierb [136]. The recent book of Hillairet and Jiao [71] is devoted to that subject.

A general study of arbitrages which can occur in an enlarged filtration is presented in Aksamit et al. [4, 5, 6], Acciao et al. [1], Fontana et al. [64]

Di Nunno et al. [46], Imkeller [73], Imkeller et al. [74], Kohatsu-Higa [94, 95] have introduced Malliavin calculus to study the insider trading problem. We shall not discuss this approach here.

Enlargement theory is also used to study asymmetric information, see, e.g. Föllmer et al. [62] and progressive enlargement is an important tool for the study of default in the reduced form approach by Bielecki et al. [24, 25, 26], Elliott et al.[53] and Kusuoka [100] among others.

Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. Our aim is to study some conditions which ensure that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales, and one can ask in a first step whether all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales. This last property is equivalent to $\mathbb{E}(\zeta|\mathcal{F}_t) = \mathbb{E}(\zeta|\mathcal{G}_t)$, for any t and $\zeta \in L^1(\mathcal{F}_\infty)$.

Let us study the simple example where $\mathbb{G} = \mathbb{F} \vee \sigma(\zeta)$ where $\zeta \in L^1(\mathcal{F}_\infty)$ and ζ is not $\mathcal{F}_0$-measurable. Obviously, $m_t := \mathbb{E}(\zeta|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. If m would be a $\mathbb{G}$-martingale, $\mathbb{E}(m_\infty|\mathcal{G}_t) = m_t$, hence $\zeta = m_t$ and, in particular $\zeta = \mathbb{E}(\zeta|\mathcal{F}_0)$ which is not the case.

In this chapter, we start with the case where $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales. In that case, there is a complete characterization so that this property holds. Then, we study a particular example: Brownian and Poisson bridges.

4.1 Immersion of Filtrations

4.1.1 Definition

The filtration $\mathbb{F}$ is said to be **immersed** in $\mathbb{G}$ if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (Tsirel'son [128], Émery [54]). This is also referred to as the $(\mathcal{H})$ hypothesis by Brémaud and Yor [30].

$(\mathcal{H})$ Every $\mathbb{F}$- martingale is a $\mathbb{G}$-martingale.

Proposition 4.1.1 *Hypothesis $(\mathcal{H})$ is equivalent to any of the following properties:*

$(\mathcal{H}1)$ $\forall\, t \geq 0$, *the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$, i.e.,* $\forall\, t \geq 0$, $\forall\, G_t \in L^2(\mathcal{G}_t), \forall\, F \in L^2(\mathcal{F}_\infty), \mathbb{E}(G_t\, F|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$.

$(\mathcal{H}2)$ $\forall\, t \geq 0$, $\forall\, G_t \in L^1(\mathcal{G}_t)$, $\mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$.

$(\mathcal{H}3)$ $\forall\, t \geq 0$, $\forall\, F \in L^1(\mathcal{F}_\infty)$, $\mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$.

In particular, $(\mathcal{H})$ holds if and only if every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale. Furthermore, if Hypothesis $(\mathcal{H})$ holds, then $\mathcal{G}_t \cap \mathcal{F}_\infty = \mathcal{F}_t$.

PROOF:
• $(\mathcal{H}) \Rightarrow (\mathcal{H}1)$. Let $F \in L^2(\mathcal{F}_\infty)$ and assume that hypothesis $(\mathcal{H})$ is satisfied. This implies that the martingale $F_t = \mathbb{E}(F|\mathcal{F}_t)$ is a $\mathbb{G}$-martingale such that $F_\infty = F$, hence $F_t = \mathbb{E}(F|\mathcal{G}_t)$. It follows that for any t and any $G_t \in L^2(\mathcal{G}_t)$:

$$\mathbb{E}(FG_t|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{G}_t)|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$$

which is exactly $(\mathcal{H}1)$.
• $(\mathcal{H}1) \Rightarrow (\mathcal{H}2)$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. Under $(\mathcal{H}1)$,

$$\mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(F|\mathcal{F}_t)\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}1}{=} \mathbb{E}(\mathbb{E}(FG_t|\mathcal{F}_t)) = \mathbb{E}(FG_t)$$

which is $(\mathcal{H}2)$.
• $(\mathcal{H}2) \Rightarrow (\mathcal{H}3)$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. If $(\mathcal{H}2)$ holds, then it is easy to prove that, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}2}{=} \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_\infty))\mathbb{E}(FG_t),$$

which implies $(\mathcal{H}3)$.
• Obviously $(\mathcal{H}3)$ implies $(\mathcal{H})$.
The proof of $\mathcal{G}_t \cap \mathcal{F}_\infty = \mathcal{F}_t$ is now simple. We have only to check that $\mathcal{G}_t \cap \mathcal{F}_\infty \subset \mathcal{F}_t$. Let $A \in \mathcal{G}_t \cap \mathcal{F}_\infty$. Then,

$$1\!\!1_A = \mathbb{E}(1\!\!1_A|\mathcal{F}_\infty) = \mathbb{E}(1\!\!1_A|\mathcal{F}_t)$$

which implies that $A \in \mathcal{F}_t$ □

In particular,if $\mathbb{F}$ is immersed in $\mathbb{G}$ and if W is an $\mathbb{F}$-Brownian motion, then it is a $\mathbb{G}$-martingale with bracket t, since such a bracket does not depend on the filtration. Hence, it is a $\mathbb{G}$-Brownian motion. It is important to note that $\int_0^t \psi_s dW_s$ is then a $\mathbb{G}$-local martingale, for a $\mathbb{G}$-adapted process ψ, satisfying some integrability conditions (see [85]).

A trivial (but useful) example for which $\mathbb{F}$ is immersed in $\mathbb{G}$ is $\mathbb{G} = \mathbb{F} \vee \widetilde{\mathbb{F}}$ where $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ are two filtrations such that $\mathcal{F}_\infty$ is independent of $\widetilde{\mathcal{F}}_\infty$.

Exercise 4.1.2 Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that W is an $\mathbb{F}$-Brownian motion. Prove that W is a $\mathbb{G}$-Brownian motion without using the bracket. ◁

Exercise 4.1.3 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{G}$, then, for any t, $\mathcal{F}_t = \mathcal{G}_t \cap \mathcal{F}_\infty$. ◁

Exercise 4.1.4 Show that, if $\tau \in \mathcal{F}_\infty$, immersion holds between $\mathbb{F}$ and $\mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is generated by $H_t = 1\!\!1_{\tau \le t}$ if and only if τ is an $\mathbb{F}$-stopping time. ◁

4.1.2 Change of probability

Of course, the notion of immersion depends strongly on the probability measure, and in particular, is not stable by change of probability. See Subsection 4.3.5 for a counter example. We now study in which setup the immersion property is preserved under change of probability.

Proposition 4.1.5 *Assume that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, and let $\mathbb{Q}$ be equivalent to $\mathbb{P}$, with $\mathbb{Q}|_{\mathcal{G}_t} = L_t\mathbb{P}|_{\mathcal{G}_t}$ where L is assumed to be $\mathbb{F}$-adapted. Then, $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{P}$ and $\mathbb{Q}$ are the same.*

PROOF: Let N be a $(\mathbb{F},\mathbb{Q})$-martingale, then $(N_tL_t, t \ge 0)$ is a $(\mathbb{F},\mathbb{P})$-martingale, and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, $(N_tL_t, t \ge 0)$ is a $(\mathbb{G},\mathbb{P})$-martingale which implies that N is a $(\mathbb{G},\mathbb{Q})$-martingale. We have for each $t \le s$

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_\mathbb{P}(L_t 1\!\!1_{\{\tau \le t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_\mathbb{P}(L_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_s) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_s),$$

where the last equality follows by another application of the Bayes formula. The assertion follows.□

Note that, if one defines a change of probability on $\mathbb{F}$ with a Radon-Nikodým density which is (as it must be) an $\mathbb{F}$-martingale L, one can not extend this change of probability to $\mathbb{G}$ by setting $\mathbb{Q}|_{\mathcal{G}_t} = L_t\mathbb{P}|_{\mathcal{G}_t}$, since, in general, L fails to be a $\mathbb{G}$-martingale.

We recall that, if X is a positive martingale, there exists N, a local martingale such that $X = \mathcal{E}(N)$. This process N is denoted $\mathcal{L}(M)$ and called the stochastic logarithm of X.

Proposition 4.1.6 *Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, and let $\mathbb{Q}$ be equivalent to $\mathbb{P}$ with $\mathbb{Q}|_{\mathcal{G}_t} = L_t\mathbb{P}|_{\mathcal{G}_t}$ where L is a $\mathbb{G}$-martingale and define $\ell_t := \mathbb{E}(L_t|\mathcal{F}_t)$. Assume that all $\mathbb{F}$-martingales are continuous and that L is continuous. Then, $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$ if and only if the $(\mathbb{G},\mathbb{P})$-local martingale*

$$\int_0^t \frac{dL_s}{L_s} - \int_0^t \frac{d\ell_s}{\ell_s} := \mathcal{L}(L)_t - \mathcal{L}(\ell)_t$$

is orthogonal to the set of all $(\mathbb{F},\mathbb{P})$-local martingales.

Proof: Every $(\mathbb{F},\mathbb{Q})$-martingale $M^{\mathbb{Q}}$ may be written as

$$M_t^{\mathbb{Q}} = M_t^{\mathbb{P}} - \int_0^t \frac{d\langle M^{\mathbb{P}}, \ell\rangle_s}{\ell_s}$$

where $M^{\mathbb{P}}$ is an $(\mathbb{F},\mathbb{P})$-martingale. By immersion hypothesis, $M^{\mathbb{P}}$ is a $(\mathbb{G},\mathbb{P})$-martingale and, from Girsanov's theorem, $M_t^{\mathbb{P}} = N_t^{\mathbb{Q}} + \int_0^t \frac{d\langle M^{\mathbb{P}}, L\rangle_s}{L_s}$ where $N^{\mathbb{Q}}$ is an $(\mathbb{G},\mathbb{Q})$-martingale. It follows that

$$\begin{aligned} M_t^{\mathbb{Q}} &= N_t^{\mathbb{Q}} + \int_0^t \frac{d\langle M^{\mathbb{P}}, L\rangle_s}{L_s} - \int_0^t \frac{d\langle M^{\mathbb{P}}, \ell\rangle_s}{\ell_s} \\ &= N_t^{\mathbb{Q}} + \int_0^t d\langle M^{\mathbb{P}}, \mathcal{L}(L) - \mathcal{L}(\ell)\rangle_s \,. \end{aligned}$$

Thus $M^{\mathbb{Q}}$ is a $(\mathbb{G},\mathbb{Q})$ martingale if and only if $\langle M^{\mathbb{P}}, \mathcal{L}(L) - \mathcal{L}(\ell)\rangle_s = 0$. □

Proposition 4.1.7 *Let $\mathbb{P}$ be a probability measure, and*

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t\mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t\mathbb{P}|_{\mathcal{F}_t} \,.$$

Then, immersion holds under $\mathbb{Q}$ if and only if:

$$\forall T, \forall X \geq 0,\ X \in \mathcal{F}_T, \forall t < T, \quad \frac{\mathbb{E}_{\mathbb{P}}(XL_T|\mathcal{G}_t)}{L_t} = \frac{\mathbb{E}_{\mathbb{P}}(X\ell_T|\mathcal{F}_t)}{\ell_t} \tag{4.1.1}$$

Proof: Note that, for $X \in \mathcal{F}_T$,

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \frac{1}{L_t}\mathbb{E}_{\mathbb{P}}(XL_T|\mathcal{G}_t) \quad , \quad \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = \frac{1}{\ell_t}\mathbb{E}_{\mathbb{P}}(X\ell_T|\mathcal{G}_t)$$

and that, from MCT, $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if, $\forall T, \forall X \in F_T, \forall t \leq T$, one has

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) \,.$$

□

Comment 4.1.8 The $(\mathcal{H})$ hypothesis (immersion hypothesis) was studied by Brémaud and Yor [30] and Mazziotto and Szpirglas [110], and in a financial setting by Kusuoka [100], Elliott et al. [53] and Jeanblanc and Rutkowski [78, 79].

Exercise 4.1.9 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$ and if $\mathbb{Q}$ is a probability equivalent to $\mathbb{P}$, then, any $(\mathbb{Q},\mathbb{F})$-semi-martingale is a $(\mathbb{Q},\mathbb{G})$-semi-martingale. Let

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t\mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t\mathbb{P}|_{\mathcal{F}_t} \,.$$

and X be a $(\mathbb{Q}, \mathbb{F})$ martingale. Assuming that $\mathbb{F}$ is a Brownian filtration and that L is continuous, prove that

$$X_t + \int_0^t \left(\frac{1}{\ell_s} d\langle X, \ell \rangle_s - \frac{1}{L_s} d\langle X, L \rangle_s \right)$$

is a $(\mathbb{G}, \mathbb{Q})$ martingale.
In a general case, prove that

$$X_t + \int_0^t \frac{L_{s-}}{L_s} \left(\frac{1}{\ell_{s^-}} d[X, \ell]_s - \frac{1}{L_{s^-}} d[X, L]_s \right)$$

is a $(\mathbb{G}, \mathbb{Q})$ martingale. See Jeulin and Yor [84]. $\triangleleft$

Exercise 4.1.10 Assume that any $\mathbb{F}$ martingale is a $\widetilde{\mathbb{F}}$ semi-martingale, with $\mathbb{F} \subset \widetilde{\mathbb{F}}$, and τ an $\widetilde{\mathbb{F}}$ stopping time. Prove that any $\mathbb{F}$ martingale is a $\mathbb{G}$ semi-martingale, where $\mathcal{G}_t = \sigma(t \wedge \tau)$ (regulariser) $\triangleleft$

Exercise 4.1.11 Assume that $\mathbb{F}$ is immersed in $\widetilde{\mathbb{F}}$ and τ is an $\widetilde{\mathbb{F}}$ stopping time. Prove that any $\mathbb{F}$ is immersed in $\mathbb{G}$ (regulariser) $\triangleleft$

Exercise 4.1.12 Assume that $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ where L is a random variable. Find under which conditions on L, immersion property holds. $\triangleleft$

Exercise 4.1.13 Construct an example where some $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, but not all $\mathbb{F}$ martingales are $\mathbb{G}$-martingales. $\triangleleft$

Exercise 4.1.14 Assume that $\mathbb{F} \subset \widetilde{\mathbb{G}}$ where $(\mathcal{H})$ holds for $\mathbb{F}$ and $\widetilde{\mathbb{G}}$.
a) Let τ be a $\widetilde{\mathbb{G}}$-stopping time. Prove that $(\mathcal{H})$ holds for $\mathbb{F}$ and $\mathbb{F}^\tau = \mathbb{F} \vee \mathbb{H}$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$.
b) Let $\mathbb{G}$ be such that $\mathbb{F} \subset \mathbb{G} \subset \widetilde{\mathbb{G}}$. Prove that $\mathbb{F}$ be immersed in $\mathbb{G}$. $\triangleleft$

Exercise 4.1.15 Assume that $\mathcal{F}_t^{(\tau)} = \mathcal{F}_t \vee \sigma(\tau)$ where τ is a positive random variable, and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$. Find under which conditions on τ the filtration $\mathbb{G}$ is immersed in $\mathbb{F}^{(\tau)}$. $\triangleleft$

4.2 Immersion in a Progressive Enlargement of Filtration

We now consider the case where a random time τ is given and where $\mathbb{G}$ is the progressively enlarged filtration. We introduce the $\mathbb{F}$-supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$.

4.2.1 Characterization of Immersion

Lemma 4.2.1 *In the progressive enlargement setting, $(\mathcal{H})$ holds between $\mathbb{F}$ and $\mathbb{G}$ if and only if one of the following equivalent conditions holds:*

$$\begin{array}{llrcl} \text{(i)} & \forall (t,s),\, s \le t, & \mathbb{P}(\tau \le s|\mathcal{F}_\infty) & = & \mathbb{P}(\tau \le s|\mathcal{F}_t), \\ \text{(ii)} & \forall t, & \mathbb{P}(\tau \le t|\mathcal{F}_\infty) & = & \mathbb{P}(\tau \le t|\mathcal{F}_t). \end{array} \tag{4.2.1}$$

PROOF: If (ii) holds, then (i) holds too. If (i) holds, $\mathcal{F}_\infty$ and $\sigma(t \wedge \tau)$ are conditionally independent given $\mathcal{F}_t$. The property follows. This result can be found in Dellacherie and Meyer [43]. $\square$

Note that, if $(\mathcal{H})$ holds, then (ii) implies that the process $\mathbb{P}(\tau \le t|\mathcal{F}_t)$ is increasing (See Section 8.7 for a study of that property).

Exercise 4.2.2 Prove that in a Cox model (see Section 2.3), immersion holds. ◁

Exercise 4.2.3 Prove that if $\mathbb{H}$ and $\mathbb{F}$ are immersed in $\mathbb{G}$, and if any $\mathbb{F}$ martingale is continuous, then τ and $\mathcal{F}_\infty$ are independent. ◁

Exercise 4.2.4 Assume that immersion property holds and let, for every u, $y_t(u)$ be an $\mathbb{F}$-martingale. Prove that, for $t > s$,

$$1\!\!1_{\tau\leq s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = 1\!\!1_{\tau\leq s}y_s(\tau)$$

◁

Exercise 4.2.5 Prove that $\mathbb{G}$ is immersed in $\mathbb{F}\vee\sigma(\tau)$ if and only if τ is constant. ◁

4.2.2 Norros's lemma

Proposition 4.2.6 *Assume that Z is continuous and $\lim_{t\to\infty} Z_t = 0$ and let Λ be the increasing predictable process such that $M_t = H_t - \Lambda_{t\wedge\tau}$ is a martingale. If $\mathbb{F}$ is immersed in $\mathbb{G}$, then, the r.v. Λ_∞ has unit exponential law and the variable Λ_τ is independent of $\mathcal{F}_\infty$.*

PROOF: Fix $z > 0$ and consider the process $X = (X_t, t \geq 0)$, defined by:

$$X_t = (1+z)^{H_t}\, e^{-z\Lambda_{t\wedge\tau}}$$

for all $t \geq 0$. Then, applying the integration by parts formula, we get:

$$dX_t = z\, e^{-z\Lambda_{t\wedge\tau}}\, dM_t\,. \tag{4.2.2}$$

Hence, by virtue of the assumption that $z > 0$, it follows from (4.2.2) that X is a $\mathbb{G}$-martingale, so that:

$$\mathbb{E}\big[(1+z)^{H_t}\, e^{-z\Lambda_{t\wedge\tau}}\,\big|\,\mathcal{G}_s\big] = (1+z)^{H_s}\, e^{-z\Lambda_{s\wedge\tau}} \tag{4.2.3}$$

holds for all $0 \leq s \leq t$. (Note that the martingale property of X follows also from Exercise **??** for $h \equiv 1$.) In view of the implied by $z > 0$ uniform integrability of X, we may let t go to infinity in (4.2.3). Setting s equal to zero in (4.2.3), we therefore obtain:

$$\mathbb{E}\big[(1+z)\, e^{-z\Lambda_\tau}\big] = 1\,.$$

This means that the Laplace transform of Λ_τ is the same as one of a standard exponential variable and thus proves the claim. Under immersion property, Z is decreasing and, under continuity assumption, $d\Lambda = dZ/Z$. Applying the change-of-variable formula, we get, for continuous Z:

$$e^{-z\Lambda_{t\wedge\tau}} = 1 + z\int_0^t e^{-z\Lambda_s}\,\frac{1\!\!1_{(\tau>s)}}{Z_s}\,dZ_s \tag{4.2.4}$$

for all $t \geq 0$ and any $z > 0$ fixed. Then, taking conditional expectations under $\mathcal{F}_t$ from both parts of expression (4.2.4) and applying Fubini's theorem, we obtain from the immersion of $\mathbb{F}$ in $\mathbb{G}$ that:

$$\begin{aligned}
\mathbb{E}\big[e^{-z\Lambda_{t\wedge\tau}}\,\big|\,\mathcal{F}_t\big] &= 1 + z\int_0^t \mathbb{E}\left[e^{-z\Lambda_s}\,\frac{1\!\!1_{(\tau>s)}}{Z_s}\,\Big|\,\mathcal{F}_t\right] dZ_s \\
&= 1 + z\int_0^t e^{-z\Lambda_s}\,\frac{\mathbb{P}(\tau > s\,|\,\mathcal{F}_t)}{Z_s}\,dZ_s \\
&= 1 + z\int_0^t e^{-z\Lambda_s}\,dZ_s
\end{aligned} \tag{4.2.5}$$

for all $t \geq 0$. Hence, using the fact that $\Lambda_t = -\ln Z_t$, we see from (4.2.5) that:

$$\mathbb{E}\big[e^{-z\Lambda_{t\wedge\tau}}\,|\,\mathcal{F}_t\big] = 1 + \frac{z}{1+z}\big((Z_t)^{1+z} - (Z_0)^{1+z}\big)$$

holds for all $t \geq 0$. Letting t go to infinity and using the assumption $Z_0 = 1$, as well as the fact that $Z_\infty = 0$ ($\mathbb{P}$-a.s.), we therefore obtain:

$$\mathbb{E}\big[e^{-z\Lambda_\tau} \,|\, \mathcal{F}_\infty\big] = \frac{1}{1+z}$$

that signifies the desired assertion. □

Comment 4.2.7 This result does not extend to the discontinuous case! As a trivial counter example, take $\mathbb{F}$ a Brownian filtration and τ be an $\mathbb{F}$ stopping time. Then, $\Lambda_t = 1\!\!1_{\tau \leq t}$.

Exercise 4.2.8 (A different proof of Norros' result) Suppose that

$$\mathbb{P}(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Prove, using the inverse of Γ that the random variable Γ_τ is independent of $\mathcal{F}_\infty$, with exponential law of parameter 1. ◁

4.2.3 $\mathbb{G}$-martingales versus $\mathbb{F}$ martingales

Proposition 4.2.9 *Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-adapted, integrable process given by the formula*

$$Y_t^{\mathbb{G}} = y_t 1\!\!1_{\tau > t} + y_t(\tau) 1\!\!1_{\tau \leq t}, \quad \forall\, t \in \mathbb{R}_+, \tag{4.2.6}$$

where:
(i) the projection of $Y^{\mathbb{G}}$ onto $\mathbb{F}$, which is defined by

$$Y_t^{\mathbb{F}} := \mathbb{E}(Y_t^{\mathbb{G}} | \mathcal{F}_t) = y_t\, \mathbb{P}(\tau > t | \mathcal{F}_t) + \mathbb{E}(y_t(\tau) 1\!\!1_{\tau \leq t} | \mathcal{F}_t),$$

is a $(\mathbb{P}, \mathbb{F})$-martingale,
(ii) for any fixed $u \in \mathbb{R}_+$, the process $(y_t(u),\, t \in [u, \infty))$ is a $(\mathbb{P}, \mathbb{F})$-martingale.
Then the process $Y^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-martingale.

Proof: Let us take $s < t$. Then

$$\begin{aligned}
\mathbb{E}(Y_t^{\mathbb{G}} | \mathcal{G}_s) &= \mathbb{E}(y_t 1\!\!1_{\tau > t} | \mathcal{G}_s) + \mathbb{E}(y_t(\tau) 1\!\!1_{s < \tau \leq t} | \mathcal{G}_s) + \mathbb{E}(y_t(\tau) 1\!\!1_{\tau \leq s} | \mathcal{G}_s) \\
&= 1\!\!1_{s < \tau} \frac{1}{Z_s} (\mathbb{E}(y_t Z_t | \mathcal{F}_s) + \mathbb{E}(y_t(\tau) 1\!\!1_{s < \tau \leq t} | \mathcal{F}_s)) + \mathbb{E}(y_t(\tau) 1\!\!1_{\tau \leq s} | \mathcal{G}_s)
\end{aligned}$$

On the one hand,

$$\mathbb{E}(y_t(\tau) 1\!\!1_{\tau \leq s} | \mathcal{G}_s) = 1\!\!1_{\tau \leq s} y_s(\tau) \tag{4.2.7}$$

Indeed, it suffices to prove the previous equality for $y_t(u) = h(u) X_t$ where X is an $\mathbb{F}$-martingale. In that case,

$$\mathbb{E}(X_t h(\tau) 1\!\!1_{\tau \leq s} | \mathcal{G}_s) = 1\!\!1_{\tau \leq s} h(\tau) \mathbb{E}_{\mathbb{P}}(X_t | \mathcal{G}_s) = 1\!\!1_{\tau \leq s} h(\tau) \mathbb{E}(X_t | \mathcal{F}_s) = 1\!\!1_{\tau \leq s} h(\tau) X_s = 1\!\!1_{\tau \leq s} y_s(\tau)$$

In the other hand, from (i)

$$\mathbb{E}(y_t Z_t + y_t(\tau) 1\!\!1_{\tau \leq t} | \mathcal{F}_s) = y_s Z_s + \mathbb{E}(y_s(\tau) 1\!\!1_{\tau \leq s} | \mathcal{F}_s)$$

It follows that

$$\mathbb{E}(Y_t^{\mathbb{G}} | \mathcal{G}_s) = 1\!\!1_{s < \tau} \frac{1}{Z_s} (y_s Z_s + \mathbb{E}((y_s(\tau) - y_t(\tau)) 1\!\!1_{\tau \leq s} | \mathcal{F}_s)) + 1\!\!1_{\tau \leq s} y_s(\tau)\,.$$

It remains to check that

$$\mathbb{E}((y_s(\tau) - y_t(\tau)) 1\!\!1_{\tau \leq s} | \mathcal{F}_s) = 0$$

which follows from

$$\mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{F}_s) = \mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s|\mathcal{F}_s) = \mathbb{E}(y_s(\tau)1\!\!1_{\tau\le s}|\mathcal{F}_s)$$

where we have used (4.2.7). □

Exercise 4.2.10 In a Cox model, for a continuous Λ, prove that τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function. ◁

Exercise 4.2.11 Prove that, if $\mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous and strictly decreasing, then there exists Θ independent of $\mathcal{F}_\infty$ such that $\tau = \inf\{t\,:\,\Lambda_t > \Theta\}$. ◁

Exercise 4.2.12 In a Cox model, write the Doob-Meyer and the multiplicative decomposition of Z. ◁

Exercise 4.2.13 Show how one can compute $\mathbb{P}(\tau > t|\mathcal{F}_t)$ when

$$\tau = \inf\{t\,:\,X_t > \Theta\}$$

where X is an $\mathbb{F}$-adapted process, not necessarily increasing, and Θ independent of $\mathcal{F}_\infty$. Does immersion property still holds? Same questions if Θ is not independent of $\mathcal{F}_\infty$. ◁

4.2.4 Martingale Representation Theorems

Theorem 4.2.14 *Suppose that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that any $\mathbb{F}$-martingale is continuous. Then the martingale $M^h_t = \mathbb{E}(h_\tau|\,\mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}|h_\tau| < \infty$, admits the following decomposition in the sum of a continuous martingale and a discontinuous martingale*

$$M^h_t := m^h_0 + \int_0^{t\wedge\tau} \frac{1}{Z_u}\,dm^h_u + \int_{]0,t\wedge\tau]} (h_u - M^h_{u-})\,dM_u, \tag{4.2.8}$$

where m^h is the continuous $\mathbb{F}$-martingale given by

$$m^h_t := -\mathbb{E}\Big(\int_0^\infty h_u\,dZ_u\,\Big|\,\mathcal{F}_t\Big)$$

and M is the discontinuous $\mathbb{G}$-martingale defined as $M_t = H_t - \Gamma_{t\wedge\tau}$, where $\Gamma = -\ln Z$.

Proof: We start by noting that

$$\begin{aligned} M^h_t &= \mathbb{E}(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{t\ge\tau\}}h_\tau - 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\,\mathbb{E}\Big(\int_t^\infty h_u\,dZ_u\,\Big|\,\mathcal{F}_t\Big) \\ &= 1\!\!1_{\{t\ge\tau\}}h_\tau + 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\Big(m^h_t + \int_0^t h_u\,dZ_u\Big). \end{aligned} \tag{4.2.9}$$

We will sketch two slightly different derivations of (4.2.8).

First derivation. Let the process J be given by the formula, for $t \in \mathbb{R}_+$,

$$J_t = e^{\Gamma_t}\Big(m^h_t + \int_0^t h_u\,dZ_u\Big).$$

Noting that Γ is a continuous increasing process and m^h is a continuous martingale, we deduce from the Itô integration by parts formula that

$$\begin{aligned} dJ_t &:= e^{\Gamma_t}\,dm^h_t - e^{\Gamma_t}h_t\,dF_t + \Big(m^h_t + \int_0^t h_u\,dZ_u\Big)e^{\Gamma_t}\,d\Gamma_t \\ &:= e^{\Gamma_t}\,dm^h_t + e^{\Gamma_t}h_t\,dZ_t + J_t\,d\Gamma_t. \end{aligned}$$

Therefore, from $dZ_t = -e^{-\Gamma_t} d\Gamma_t$,

$$dJ_t = e^{\Gamma_t}\, dm_t^h + (J_t - h_t)\, d\Gamma_t$$

or, in the integrated form,

$$J_t = M_0^h + \int_0^t e^{\Gamma_u}\, dm_u^h + \int_0^t (J_u - h_u)\, d\Gamma_u.$$

Note that $J_t = M_t^h = M_{t-}^h$ on the event $\{t < \tau\}$. Therefore, on the event $\{t < \tau\}$,

$$M_t^h = M_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u}\, dm_u^h + \int_0^{t\wedge\tau} (M_{u-}^h - h_u)\, d\Gamma_u.$$

From (4.2.9), the jump of M^h at time τ equals

$$h_\tau - J_\tau = h_\tau - M_{\tau-}^h = M_\tau^h - M_{\tau-}^h.$$

Equality (4.2.8) now easily follows.

Second derivation. Equality (4.2.9) can be re-written as follows

$$M_t^h = \int_0^t h_u\, dH_u + (1 - H_t) e^{\Gamma_t} \Big(m_t^h - \int_0^t h_u\, dF_u \Big).$$

Hence formula (4.2.8) can be obtained directly by the integration by parts formula. □

Corollary 4.2.15 *Suppose that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that $\mathbb{F}$ is a Brownian filtration generated by B. Then, any $\mathbb{G}$ martingale Y admits a representation as*

$$Y_t := Y_0 + \int_0^{t\wedge\tau} \varphi_u dB_u + \int_{]0,t\wedge\tau]} (Y_u - Y_u^{\mathbb{F}})\, dM_u, \qquad (4.2.10)$$

4.2.5 Stability under Change of Probability

In this section, we extend the results obtained in the Cox setting (see Section 2.3.4).

Case of the Brownian filtration

Let W be a Brownian motion under $\mathbb{P}$ and $\mathbb{F}$ its natural filtration. Since we work under immersion hypothesis, W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$. Our goal is to show that immersion is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures on $(\Omega, \mathcal{G})$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. In our set-up, Kusuoka's representation theorem 4.2.15 implies that there exist $\mathbb{G}$-predictable processes θ and $\zeta > -1$, such that the Radon-Nikodým density L of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$dL_t = L_{t-}\big(\theta_t\, dW_t + \zeta_t\, dM_t\big) \qquad (4.2.11)$$

with the initial value $L_0 = 1$. More explicitly, the process η equals

$$L_t = \mathcal{E}_t\left(\int_0^\cdot \theta_u\, dW_u\right) \mathcal{E}_t\left(\int_0^\cdot \zeta_u\, dM_u\right) = L_t^{(1)} L_t^{(2)}, \qquad (4.2.12)$$

where we write

$$L_t^{(1)} = \mathcal{E}_t\left(\int_0^\cdot \theta_u\, dW_u\right) = \exp\left(\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du\right),$$

and

$$L_t^{(2)} = \mathcal{E}_t\left(\int_0^\cdot \zeta_u\, dM_u\right) = \exp\left(\int_0^t \ln(1+\zeta_u)\, dH_u - \int_0^{t\wedge\tau} \zeta_u \gamma_u\, du\right). \qquad (4.2.13)$$

Proposition 4.2.16 *Assume that immersion holds under $\mathbb{P}$. Let $\mathbb{Q}$ be a probability measure locally equivalent to $\mathbb{P}$ with the associated Radon-Nikodým density process L given by formula (4.2.12) . If the process θ is $\mathbb{F}$-adapted then immersion is valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensity of τ under $\mathbb{Q}$ equals $\widehat{\gamma}_t = (1+\widetilde{\zeta}_t)\gamma_t$, where $\widetilde{\zeta}$ is the unique $\mathbb{F}$-predictable process such that the equality $\widetilde{\zeta}_t \mathbb{1}_{\{t\le\tau\}} = \zeta_t \mathbb{1}_{\{t\le\tau\}}$ holds for every $t \in \mathbb{R}_+$.*

Proof: Let $\mathbb{P}^*$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$, given by

$$d\mathbb{P}^* \,|\, _{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^\cdot \zeta_u \, dM_u \right) d\mathbb{P} \,|\, _{\mathcal{G}_t} = L_t^{(2)} \, d\mathbb{P} \,|\, _{\mathcal{G}_t}. \tag{4.2.14}$$

We claim that immersion holds under $\mathbb{P}^*$. From Girsanov's theorem, the process W follows a Brownian motion under $\mathbb{P}^*$ with respect to both $\mathbb{F}$ and $\mathbb{G}$. Moreover, from the predictable representation property of W under $\mathbb{P}^*$, we deduce that any $\mathbb{F}$-local martingale L under $\mathbb{P}^*$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbb{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u \, dW_u.$$

This shows that L is also a $\mathbb{G}$-local martingale, and thus immersion holds under $\mathbb{P}^*$. Since

$$d\mathbb{Q} \,|\, _{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^\cdot \theta_u \, dW_u \right) d\mathbb{P}^* \,|\, _{\mathcal{G}_t},$$

by virtue of Proposition 4.1.5, immersion is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that immersion holds under $\mathbb{Q}$ and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}} \gamma_u \zeta_u \, du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}} (1 + \widetilde{\zeta}_u) \gamma_u \, du$$

is a $\mathbb{Q}$-martingale. □

We claim that the equality $\mathbb{P}^* = \mathbb{P}$ holds on the filtration $\mathbb{F}$. Indeed, we have $d\mathbb{P}^* \,|\, _{\mathcal{F}_t} = \widetilde{L}_t \, d\mathbb{P} \,|\, _{\mathcal{F}_t}$, where we write $\widetilde{L}_t = \mathbb{E}_{\mathbb{P}}(L_t^{(2)} \,|\, \mathcal{F}_t)$, and

$$\mathbb{E}_{\mathbb{P}}(L_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}} \left(\mathcal{E}_t \left(\int_0^\cdot \zeta_u \, dM_u \right) \Big| \mathcal{F}_\infty \right) = 1, \quad \forall\, t \in \mathbb{R}_+, \tag{4.2.15}$$

where the first equality follows immersion.

To establish the second equality in (4.2.15), we first note that since the process M is stopped at τ, we may assume, without loss of generality, that $\zeta = \widetilde{\zeta}$ where the process $\widetilde{\zeta}$ is $\mathbb{F}$-predictable. Moreover,the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1 - \exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes ζ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (4.2.13)),

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big(\mathcal{E}_t\Big(\int_0^{\cdot}\widetilde{\zeta}_u\,dM_u\Big)\,\Big|\,\mathcal{F}_\infty\Big) &= \mathbb{E}_{\mathbb{P}}\Big(\big(1+1\!\!1_{\{t\geq\tau\}}\widetilde{\zeta}_\tau\big)\exp\Big(-\int_0^{t\wedge\tau}\widetilde{\zeta}_u\gamma_u\,du\Big)\,\Big|\,\mathcal{F}_\infty\Big)\\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty\big(1+1\!\!1_{\{t\geq u\}}\widetilde{\zeta}_u\big)\exp\Big(-\int_0^{t\wedge u}\widetilde{\zeta}_v\gamma_v\,dv\Big)\gamma_u e^{-\int_0^u\gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\Big)\\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_0^t(1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\,\Big|\,\mathcal{F}_\infty\Big)\\
&+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^\infty\gamma_u e^{-\int_0^u\gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\Big)\\
&= \int_0^t(1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\\
&+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\int_t^\infty\gamma_u e^{-\int_0^u\gamma_v\,dv}du\\
&= 1-\exp\Big(-\int_0^t(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\exp\Big(-\int_0^t\gamma_v\,dv\Big)=1,
\end{aligned}
$$

where the second last equality follows by an application of the chain rule.

Extension to orthogonal martingales

Equality (4.2.15) suggests that Proposition 4.2.16 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$. It is also a $\mathbb{G}$-local martingale, since we maintain the assumption that immersion holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$. We assume that the Radon-Nikodým density process L of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$dL_t = L_{t-}\big(\theta_t\,dN_t+\zeta_t\,dM_t\big) \tag{4.2.16}$$

for some $\mathbb{G}$-predictable processes θ and $\zeta>-1$ (the properties of the process θ depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, so that the product MN follows a $\mathbb{G}$-local martingale.

Proposition 4.2.17 *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbb{F}$, in the sense that any $\mathbb{F}$-local martingale L under $\mathbb{P}$ can be written as

$$L_t = L_0+\int_0^t\xi_u\,dN_u,\quad \forall\,t\in\mathbb{R}_+,$$

for some $\mathbb{F}$-predictable process ξ,
(c) $\mathbb{P}^*$ is a probability measure on $(\Omega,\mathcal{G})$ such that (4.2.14) holds.
Then we have:
(i) immersion is valid under $\mathbb{P}^*$,
(ii) if the process θ is $\mathbb{F}$-adapted then immersion is valid under $\mathbb{Q}$.

The proof of the proposition hinges on the following simple lemma.

Lemma 4.2.18 *Under the assumptions of Proposition 4.2.17, we have:*
(i) N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$,
(ii) N has the predictable representation property for $\mathbb{F}$-local martingales under $\mathbb{P}^*$.

PROOF: In view of (c), we have $d\mathbb{P}^*\,|\,\mathcal{G}_t = L^{(2)}_t\, d\mathbb{P}\,|\,\mathcal{G}_t$, where the density process $L^{(2)}$ is given by (4.2.13), so that $dL^{(2)}_t = L^{(2)}_{t-}\zeta_t\, dM_t$. From the assumed orthogonality of N and M, it follows that N and $L^{(2)}$ are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, and thus $NL^{(2)}$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{L}$ by setting $\widetilde{L}_t = \mathbb{E}_{\mathbb{P}}(L^{(2)}_t\,|\,\mathcal{F}_t)$. Then manifestly $d\mathbb{P}^*\,|\,\mathcal{F}_t = \widetilde{L}_t\, d\mathbb{P}\,|\,\mathcal{F}_t$, and thus in order to show that any $\mathbb{F}$-local martingale under $\mathbb{P}^*$ follows an $\mathbb{F}$-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t \in \mathbb{R}_+$, so that $\mathbb{P}^* = \mathbb{P}$ on $\mathbb{F}$. To this end, we note that

$$\mathbb{E}_{\mathbb{P}}(L^{(2)}_t\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u\, dM_u\right)\Big|\,\mathcal{F}_\infty\right) = 1, \quad \forall\, t \in \mathbb{R}_+,$$

where the first equality follows from immersion property, and the second one can established similarly as the second equality in (4.2.15).

We are in a position to prove (ii). Let L be an $\mathbb{F}$-local martingale under $\mathbb{P}^*$. Then it follows also an $\mathbb{F}$-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\mathbb{P}^*$. This shows that N has the predictable representation property with respect to $\mathbb{F}$ under $\mathbb{P}^*$. □

We now proceed to the proof of Proposition 4.2.17.

Proof of Proposition 4.2.17. We shall argue along the similar lines as in the proof of Proposition 4.2.16. To prove (i), note that by part (ii) in Lemma 4.2.18 we know that any $\mathbb{F}$-local martingale under $\mathbb{P}^*$ admits the integral representation with respect to N. But, by part (i) in Lemma 4.2.18, N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$. We conclude that L is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$, and thus the immersion is valid under $\mathbb{P}^*$. Assertion (ii) now follows from Proposition 4.1.5. □

Remark 4.2.19 It should be stressed that Proposition 4.2.17 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modeling dependent default times through an equivalent change of a probability measure (see Kusuoka [100]).

Example 4.2.20 Kusuoka [100] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M^i_t = H^i_t - \int_0^{t\wedge\tau_i} \gamma_i(u)\, du$, where $H^i_t = \mathbb{1}_{\{t\geq\tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|\,\mathcal{G}_t = L_t\, d\mathbb{P}\,|\,\mathcal{G}_t$, where $L_t = L^{(1)}_t L^{(2)}_t$ and, for $i = 1, 2$ and every $t \in \mathbb{R}_+$,

$$L^{(i)}_t = 1 + \int_0^t L^{(i)}_{u-}\zeta^{(i)}_u\, dM^i_u = \mathcal{E}_t\left(\int_0^{\cdot} \zeta^{(i)}_u\, dM^i_u\right)$$

for some $\mathbb{G}$-predictable processes $\zeta^{(i)}$, $i = 1, 2$, where $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Manifestly, the immersion holds under $\mathbb{P}$. Moreover, in view of Proposition 4.2.17, it is still valid under the equivalent probability measure $\mathbb{P}^*$ given by

$$d\mathbb{P}^*\,|\,\mathcal{G}_t = \mathcal{E}_t\left(\int_0^{\cdot} \zeta^{(2)}_u\, dM^2_u\right) d\mathbb{P}\,|\,\mathcal{G}_t.$$

It is clear that $\mathbb{P}^* = \mathbb{P}$ on $\mathbb{F}$, since

$$\mathbb{E}_{\mathbb{P}}(L^{(2)}_t\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta^{(2)}_u\, dM^2_u\right)\Big|\,\mathcal{H}^1_t\right) = 1, \quad \forall\, t \in \mathbb{R}_+.$$

However, immersion is not necessarily valid under $\mathbb{Q}$ if the process $\zeta^{(1)}$ fails to be $\mathbb{F}$-adapted. In Kusuoka's counter-example, the process $\zeta^{(1)}$ was chosen to be explicitly dependent on both random times, and it was shown that immersion does not hold under $\mathbb{Q}$. For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne *et al.* [36].

4.3 Successive Enlargements

4.3.1 Immersion

Proposition 4.3.1 *Let $\tau_1 < \tau_2$ a.s., $\mathbb{H}^i$ be the filtration generated by the default process $H^i_t = 1\!\!1_{\tau_i \le t}$, and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$. Then, the two following assertions are equivalent:*
(i) $\mathbb{F}$ is immersed in $\mathbb{G}$
(ii) $\mathbb{F}$ is immersed in $\mathbb{F} \vee \mathbb{H}^1$ and $\mathbb{F} \vee \mathbb{H}^1$ is immersed in $\mathbb{G}$.

PROOF: (this result was obtained by Ehlers and Schönbucher [47], we give here a slightly different proof.) The only fact to check is that if $\mathbb{F}$ is immersed in $\mathbb{G}$, then $\mathbb{F} \vee \mathbb{H}^1$ is immersed in $\mathbb{G}$, or that

$$\mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}^1_t) = \mathbb{P}(\tau_2 > t | \mathcal{F}_\infty \vee \mathcal{H}^1_\infty)$$

This is equivalent to, for any h, and any $A_\infty \in \mathcal{F}_\infty$

$$\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}^1_t))$$

We split this equality in two parts. The first equality

$$\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}^1_t))$$

is obvious since $1\!\!1_{\tau_1 > t} 1\!\!1_{\tau_2 > t} = 1\!\!1_{\tau_1 > t}$ and $1\!\!1_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}^1_t) = 1\!\!1_{\tau_1 > t}$.

Since $\mathbb{F}$ is immersed in $\mathbb{G}$, one has $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t)$ and it follows (WHY?) that $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}^1_t)$, therefore

$$\begin{aligned}
\mathbb{E}(A_\infty h(\tau_1) 1\!\!1_{\tau_2 > t \ge \tau_1}) &= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{G}_t) h(\tau_1) 1\!\!1_{\tau_2 > t \ge \tau_1}) \\
&= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}^1_t) h(\tau_1) 1\!\!1_{\tau_2 > t \ge \tau_1}) \\
&= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}^1_t) \mathbb{E}(h(\tau_1) 1\!\!1_{\tau_2 > t \ge \tau_1} | \mathcal{F}_t \vee \mathcal{H}^1_t)) \\
&= \mathbb{E}(A_\infty \mathbb{E}(h(\tau_1) 1\!\!1_{\tau_2 > t \ge \tau_1} | \mathcal{F}_t \vee \mathcal{H}^1_t))
\end{aligned}$$

Exercise 4.3.2 Prove that $\mathbb{H}^i, i = 1, 2$ are immersed in $\mathbb{H}^1 \vee \mathbb{H}^2$ if and only if $\tau_i, i = 1, 2$ are independent. ◁

4.3.2 Various immersion

Lemma 4.3.3 *Let $\mathbb{F}$ be generated by a Brownian motion. Assume that $\mathbb{F}$ is immersed in $\mathbb{G}^1 = \mathbb{F} \vee \mathbb{H}^1$ and in $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$ and that there exists an $\mathbb{F}$ predictable increasing process Λ^1 such that $M^1_t = H^1_t - \Lambda^1_{t \wedge \tau_1}$ is a $\mathbb{G}$ martingale. Then $\mathbb{G}^1$ is immersed in $\mathbb{G}$*

PROOF: Any $\mathbb{G}^1$ martingale admits a decomposition as $Y_t = y + \int_0^t y_s dW_s + \int_0^t \widetilde{y}_s dM^1_s$. The result follows since W and M^1 are assumed to be $\mathbb{G}$ martingales □

This result extends to the case of an arbitrary filtration $\mathbb{F}$. Indeed, for $X \in b\mathcal{F}_T$ and h bounded Borel function

$$\mathbb{E}(X h(\tau_1) | \mathcal{G}^1_t) = h(\tau_1) 1\!\!1_{\tau_1 \le t} \mathbb{E}(X | \mathcal{G}^1_t) + 1\!\!1_{t < \tau_1} \frac{1}{Z^1_t} \mathbb{E}(X \int_t^\infty h(u) dF_u | \mathcal{F}_t)$$

can be written as a sum of stochastic integrals wrt M^1 and to some $\mathbb{F}$ martingales (note that, from immersion $\mathbb{E}(X | \mathcal{G}^1_t) = \mathbb{E}(X | \mathcal{F}_t)$.

4.3.3 Norros' lemma

Lemma 4.3.4 *Norros Lemma.*
Let $\tau_i, i = 1, \cdots, n$ be n finite-valued random times and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \cdots \vee \mathcal{H}_t^n$. Assume that

(i) $P(\tau_i = \tau_j) = 0, \forall i \neq j$

(ii) there exists continuous increasing processes Λ^i such that $M_t^i = H_t^i - \Lambda_{t\wedge\tau_i}^i$ are $\mathbb{G}$-martingales

then, the r.v's $\Lambda_{\tau_i}^i$ are independent with exponential law.

Proof: For any $\mu_i > -1$, the processes $L_t^i = (1+\mu_i)^{H_t^i} e^{-\mu_i \Lambda_{t\wedge\tau_i}^i}$, solution of

$$dL_t^i = L_{t-}^i \mu_i dM_t^i$$

are uniformly integrable martingales. Moreover, these martingales have no common jumps, and are orthogonal. Hence $E(\prod_i (1+\mu_i) e^{-\mu_i \Lambda_{\tau_i}^i}) = 1$, which implies

$$E(\prod_i e^{-\mu_i \Lambda_{\tau_i}^i}) = \prod_i (1+\mu_i)^{-1}$$

hence the independence property. □

Application: Let us study the particular case of Poisson process. Let τ_1 and τ_2 are the two first jumps of a Poisson process, we have

$$G(t,s) = \begin{cases} e^{-\lambda t} \text{ for } s < t \\ e^{-\lambda s}(1 + \lambda(s-t)) \text{ for } s > t \end{cases}$$

with partial derivatives

$$\partial_1 G(t,s) = \begin{cases} -\lambda e^{-\lambda t} \text{ for } t > s \\ -\lambda e^{-\lambda s} \text{ for } s > t \end{cases}, \quad \partial_2 G(t,s) = \begin{cases} 0 \text{ for } t > s \\ -\lambda^2 e^{-\lambda s}(s-t) \text{ for } s > t \end{cases}$$

and

$$h(t,s) = \begin{cases} 1 \text{ for } t > s \\ \frac{t}{s} \text{ for } s > t \end{cases}, \quad \partial_1 h(t,s) = \begin{cases} 0 \text{ for } t > s \\ \frac{1}{s} \text{ for } s > t \end{cases}$$

$$k(t,s) = \begin{cases} 0 \text{ for } t > s \\ 1 - e^{-\lambda(s-t)} \text{ for } s > t \end{cases}, \quad \partial_2 k(t,s) = \begin{cases} 0 \text{ for } t > s \\ \lambda e^{-\lambda(s-t)} \text{ for } s > t \end{cases}$$

Then, one obtains $\Lambda_{\tau_1} = \tau_1$ et $\Lambda_{\tau_2} = \tau_2 - \tau_1$

4.3.4 Several Defaults in a Cox model

Proposition 4.3.5 *Let $\tau_i := \inf\{t : \Lambda_t^i \geq \Theta_i\}$, where the Θ_i's are independent from $\mathbb{F}$ and Λ^i's are $\mathbb{F}$ adapted increasing processes. Let $\mathbb{H}^i$ be the natural filtration of H^i, where $H_t^i = \mathbb{1}_{\tau_i \leq t}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$ be the full observation filtration. Then $\mathbb{F}$ is immersed in $\mathbb{G}$.*

Proof: Observe that, $\mathbb{G} \subset \mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$ and that, from the independence hypothesis, obviously $\mathbb{F}$ is immersed in $\mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$. □

Corollary 4.3.6 *In the case where Θ^i are independent, $\mathbb{G}^i := \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$ and the $\mathbb{G}^i$ intensity of τ_i is the $\mathbb{G}$ intensity. The filtration $\mathbb{F}^i := \mathbb{F} \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$ and the $\mathbb{F}^i$ intensity of τ_i is the $(\mathbb{F}, \mathbb{G})$ intensity.*

PROOF: $\mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^i$ is immersed in $\mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$. □

It is important to note that in the case of Proposition 4.3.5, the $(\mathbb{F}, \mathbb{G}^i)$ intensity of τ_i is not equal to its $(\mathbb{F}, \mathbb{G})$ intensity. In other terms, $\mathbb{G}^i$ is not immersed in $\mathbb{G}$ in that general setting.

We can extend the characterization of Cox model with immersion property as follows. We keep the notation of the previous Proposition.

Proposition 4.3.7 *We assume that $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. If, for any $i = 0, \dots, n$, $\mathbb{G}^i$ is immersed in $\mathbb{G}$ and if there exists $\mathbb{F}$-adapted processes Λ^i such that $M^i_t := H^i_t - \Lambda^i_{t \wedge \tau_i}$ are $\mathbb{G}^i$ martingales, then, there exist independent random variables Θ^i, independent from $\mathbb{F}$ such that $\tau_i = \inf\{t : \Lambda^i_t \geq \Theta^i\}$.*

PROOF: The fact that $\Theta^i := \Lambda_{\tau_i}$ are independent follows from Norros'lemma. The Θ^i are independent from $\mathbb{F}$ from the single default case. Note that our hypothesis implies that M^i are $\mathbb{F} \vee \mathbb{H}^i$ martingales and $\mathbb{G}$ martingales and that, from Corollary 4.3.6, $\mathbb{F} \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$. □

4.3.5 Kusuoka counter example

Kusuoka [100] presents a counter-example of the stability of $\mathcal{H}$ hypothesis under a change of probability, based on two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and admitting a density w.r.t. Lebesgue's measure. The process $M^1_t = H^1_t - \int_0^{t \wedge \tau_1} \lambda_1(u)\, du$, where $H^1_t = 1\!\!1_{\{t \geq \tau_1\}}$ and λ_i is the deterministic intensity function of τ_i under $\mathbb{P}$, is a $(\mathbb{P}, \mathbb{H}^i)$ and a $(\mathbb{P}, \mathbb{G})$-martingale, where $\mathbb{G} =$
$hh^1 \vee \mathbb{H}^2$. (Recall that $\lambda_i(s)ds = \frac{\mathbb{P}(\tau_i \in ds)}{\mathbb{P}(\tau_i > s)}$). Manifestly, immersion hypothesis holds under $\mathbb{P}$ between $\mathbb{H}^1$ and $\mathbb{G}$. Let us set $d\mathbb{Q}\,|_{\mathcal{G}_t} = L_t\, d\mathbb{P}\,|_{\mathcal{G}_t}$, where

$$L_t = 1 + \int_0^t L_{u-} \kappa_u \, dM^1_u$$

for some $\mathbb{G}$-predictable process κ satisfying $\kappa > 1$ (WHY?). We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Let

$$\begin{aligned} \widehat{M}^1_t &= H^1_t - \int_0^{t \wedge \tau_1} \widehat{\lambda}_1(u)\, du \\ \widetilde{M}^1_t &= H^1_t - \int_0^{t \wedge \tau_1} \lambda_1(u)(1 + \kappa_u)\, du \end{aligned}$$

where $\widehat{\lambda}(u)du = \mathbb{Q}(\tau_1 \in du)/\mathbb{Q}(\tau_1 > u)$ is deterministic. It is easy to see that, under $\mathbb{Q}$, the process $\widehat{M}^1$ is a $(\mathbb{Q}, \mathbb{H}^1)$-martingale and $\widetilde{M}^1$ is a $(\mathbb{Q}, \mathbb{G})$ martingale. The process $\widehat{M}^1$ is not a $(\mathbb{Q}, \mathbb{G})$-martingale (WHY?), hence, immersion does not hold under $\mathbb{Q}$.

Exercise 4.3.8 Compute $\mathbb{Q}(\tau_1 > t | \mathcal{H}^2_t)$. ◁

4.3.6 Ordered times

Assume that $\tau_i, i = 1, \dots, n$ are n random times. Let $\sigma_i, i = 1, \dots, n$ be the sequence of ordered random times and $\mathbb{G}^{(k)} = \mathbb{F} \vee \mathbb{H}^{(1)} \cdots \vee \mathbb{H}^{(k)}$ where $\mathbb{H}^{(i)} = (\mathcal{H}^{(i)}_t = \sigma(t \wedge \sigma_i), t \geq 0)$. The $\mathbb{G}^{(k)}$-intensity of σ_k is the positive $\mathbb{G}^{(k)}$-adapted process λ^k such that $(M^{(k)}_t := 1\!\!1_{\{\sigma_k \leq t\}} - \int_0^t \lambda^k_s ds, t \geq 0)$ is a $\mathbb{G}^{(k)}$-martingale. The $\mathbb{G}^{(k)}$-martingale $M^{(k)}$ is stopped at σ_k and the $\mathbb{G}^{(k)}$-intensity of σ_k satisfies $\lambda^k_t = 0$ on $\{t \geq \sigma_k\}$. The following lemma shows the $\mathbb{G}^{(k)}$-intensity of σ_k coincides with its $\mathbb{G}^{(n)}$-intensity.

Lemma 4.3.9 *For any k, a $\mathbb{G}^{(k)}$-martingale stopped at σ_k is a $\mathbb{G}^{(n)}$-martingale.*

PROOF: We prove that any $\mathbb{G}^{(1)}$-martingale stopped at σ_1 is a $\mathbb{G}^{(2)}$-martingale. The result will follow. Let X be a $\mathbb{G}^{(1)}$-martingale stopped at σ_1, i.e. $X_t = X_{t\wedge\sigma_1}$ for any t. For $s < t$,

$$\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(2)}] = 1\!\!1_{\{\sigma_2\leq s\}}X_{\sigma_1} + 1\!\!1_{\{s<\sigma_2\}}\frac{\mathbb{E}[X_{t\wedge\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]}{\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)})}$$

It remains to note that

$$\mathbb{E}[X_{t\wedge\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{s<\sigma_1\}}\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] + 1\!\!1_{\{\sigma_1\leq s\}}\mathbb{E}[X_{\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]\,.$$

Since $\sigma_2 > s$ on $\{\sigma_1 > s\}$, we obtain $1\!\!1_{\{s<\sigma_1\}}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}) = 1\!\!1_{\{s<\sigma_1\}}$. The martingale property of X yields to

$$1\!\!1_{\{s<\sigma_1\}}\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{s<\sigma_1\}}X_{s\wedge\sigma_1}$$

It is obvious that

$$1\!\!1_{\{\sigma_1\leq s\}}\mathbb{E}[X_{\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{\sigma_1\leq s\}}X_{\sigma_1}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}).$$

The result follows. □

The following is a familiar result in the literature.

Proposition 4.3.10 *Assume that the $\mathbb{G}^{(k)}$-intensity λ^k of σ_k exists for all $k\in\Theta$. Then the intensity of the loss process $\sum_{k=1}^n 1\!\!1_{\sigma_k\leq t}$ is the sum of the intensities of σ_k, i.e.*

$$\lambda^L = \sum_{k=1}^n \lambda^k,\ a.s.. \tag{4.3.1}$$

PROOF: Since $(1\!\!1_{\{\sigma_k\leq t\}} - \int_0^t \lambda_s^k ds, t\geq 0)$ is a $\mathbb{G}^{(k)}$-martingale stopped at σ_k, it is a $\mathbb{G}^{(n)}$-martingale. We have by taking the sum that $(L_t - \int_0^t\sum_{k=1}^n \lambda_s^k ds, t\geq 0)$ is a $\mathbb{G}^{(n)}$-martingale. So $\lambda_t^L = \sum_{k=1}^n \lambda_t^k$ for all $t\geq 0$. □

Chapter 5

Bridges and utility maximization

The first applications of enlargement of filtration in Finance concerns an insider who has, at time 0, some information about the value of the asset's price at some date in the future.

5.1 The Brownian Bridge

Rather than studying ab initio the general problem of initial enlargement, we discuss an interesting example. Let us start with a BM $(B_t, t \geq 0)$ and its natural filtration $\mathbb{F}^B$. Define a new filtration as $\mathcal{F}_t^{(B_1)} = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(\mathbb{E}(B_1|\mathcal{F}_t^{(B_1)}), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t, t \geq 0)$ is not a $\mathbb{G}$-martingale. However, $(B_t, t \geq 0)$ is a $\mathbb{F}^{(B_1)}$-semi-martingale, as follows from the next proposition 5.1.2.

Before giving this proposition, we recall some facts on Brownian bridge.

The **Brownian bridge** $(b_t, 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t, t \leq 1 | B_1 = 0)$. Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1, t \leq 1)$ and the random variable B_1 are independent. Hence $(b_t, 0 \leq t \leq 1) \stackrel{\text{law}}{=} (B_t - tB_1, \, 0 \leq t \leq 1)$. The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t), s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.

We can represent the Brownian bridge between 0 and y during the time interval $[0, 1]$ as

$$(B_t - tB_1 + ty; \; t \leq 1) \, .$$

More generally, the Brownian bridge between x and y during the time interval $[0, T]$ may be expressed as

$$\left(x + B_t - \frac{t}{T} B_T + \frac{t}{T}(y - x); \; t \leq T \right),$$

where $(B_t; t \leq T)$ is a standard BM starting from 0.

Exercise 5.1.1 a) Prove that the Riemann integral $\int_0^{t \wedge 1} \frac{B_1 - B_s}{1-s} ds$ is absolutely convergent.
b) Prove that, for $0 \leq s < t \leq 1$, $\mathbb{E}(B_t - B_s | B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s)$ ◁

5.1.1 Decomposition of the BM in the enlarged filtration $\mathbb{F}^{(B_1)}$

Proposition 5.1.2 *Let $\mathcal{F}_t^{(B_1)} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. The process*

$$\beta_t := B_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is an $\mathbb{F}^{(B_1)}$-martingale, and an $\mathbb{F}^{(B_1)}$ Brownian motion. In other words,

$$B_t = \beta_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is the decomposition of B as an $\mathbb{F}^{(B_1)}$-semi-martingale.

PROOF: Note that the definition of $\mathbb{F}^{(B_1)}$ is done to satisfy the right-continuity assumption. We shall note, as a short cut, $\mathcal{F}_t^{(B_1)} := \mathcal{F}_t \vee \sigma(B_1) = \mathcal{F}_t \vee \sigma(B_1 - B_t)$. Then, since $\mathcal{F}_s$ is independent of $(B_{s+h} - B_s, h \geq 0)$, one has, for $s < t$:

$$\mathbb{E}(B_t - B_s | \mathcal{F}_s^{(B_1)}) = \mathbb{E}(B_t - B_s | B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s)\,.$$

For $s < t < 1$,

$$\begin{aligned}
\mathbb{E}(\int_s^t \frac{B_1 - B_u}{1-u} du | \mathcal{F}_s^{(B_1)}) &= \int_s^t \frac{1}{1-u} \mathbb{E}(B_1 - B_u | B_1 - B_s)\, du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \mathbb{E}(B_u - B_s | B_1 - B_s)\right) du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \frac{u-s}{1-s}(B_1 - B_s)\right) du \\
&= \frac{1}{1-s}(B_1 - B_s) \int_s^t du = \frac{t-s}{1-s}(B_1 - B_s)
\end{aligned}$$

It follows that

$$\mathbb{E}(\beta_t - \beta_s | \mathcal{F}_s^{(B_1)}) = 0$$

hence, β is an $\mathbb{F}^{(B_1)}$-martingale (and an $\mathbb{F}^{(B_1)}$-Brownian motion). □

It follows that if M is an $\mathbb{F}$-local martingale such that $\int_0^1 \frac{1}{\sqrt{1-s}} d|\langle M, B\rangle|_s$ is finite, then

$$\widehat{M}_t = M_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} d\langle M, B\rangle_s$$

is a $\mathbb{F}^{(B_1)}$-local martingale.

Comment 5.1.3 The singularity of $\frac{B_1 - B_t}{1-t}$ at $t = 1$, i.e., the fact that $\frac{B_1 - B_t}{1-t}$ is not square integrable between 0 and 1 prevents a Girsanov measure change transforming the $(\mathbb{P}, \mathbb{F}^{(B_1)})$ semi-martingale B into a $(\mathbb{Q}, \mathbb{F}^{(B_1)})$ martingale.

Comment 5.1.4 We obtain that the standard Brownian bridge b is a solution of the following stochastic equation (take care about the change of notation)

$$\left\{ \begin{array}{lll} db_t &=& -\dfrac{b_t}{1-t}\, dt + dW_t\,;\, 0 \leq t < 1 \\ \\ b_0 &=& 0\,. \end{array} \right.$$

The solution of the above equation is $b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s$ which is a Gaussian process with zero mean and covariance $s(1-t), s \leq t$.

Exercise 5.1.5 Using the notation of Proposition 5.1.2, prove that B_1 and β are independent. Check that the projection of β on $\mathbb{F}^B$ is equal to B.
◁

Exercise 5.1.6 Consider the SDE

$$\left\{\begin{array}{rcl} dX_t & = & -\dfrac{X_t}{1-t}\,dt + dW_t\,;\, 0 \le t < 1 \\ X_0 & = & 0 \end{array}\right.$$

1. Prove that
$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}\,;\, 0 \le t < 1\,.$$
2. Prove that $(X_t, t \ge 0)$ is a Gaussian process. Compute its expectation and its covariance.
3. Prove that $\lim_{t\to 1} X_t = 0$.

◁

Exercise 5.1.7 (See Jeulin and Yor [85]) Let $X_t = \int_0^t \varphi_s dB_s$ where φ is predictable such that $\int_0^t \varphi_s^2 ds < \infty$. Prove that the following assertions are equivalent

1. X is an $\mathbb{F}^{(B_1)}$-semimartingale with decomposition
$$X_t = \int_0^t \varphi_s d\beta_s + \int_0^{t\wedge 1} \frac{B_1 - B_s}{t-s}\varphi_s ds$$
2. $\int_0^1 |\varphi_s| \frac{|B_1 - B_s|}{1-s} ds < \infty$
3. $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}} ds < \infty$

◁

5.2 Poisson Bridge

Let N be a Poisson process with constant intensity λ, $\mathcal{F}_t^N = \sigma(N_s, s \le t)$ its natural filtration and $T > 0$ a fixed time. The process $M_t = N_t - \lambda t$ is a martingale. Let $\mathcal{G}_t^* = \sigma(N_s, s \le t; N_T)$ be the natural filtration of N enlarged with the terminal value N_T of the process N.

Proposition 5.2.1 *Assume that $\lambda = 1$. The process*

$$\eta_t = M_t - \int_0^{t\wedge T} \frac{M_T - M_s}{T-s} ds,$$

is a $\mathbb{G}^*$-martingale with predictable bracket, for $t \le T$,

$$\Lambda_t = \int_0^t \frac{N_T - N_s}{T-s} ds\,.$$

PROOF: For $0 < s < t < T$,

$$\mathbb{E}(N_t - N_s|\mathcal{G}_s^*) = \mathbb{E}(N_t - N_s|N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)$$

where the last equality follows from the fact that, if X and Y are independent with Poisson laws with parameters μ and ν respectively, then

$$\mathbb{P}(X = k|X+Y = n) = \frac{n!}{k!(n-k)!}\alpha^k(1-\alpha)^{n-k}$$

where $\alpha = \dfrac{\mu}{\mu+\nu}$. Hence,

$$\begin{aligned}
\mathbb{E}\left(\int_s^t du \frac{N_T - N_u}{T-u} | \mathcal{G}_s^*\right) &= \int_s^t \frac{du}{T-u} \left(N_T - N_s - \mathbb{E}(N_u - N_s | \mathcal{G}_s^*)\right) \\
&= \int_s^t \frac{du}{T-u} \left(N_T - N_s - \frac{u-s}{T-s}(N_T - N_s)\right) \\
&= \int_s^t \frac{du}{T-s}(N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)\,.
\end{aligned}$$

Therefore,

$$\mathbb{E}(N_t - N_s - \int_s^t \frac{N_T - N_u}{T-u} du | \mathcal{G}_s^*) = \frac{t-s}{T-s}(N_T - N_s) - \frac{t-s}{T-s}(N_T - N_s) = 0$$

and the result follows.

□

Comment 5.2.2 Poisson bridges are studied in Jeulin and Yor [85]. This kind of enlargement of filtration is used for modelling insider trading in Elliott and Jeanblanc [52], Grorud and Pontier [66] and Kohatsu-Higa and Øksendal [96].

Exercise 5.2.3 Prove that, for any enlargement of filtration the compensated martingale M remains a semi-martingale. ◁

Exercise 5.2.4 Prove that any $\mathbb{F}^N$-martingale is a $\mathbb{G}^*$-semimartingale. ◁

Exercise 5.2.5 Prove that

$$\eta_t = N_t - \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds - (t-T)^+,$$

Prove that

$$\langle \eta \rangle_t = \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds + (t-T)^+\,.$$

Therefore, $(\eta_t, t \leq T)$ is a compensated $\mathbb{G}^*$-Poisson process, time-changed by $\int_0^t \frac{N_T - N_s}{T-s} ds$, i.e., $\eta_t = \widetilde{M}(\int_0^t \frac{N_T - N_s}{T-s} ds)$ where $(\widetilde{M}(t), t \geq 0)$ is a compensated Poisson process. ◁

Exercise 5.2.6 A process X fulfills the **harness property** if

$$\mathbb{E}\left(\frac{X_t - X_s}{t-s} \Big| \mathcal{F}_{s_0], [T}\right) = \frac{X_T - X_{s_0}}{T - s_0}$$

for $s_0 \leq s < t \leq T$ where $\mathcal{F}_{s_0], [T} = \sigma(X_u, u \leq s_0, u \geq T)$. Prove that a process with the harness property satisfies

$$\mathbb{E}\left(X_t \Big| \mathcal{F}_{s], [T}\right) = \frac{T-t}{T-s} X_s + \frac{t-s}{T-s} X_T\,,$$

and conversely. Prove that, if X satisfies the harness property, then, for any fixed T,

$$M_t^T = X_t - \int_0^t du \frac{X_T - X_u}{T-u},\ t < T$$

is an $\mathcal{F}_{t], [T}$-martingale and conversely. See [3M] for more comments. ◁

5.3 Insider trading

In this section, we study a simple case of insider trading. We assume, that, in a BLack and Scholes model, an insider knows, at time 0 the value of the price at time 1. If the maturity of the market is 1, there are obviously arbitrage opportunity. We show how this insider can increase his wealth if the market terminates before date 1. We then study the same problem in a Poisson case.

5.3.1 Brownian Bridge

Let

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where μ and σ are constants, be the price of a risky asset. Assume that the riskless asset has an constant interest rate r.

The wealth of an agent holding ϑ^0 shares of the savings account and ϑ shares of the underlying risky process is $X_t = \vartheta_t^0 e^{rt} + \vartheta_t S_t$. The self financing condition is that

$$dX_t = \vartheta_t^0 de^{rt} + \vartheta_t dS_t = rX_t dt + \vartheta_t(dS_t - rS_t dt)$$

With the change of notation $\pi_t = \vartheta_t S_t / X_t$ (so that the wealth remains non negative) one has

$$dX_t = rX_t dt + \pi_t \sigma X_t(dW_t + \theta dt),\ X_0 = x$$

Here ϑ is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that

$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma)ds + \int_0^T \sigma\pi_s dW_s$$

Then, assuming that the local martingale represented by the stochastic integral is in fact a martingale,

$$\mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + \int_0^T \mathbb{E}\left(r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma\right) ds$$

The portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and

$$\sup \mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$

Note that, if the coefficients r, μ and σ are $\mathbb{F}$-adapted, the same computation leads to

$$\sup \mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + \int_0^T \mathbb{E}\left(r_t + \frac{1}{2}\theta_t^2\right) dt$$

where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$.

We come back to the case of constant coefficients. We now enlarge the filtration with S_1 (or equivalently, with B_1. In the enlarged filtration, setting, for $t < 1$, $\alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma\alpha_t)dt + \sigma d\beta_t)\,,$$

where β is defined in Proposition 5.1.2 and the dynamics of the wealth are

$$dX_t = rX_t dt + \pi_t\sigma X_t(d\beta_t + \widetilde{\theta}_t dt),\ X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu - r}{\sigma} + \alpha_t = \frac{\mu - r}{\sigma} + \frac{B_1 - B_s}{1-s}$. Assuming again that the stochastic integral which appears is a martingale, the portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.
Then, for $T < 1$,

$$\begin{aligned}
\ln(X_T^{\pi,x,*}) &= \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2)ds + \int_0^T \sigma \pi_s d\beta_s \\
\mathbb{E}(\ln(X_T^{\pi,x,*})) &= \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + \mathbb{E}(\alpha_s^2) + 2\theta\mathbb{E}(\alpha_s))ds = \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T \mathbb{E}(\alpha_s^2)ds
\end{aligned}$$

where we have used the fact that $\mathbb{E}(\alpha_t) = 0$ (if the coefficients r, μ and σ are $\mathbb{F}$ adapted, α is orthogonal to $\mathcal{F}_t$, hence $\mathbb{E}(\alpha_t \theta_t) = 0$). Let

$$\begin{aligned}
V^{\mathbb{F}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{F} \text{ adapted} \\
V^{\mathbb{G}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{G} \text{ adapted}
\end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}\mathbb{E}\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1-T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there does not exist an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \leq 1)$ is a $\mathbb{G}$-martingale. However, for any $\epsilon \in]0,1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\zeta_t}{\sigma} L_t d\beta_t,\ t \leq 1 - \epsilon, \quad L_0 = 1\,,$$

such that, setting $d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}$, the process $(e^{-rt}S_t, t \leq 1-\epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity, which is effective at time 1.

It is important to mention, that in both cases, the wealth of the investor is $X_t e^{-rt} = x + \int_0^t \pi_s d(S_s e^{-rs})$. The insider has a larger class of portfolio, and in order to give a meaning to the stochastic integral for processes π which are not adapted with respect to the semi-martingale S, one has to give the decomposition of this semi-martingale in the larger filtration.

Exercise 5.3.1 Prove carefully that there does not exist any emm in the enlarged filtration. Make precise the arbitrage opportunity. ◁

5.3.2 Poisson Bridge

We suppose that the interest rate is null and that the risky asset has dynamics

$$dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \phi dM_t\right)$$

where M is the compensated martingale of a standard Poisson process. Let $(X_t, t \geq 0)$ be the wealth of an un-informed agent whose portfolio is described by (π_t), the proportion of wealth invested in the asset S at time t. Then

$$dX_t = \pi_t X_{t-}(\mu dt + \sigma dW_t + \phi dM_t) \tag{5.3.1}$$

Then,

$$X_t = x \exp\left(\int_0^t \pi_s(\mu - \phi\lambda)ds + \int_0^t \sigma\pi_s dW_s + \frac{1}{2}\int_0^t \sigma^2\pi_s^2 ds + \int_0^t \ln(1 + \pi_s\phi)dN_s\right)$$

Assuming that the stochastic integrals with respect to W and M are martingales,

$$\mathbb{E}[\ln(X_T)] = \ln(x) + \int_0^T \mathbb{E}(\mu\pi_s - \frac{1}{2}\sigma^2\pi_s^2 + \lambda(\ln(1 + \phi\pi_s) - \phi\pi_s)ds\,.$$

Our aim is to solve

$$V(x) = \sup_\pi \mathbb{E}\left(\ln(X_T^{x,\pi})\right)$$

We can then maximize the quantity under the integral sign for each s and ω.
The maximum attainable wealth for the uninformed agent is obtained using the constant strategy $\tilde\pi$ for which

$$\tilde\pi\mu + \lambda[\ln(1+\tilde\pi\phi) - \tilde\pi\phi] - \frac{1}{2}\tilde\pi^2\sigma^2 = \sup_\pi[\pi\mu + \lambda[\ln(1+\pi\phi) - \pi\phi] - \frac{1}{2}\pi^2\sigma^2] .$$

Hence

$$\tilde\pi = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda - \sigma^2)^2 + 4\sigma^2\phi\mu}\right) .$$

The quantity under the square root is $(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\lambda$ and is non-negative.
The sign to be used depends on the sign of quantities related to the parameters. The optimal $\widetilde\pi$ is the only one such that $1+\phi\widetilde\pi > 0$. Solving the equation (5.3.1), it can be proved that the optimal wealth is $\widetilde X_t = x(\widetilde L_t)^{-1}$ where $d\widetilde L_t = \widetilde L_{t-}(-\sigma\widetilde\pi dW_t + (\frac{1}{1+\phi\widetilde\pi} - 1)dM_t)$ is a Radon Nikodym density of an equivalent martingale measure. In this incomplete market, we thus obtain the utility equivalent martingale measure defined by Davis [38] and duality approach (See Kramkov and Schachermayer).

We assume now that the informed agent knows N_T from time 0. Therefore, his wealth evolves according to the dynamics

$$dX_t^* = \pi_t X_{t-}^*[(\mu + \phi(\Lambda_t - \lambda)]dt + \sigma dW_t + \phi dM_t^*]$$

where Λ is given in Proposition 5.2.1. Exactly the same computations as above can be carried out. In fact these only require changing μ to $(\mu + \phi(\Lambda_t - \lambda))$ and the intensity of the jumps from λ to Λ_t. The optimal portfolio π^* is now such that $\mu - \lambda\phi + \phi\Lambda_s[\frac{1}{1+\pi^*\phi}] - \pi^*\sigma^2 = 0$ and is given by

$$\pi_s^* = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\Lambda_s}\right) ,$$

The optimal wealth is $X_t^* = x(L_t^*)^{-1}$ where

$$dL_t^* = L_{t-}^*(-\sigma\pi_s^* dW_t + (\frac{1}{1+\phi\pi_s^*} - 1)dM_t^*) .$$

Whereas the optimal portfolio of the uninformed agent is constant, the optimal portfolio of the informed agent is time-varying and has a jump as soon as a jump occurs for the prices.
The informed agent must maximize at each (s,ω) the quantity

$$\pi\mu + \Lambda_s(\omega)\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2 .$$

Consequently,

$$\sup_\pi \pi\mu + \Lambda_s\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2 \geq \tilde\pi\mu + \Lambda_s\ln(1+\tilde\pi\phi) - \lambda\tilde\pi\phi - \frac{1}{2}\tilde\pi^2\sigma^2$$

Now, $\mathbb{E}[\Lambda_s] = \lambda$, so

$$\begin{aligned}
\sup_\pi \mathbb{E}(\ln X_T^*) &= \ln x + \sup_\pi \int_0^T \mathbb{E}(\pi\mu + \Lambda_s\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2)ds \\
&\geq \ln x + \int_0^T \tilde\pi(\mu + \lambda\ln(1+\tilde\pi\phi) - \lambda\tilde\pi\phi - \frac{1}{2}\tilde\pi^2\sigma^2)ds = \mathbb{E}(\ln \widetilde X_T)
\end{aligned}$$

Therefore, the maximum expected wealth for the informed agent is greater than that of the uninformed agent. This is obvious because the informed agent can use any strategy available to the uninformed agent.

Exercise 5.3.2 Solve the same problem for power utility function. ◁

5.4 Drift Information in a Progressive Enlargement in a Brownian Setting

We assume in this part that W is a Brownian motion with natural filtration $\mathbb{F}$ and $\mathbb{G}$ is a filtration larger than $\mathbb{F}$ and that there there exists an integrable $\mathbb{G}$-adapted process $\mu^{\mathbb{G}}$ such that $dW_t = dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt$ where $W^{\mathbb{G}}$ is a $\mathbb{G}$-BM. We study a financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset $S^0 \equiv 1$ are traded is arbitrage free. More precisely, we assume w.l.g. that S is a $(\mathbb{P}, \mathbb{F})$ (local) martingale, $dS_t = S_t \sigma_t dW_t$.

Let X be the wealth process associated with a $\mathbb{G}$ predictable strategy

$$dX_t = \vartheta_t dS_t = \vartheta_t S_t dW_t = \pi_t X_t dW_t = \pi_t X_t (dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt)$$

(where the change of parameter is due to the fact that we restrict our attention to positive wealth) so that

$$X_t = x \exp \left(\int_0^t \pi_s dW_s^{\mathbb{G}} - \frac{1}{2} \int_0^t \pi_s^2 ds + \int_0^t \pi_s \mu_s^{\mathbb{G}} ds \right)$$

Our goal is to solve $\sup(\mathbb{E}(\ln X_T), \pi \in \mathbb{F})$ and $\sup(\mathbb{E}(\ln X_T), \pi \in \mathbb{G})$. It is then easy to see that the optimal π is $\pi^* = \mu^{\mathbb{G}}$ and that

$$\ln X_t^* = \ln x + \int_0^t \pi_s^* dW_s^{\mathbb{G}} + \frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds$$

so that, assuming that $\mathbb{E} \left(\int_0^t (\mu_s^{\mathbb{G}})^2 ds \right) < \infty$, one finds

$$\sup_{\pi \in \mathbb{F}} \mathbb{E}(\ln X_T) = \ln x < \sup_{\pi \in \mathbb{G}} \mathbb{E}(\ln X_T) = \ln x + \mathbb{E} \left(\frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds \right)$$

Note that, if $L_t := \mathcal{E}(-\mu^{\mathbb{G}} W^{\mathbb{G}})_t$ is a martingale, NFLVR holds, and if L is a local martingales, the No arbitrages of the first kind holds (see Section 1.5.1).

Chapter 6

Initial Enlargement

In this chapter, we study initial enlargement, where the enlarged filtration is $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ for a random variable L. The goal is to give conditions such that $\mathbb{F}$-martingales remain $\mathbb{F}^{(L)}$-semi-martingales and, in that case, to give the $\mathbb{F}^{(L)}$-semi-martingale decomposition of the $\mathbb{F}$-martingales.

More precisely, in order to satisfy the usual hypotheses, define

$$\mathcal{F}_t^{(L)} = \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(L)\} \ .$$

In this chapter, we study the $(\mathcal{H}')$ hypothesis between $\mathbb{F}$ and $\mathbb{F}^{(L)}$

- We give Jacod's criteria
- We present Yor's methodology in a Brownian setting
- We give some examples

6.1 General Facts

We denote $\mathcal{P}(\mathbb{F})$ the predictable σ-algebra (see Subsection 1.1.3).

Proposition 6.1.1 *One has*
(i) Every $\mathcal{F}_t^{(L)}$-measurable r.v. Y_t is of the form $Y_t(\omega) = y_t(\omega, L(\omega))$ for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable random variable $(y_t(\omega, u), t \geq 0)$.
(ii) Every $\mathbb{F}^{(L)}$-predictable process Y is of the form $Y_t(\omega) = y_t(\omega, L(\omega))$ where $(t, \omega, u) \mapsto y_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable function.

PROOF: The proof of part (i) is based on the fact that $\mathcal{F}_t^{(L)}$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega) f\big(L(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f bounded Borel function on $\mathbb{R}$.

(ii) It suffices to notice that processes of the form $X_t := x_t f(L)$, $t \geq 0$, where x is $\mathbb{F}$-predictable and f is a bounded Borel function on $\mathbb{R}$, generate the $\mathcal{F}^{(L)}$-predictable σ-field. □

We shall now simply write $y_t(L)$ for $y_t(\omega, L(\omega))$.

6.2 An absolute continuity result

We recall that there exists a family of regular conditional distributions $P_t(\omega, dx)$ such that $P_t(\cdot, A)$ is a version of $\mathbb{P}(L \in A|\mathcal{F}_t)$.

6.2.1 Jacod's criterion

In what follows, for $y(u)$ a family of martingales and X a martingale, we shall write $\langle y(L), X\rangle$ for $\langle y(u), X\rangle|_{u=L}$.

Proposition 6.2.1 (Jacod's Criterion.) *Suppose that, for each $t \geq 0$, $P_t(\omega, dx) \ll \nu(dx)$ where ν is the law of L. Then, every $\mathbb{F}$-semi-martingale $(X_t, t < T)$ is also an $\mathbb{F}^{(L)}$-semi-martingale.*
If X is an $\mathbb{F}$-martingale, the process

$$\widetilde{X}_t = X_t - \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s^-}(L)}\,, t < T$$

is an $\mathbb{F}^{(L)}$-martingale. In other words, the decomposition of the $\mathbb{F}^{(L)}$-semi-martingale X is

$$X_t = \widetilde{X}_t + \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s^-}(L)}\,.$$

PROOF: In a first step, we show that, for any θ, the process $p(\theta) = (p_t(\theta), t \geq 0)$ is an $\mathbb{F}$-martingale. One has to show that, for a bounded r.v. $\zeta_s \in \mathcal{F}_s$ and $s < t$

$$\mathbb{E}(p_t(\theta)\zeta_s) = \mathbb{E}(p_s(\theta)\zeta_s)$$

This follows from

$$\mathbb{E}(\mathbb{E}(\mathbb{1}_{\tau>\theta}|\mathcal{F}_t)\zeta_s) = \mathbb{E}(\mathbb{E}(\mathbb{1}_{\tau>\theta}|\mathcal{F}_s)\zeta_s)\,.$$

In a second step, we assume that $\mathbb{F}$-martingales are continuous (condition **(C)**), and that X and p are square integrable. In that case, $\langle p.(L), X\rangle$ exists. Let F_s be a bounded $\mathcal{F}_s$-measurable random variable and $h : \mathbb{R}^+ \to \mathbb{R}$, be a bounded Borel function. Then the variable $F_s h(L)$ is $\mathcal{F}_s^{(L)}$-measurable and if a decomposition of the form $X_t = \widetilde{X}_t + \int_0^t dK_u(L)$ holds, the martingale property of $\widetilde{X}$ should imply that $\mathbb{E}\left(F_s h(L)\left(\widetilde{X}_t - \widetilde{X}_s\right)\right) = 0$, hence

$$\mathbb{E}\left(F_s h(L)\left(X_t - X_s\right)\right) = \mathbb{E}\left(F_s h(L)\int_s^t dK_u(L)\right).$$

We can write:

$$\begin{aligned}
\mathbb{E}\left(F_s h(L)\left(X_t - X_s\right)\right) &= \mathbb{E}\left(F_s\left(X_t - X_s\right)\int_{-\infty}^{\infty} h(\theta)p_t(\theta)\nu(d\theta)\right) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s\left(X_t p_t(\theta) - X_s p_s(\theta)\right)\right)\nu(d\theta) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s \int_s^t d\left\langle X, p(\theta)\right\rangle_v\right)\nu(d\theta)
\end{aligned}$$

where the first equality comes from a conditioning w.r.t. $\mathcal{F}_t$, the second from the martingale property of $p(\theta)$, and the third from the fact that both X and $p(\theta)$ are square-integrable $\mathbb{F}$-martingales. Moreover:

$$\begin{aligned}
\mathbb{E}\left(F_s h(L)\int_s^t dK_v(L)\right) &= \mathbb{E}\left(F_s \int_{\mathbb{R}} h(\theta)\int_s^t dK_v(\theta)p_t(\theta)\nu(d\theta)\right) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s\int_s^t p_v(\theta)dK_v(\theta)\right)\nu(d\theta)
\end{aligned}$$

where the first equality comes from the definition of p, and the second from the martingale property of $p(\theta)$. By equalization of these two quantities, we obtain that it is necessary to have $dK_u(\theta) = d\left\langle X, p(\theta)\right\rangle_u / p_u(\theta)$. □

For the general case, we refer the reader to Jacod. If $P_t(\omega, dx) = p_t(\omega, x)\nu(dx)$, the process $p(L)$ does not vanish on $[0, T[$.

Remark 6.2.2 Of course, if for each $t \leq T$, $P_t(\omega, dx) \ll \nu(dx)$ where ν is the law of L, every $\mathbb{F}$-semi-martingale $(X_t, t \leq T)$ is also an $\mathbb{F}^{(L)}$-semi-martingale. In many cases, the hypothesis is not satisfied for T (see the Brownian bridge case).

Definition 6.2.3 *We shall say that L satisfies absolutely continuity hypothesis if*

$$\mathbb{P}(L \in dx|\mathcal{F}_t) = P_t(dx) = p_t(x)\nu(dx)$$

The stability of absolutely continuity hypothesis under a change of probability is rather obvious.

Corollary 6.2.4 *Let Z be a random variable taking only a countable number of values. Then every $\mathbb{F}$ semimartingale is a $\mathbb{F}^{(Z)}$ semimartingale.*

Proof: If we note

$$\eta\left(dx\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(Z = x_k\right) \delta_{x_k}\left(dx\right),$$

where $\delta_{x_k}(dx)$ is the Dirac measure at x_k, the law of Z, then $P_t(\omega, dx)$ is absolutely continuous with respect to η with Radon-Nikodym density:

$$\sum_{k=1}^{\infty} \frac{\mathbb{P}\left(Z = x_k|\mathcal{F}_t\right)}{\mathbb{P}\left(Z = x_k\right)} 1\!\!1_{x=x_k}.$$

Now the result follows from Jacod's theorem.□

Exercise 6.2.5 Assume that $\mathbb{F}$ is a Brownian filtration. Then, check directly that $\mathbb{E}(\int_0^t \frac{d\langle p_\cdot(L), X\rangle_s}{p_{s-}(L)}|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. ◁

6.2.2 Regularity Conditions

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of the family of densities, which is important in order to avoid difficulties with negligible sets. Fortunately, results of Jacod [75] or Stricker and Yor [126] help us to solve this technical problem. See also [9] for a detailed discussion. We emphazise that these results are the most important part of enlargement of filtration theory.

Jacod ([75], Lemme 1.8 and 1.10) establishes the existence of a universal càdlàg version of the density process in the following sense: there exists a non negative function $p_t(\omega, \theta)$ càdlàg in t, optional w.r.t. the filtration $\widehat{\mathbb{F}}$ on $\widehat{\Omega} = \Omega \times \mathbb{R}^+$, generated by $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$, such that

- for any θ, $p_\cdot(\theta)$ is an $\mathbb{F}$-martingale; moreover, denoting $\zeta^\theta = \inf\{t : p_{t-}(\theta) = 0\} \wedge T$, then $p_\cdot(\theta) > 0$, and $p_-(\theta) > 0$ on $[0, \zeta^\theta)$, and $p_\cdot(\theta) = 0$ on $[\zeta^\theta, T)$. Furthermore, $\zeta^L = T$, $\mathbb{P}$-a.s.
- For any bounded family $(Y_t(\omega, \theta), t \geq 0)$ measurable w.r.t. $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$, the $\mathbb{F}$-predictable projection of the process $Y_t(\omega, L(\omega))$ is the process $Y_t^{(p)} = p_{t-}(\theta)Y_t(\theta)\nu(d\theta)$.
- If $(\omega, t, \theta) \to)Y_t(\omega, \theta)$ is non negative and $\mathcal{O}(\mathbb{F})) \otimes \mathcal{B}$ measurable, the optional projection of the process $Y(L)$ is $\int Y_t(\theta)p_t(\theta)\nu(d\theta)$.
- Let m be a local $\mathbb{F}$-martingale. There exists a predictable increasing process A and a $\widehat{\mathbb{F}}$-predictable function k such that

$$\langle p(\theta), m\rangle_t = \int_0^t k_s(\theta)p_{s-}(\theta)dA_s\,.$$

  If m is locally square integrable, one can chose $A = \langle m\rangle$.

Exercise 6.2.6 Prove that if there exists a probability $\mathbb{Q}^*$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}^*$, the r.v. L is independent of $\mathcal{F}_\infty$, then every $(\mathbb{P}, \mathbb{F})$-semi-martingale X is also an $(\mathbb{P}, \mathbb{F}^{(L)})$-semi-martingale. See Chapter 9 for a more exhaustive study. ◁

6.3 Yor's Method

We follow here Yor [135] (see also [134]). We assume that $\mathbb{F}$ is a Brownian filtration. For a bounded Borel function f, let $(\lambda_t(f), t \geq 0)$ be the continuous version of the martingale $(\mathbb{E}(f(L)|\mathcal{F}_t), t \geq 0)$. There exists a predictable kernel $\lambda_t(dx)$ such that

$$\lambda_t(f) = \int_{\mathbb{R}} \lambda_t(dx) f(x)\,.$$

From the predictable representation property applied to the martingale $\mathbb{E}(f(L)|\mathcal{F}_t)$, there exists a predictable process $\widehat{\lambda}(f)$ such that

$$\lambda_t(f) = \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s\,.$$

Proposition 6.3.1 *We assume that there exists a predictable kernel $\widehat{\lambda}_t(dx)$ such that*

$$dt \text{ a.s.}, \quad \widehat{\lambda}_t(f) = \int_{\mathbb{R}} \widehat{\lambda}_t(dx) f(x)\,.$$

Assume furthermore that $dt \times d\mathbb{P}$ a.s. the measure $\widehat{\lambda}_t(dx)$ is absolutely continuous with respect to $\lambda_t(dx)$:

$$\widehat{\lambda}_t(dx) = \rho(t,x)\lambda_t(dx)\,.$$

Then, if X is an $\mathbb{F}$-martingale, there exists a $\mathbb{F}^{(L)}$-martingale $\widehat{X}$ such that

$$X_t = \widehat{X}_t + \int_0^t \rho(s,L) d\langle X, B\rangle_s\,.$$

Sketch of the proof: Let X be an $\mathbb{F}$-martingale, f a given bounded Borel function and $F_t = \mathbb{E}(f(L)|\mathcal{F}_t)$. From the hypothesis

$$F_t = \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s\,.$$

Then, for $A_s \in \mathcal{F}_s$, $s < t$:

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) &= \mathbb{E}(\mathbb{1}_{A_s}(F_t X_t - F_s X_s)) = \mathbb{E}(\mathbb{1}_{A_s}(\langle F, X\rangle_t - \langle F, X\rangle_s)) \\
&= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \, \widehat{\lambda}_u(f)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \int_{\mathbb{R}} \lambda_u(dx) f(x) \rho(u,x)\right)\,.
\end{aligned}$$

Therefore, $V_t = \int_0^t \rho(u,L)\, d\langle X, B\rangle_u$ satisfies

$$\mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{A_s} f(L)(V_t - V_s))\,.$$

It follows that, for any $G_s \in \mathcal{F}_s^{(L)}$,

$$\mathbb{E}(\mathbb{1}_{G_s}(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{G_s}(V_t - V_s))\,,$$

hence, $(X_t - V_t, t \geq 0)$ is an $\mathbb{F}^{(L)}$-martingale. □

Let us write the result of Proposition 6.3.1 in terms of Jacod's criterion. If $\lambda_t(dx) = p_t(x)\nu(dx)$, then

$$\lambda_t(f) = \int p_t(x) f(x) \nu(dx)\,.$$

Hence,

$$d\langle \lambda.(f), B\rangle_t \quad = \quad \widehat{\lambda}_t(f)dt = \int dx f(x)\, d_t\langle p.(x), B\rangle_t$$

and

$$\widehat{\lambda}_t(dx) = d_t\langle p.(x), B\rangle_t = \frac{d_t\langle p.(x), B\rangle_t}{p_t(x)} p_t(x)dx$$

therefore,

$$\widehat{\lambda}_t(dx)dt = \frac{d_t\langle p.(x), B\rangle_t}{p_t(x)} \lambda_t(dx)\,.$$

In the case where $\lambda_t(dx) = \Phi(t,x)dx$, with $\Phi > 0$, it is possible to find ψ such that

$$\Phi(t,x) = \Phi(0,x) \exp\left(\int_0^t \psi(s,x)dB_s - \frac{1}{2}\int_0^t \psi^2(s,x)ds\right)$$

and it follows that $\widehat{\lambda}_t(dx) = \psi(t,x)\lambda_t(dx)$. Then, if X is an $\mathbb{F}$-martingale of the form $X_t = x + \int_0^t x_s dB_s$, the process $(X_t - \int_0^t ds\, x_s\, \psi(s,L), t \geq 0)$ is an $\mathbb{F}^{(L)}$-martingale.

6.3.1 Faux amis

Theorem 6.3.2 *Let X be an $\mathbb{F}$-local martingale with representation $X_t = X_0 + \int_0^t \varphi_s dB_s$ for an $\mathbb{F}$-predictable process φ satisfying $\int_0^1 \varphi_s^2 ds < \infty$ a.s. Then, the following conditions are equivalent:*
a) the process X is an $\mathbb{F}^{\sigma(B_1)}$-semimartingale;
b) $\int_0^1 |\varphi_s| \frac{|B_1 - B_s|}{1-s} ds < \infty$ $\mathbb{P}$-a.s.;
c) $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}} ds < \infty$ $\mathbb{P}$-a.s.
If these conditions are satisfied, the $\mathbb{F}^{\sigma(B_1)}$-semimartingale decomposition of X is

$$X_t = X_0 + \int_0^{t\wedge 1} \varphi_s d\beta_s + \int_0^{t\wedge 1} \varphi_s \frac{B_1 - B_s}{1-s} ds. \tag{6.3.1}$$

This is an example where hypothesis $(\mathcal{H}')$ fails: some $\mathbb{F}$-martingales are $\mathbb{F}^{\sigma(B_1)}$-semimartingales, but not all of them.

6.4 Examples

We now give some examples taken from Mansuy & Yor [108] in a Brownian set-up for which we use the preceding. Here, B is a standard Brownian motion.
See Jeulin [82] and Mansuy & Yor [108] for more examples.

6.4.1 Enlargement with B_1.

We compare the results obtained in Subsection 5.1 and the method presented in Subsection 6.3. Let $L = B_1$. Note that,we can not apply directly Jacod's results, since, at time $t = 1$, the conditional law of B_1 given $\mathcal{F}_1$ is not absolutely continuous w.r.t. the law of B_1. From the Markov property

$$\mathbb{E}(g(B_1)|\mathcal{F}_t) = \mathbb{E}(g(B_1 - B_t + B_t)|\mathcal{F}_t) = F_g(B_t, 1-t)$$

where $F_g(y, 1-t) = \int g(x)P(1-t; y, x)dx$ and $P(s; y, x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-y)^2}{2s}\right)$. It follows that $\lambda_t(dx) = \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right) dx$. Then

$$\lambda_t(dx) = p_t(x)\mathbb{P}(B_1 \in dx) = p_t(x)\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

with

$$p_t(x) = \frac{1}{\sqrt{(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)} + \frac{x^2}{2}\right).$$

From Itô's formula,

$$d_t p_t(x) = p_t(x) \frac{x-B_t}{1-t} dB_t .$$

(This can be considered as a partial check of the martingale property of $(p_t(x), t \geq 0)$.) It follows that $d\langle p(x), B\rangle_t = p_t(x) \frac{x-B_t}{1-t} dt$, hence

$$B_t = \widetilde{B}_t + \int_0^t \frac{B_1 - B_s}{1-s} ds .$$

Note that, in the notation of Proposition 6.3.1, one has

$$\widehat{\lambda}_t(dx) = \frac{x-B_t}{1-t} \frac{1}{\sqrt{2\pi(1-t)}} \exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right) dx .$$

6.4.2 Enlargement with $M^B = \sup_{s\leq 1} B_s$.

From Exercise 1.6.1,

$$\mathbb{E}(f(M^B)|\mathcal{F}_t) = F(1-t, B_t, M_t^B)$$

where $M_t^B = \sup_{s\leq t} B_s$ with

$$F(s,a,b) = \sqrt{\frac{2}{\pi s}} \left(f(b) \int_0^{b-a} e^{-u^2/(2s)} du + \int_b^\infty f(u) e^{-(u-a)^2/(2s)} du \right)$$

and, denoting by δ_y the Dirac measure at y,

$$\lambda_t(dy) = \sqrt{\frac{2}{\pi(1-t)}} \left\{ \delta_y(M_t^B) \int_0^{M_t^B - B_t} \exp\left(-\frac{u^2}{2(1-t)}\right) du + \mathbb{1}_{\{y > M_t^B\}} \exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right) dy \right\} .$$

Hence, by applying Itô's formula

$$\widehat{\lambda}_t(dy) = \sqrt{\frac{2}{\pi(1-t)}} \left\{ \delta_y(M_t^B) \exp\left(-\frac{(M_t^B - B_t)^2}{2(1-t)}\right) + \mathbb{1}_{\{y > M_t^B\}} \frac{y-B_t}{1-t} \exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right) \right\} .$$

It follows that

$$\rho(t,x) = \mathbb{1}_{\{x > M_t^B\}} \frac{x-B_t}{1-t} + \mathbb{1}_{\{M_t^B = x\}} \frac{1}{\sqrt{1-t}} \frac{\Phi'}{\Phi} \left(\frac{x-B_t}{\sqrt{1-t}}\right)$$

with $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}} du$.

6.4.3 Enlargement with $\int_0^\infty e^{2B_s^{(-\mu)}} ds$

Consider $A_t^{(\mu)} := \int_0^t e^{2B_s^{(\mu)}} ds$ where $B_t^{(\mu)} = B_t + \mu t$, μ being a positive constant. Matsumoto and Yor [109] have established that $A_\infty^{(-\mu)} = A_t^{(-\mu)} + e^{2B_t^{(-\mu)}} \widetilde{A}_\infty^{(-\mu)}$ where $\widetilde{A}_\infty^{(-\mu)}$ is independent of $\mathcal{F}_t$, with the same law as $A_\infty^{(-\mu)}$. The law of $A_\infty^{(-\mu)}$ is proved to be the law of $1/(2\gamma_\mu)$, γ_μ being a Gamma random variable with parameter μ, i.e., admits the survival probability of $\Upsilon(x) = \frac{1}{\Gamma(\mu)} \int_0^{1/(2x)} y^{\mu-1} e^{-y} dy$, where Γ is the Gamma function. Then, one obtains

$$G_t(\theta) = P(A_\infty^{(-\mu)} > \theta | \mathcal{F}_t) = \Upsilon\Big(\frac{\theta - A_t^{(-\mu)}}{e^{2B_t^{(-\mu)}}}\Big) \mathbb{1}_{\theta > A_t^{(-\mu)}} + \mathbb{1}_{\theta \leq A_t^{(-\mu)}} .$$

This gives a family of martingale survival processes G with gamma structure. It follows that, on $\{\theta > A_t^{(-\mu)}\}$

$$dG_t(\theta) = \frac{1}{2^{\mu-1}\Gamma(\mu)} e^{-\frac{1}{2}Z_t(\theta)}(Z_t(\theta))^{\mu} dB_t$$

where $Z_t(\theta) = \frac{e^{2B_t^{(-\mu)}}}{\theta - A_t^{(-\mu)}}$ (to have light notation, we do not specify that this process Z depends on μ). One can check that $G_t(\cdot)$ is differentiable w.r.t. θ, so that $G_t(\theta) = \int_\theta^\infty g_t(u)du$, where

$$g_t(u) = 1_{u > A_t^{(-\mu)}} \frac{1}{2^{\mu}\Gamma(\mu)} (Z_t(u))^{\mu+1} e^{-\frac{1}{2}Z_t(u) - 2B_t^{(-\mu)}}.$$

6.4.4 Enlargement with $L := \int_0^\infty f(s)dB_s$

Let B be a Brownian motion with natural filtration $\mathbb{F}$ and $L = \int_0^\infty f(s)dB_s$ where f is a deterministic function such that $\int_0^\infty f^2(s)ds < \infty$ and $\int_t^\infty f^2(s) \neq 0$. The above method applies step by step: it is easy to compute $\lambda_t(dx)$, since conditionally on $\mathcal{F}_t$, L is Gaussian, with mean $m_t = \int_0^t f(s)\, dB_s$, and variance $\sigma^2(t) = \int_t^\infty f^2(s)\, ds$. Since $\mathbb{P}(L \le x|\mathcal{F}_t) = \Phi(\frac{x - m_t}{\sigma(t)})$, where Φ is the cumulative distribution function of a standard gaussian law, the absolute continuity requirement is satisfied with:

$$p_t(x)\nu(dx) = \frac{1}{\sigma(t)} \varphi(\frac{x - m_t}{\sigma(t)})dx,$$

where φ is the density of a standard Gaussian law, and ν the law of Z (a centered Gaussian law with variance $\sigma^2(0)$). Note that, from Itô's calculus,

$$dp_t(x) = p_t(x) \frac{x - m_t}{\sigma^2(t)} dm_t$$

But here, we have to impose an extra integrability condition. For example, if we assume that

$$\int_0^t \frac{|f(s)|}{\sigma(s)} ds < \infty,$$

then B is a $\mathbb{F}^{(L)}$-semimartingale with canonical decomposition:

$$B_t = \widetilde{B}_t + \int_0^t ds \frac{f(s)}{\sigma^2(s)} \left(\int_s^\infty f(u)\, dB_u \right),$$

As a particular case, taking care of the fact that σ vanishes after t_0, we may take $L = B_{t_0}$, for some fixed t_0 and we recover the results for the Brownian bridge.

6.4.5 Enlargement with $S_\infty = \sup_t N_t$

We start with a generalization of the result presented in Exercise 1.6.2.

Proposition 6.4.1 ***Azéma-Yor formula*** *Let N be a local continuous martingale and $S_t = \sup_{s \le t} N_s$. Let f be a locally bounded Borel function and define $F(x) = \int_0^x dy f(y)$. Then, $X_t := F(S_t) - f(S_t)(S_t - N_t)$ is a local martingale and:*

$$F(S_t) - f(S_t)(S_t - N_t) = \int_0^t f(S_s)\, dN_s + F(S_0), \tag{6.4.1}$$

PROOF: If F is C^2,

$$\begin{aligned} F(S_t) - f(S_t)(S_t - N_t) &= F(S_t) - \int_0^t f(S_s)\, dS_s + \int_0^t f(S_s)\, dN_s \\ &\quad + \int_0^t (S_s - N_s) f'(S_s) dS_s \end{aligned}$$

The last integral is null, because dS is carried by $\{S - N = 0\}$ and $\int_0^t f(S_s)\, dS_s = F(S_t) - F(S_0)$. For the general case, we refer the reader to [115]. □

The result can be extended to the case where N be a local martingale with a continuous running maximum (see [115]).

Let N be a positive continuous local martingale such that N_t goes to 0 when $t \to \infty$. Let us introduce $\mathcal{F}_t^{(S_\infty)} = \mathcal{F}_t \vee \sigma(S_\infty))$ and set $g = \sup\{t:\ N_t = S_\infty\}$. Obviously, the random variable g is an $\mathbb{F}^{(S_\infty)}$-stopping time. Consequently $\mathcal{F}_t^g \subset \mathcal{F}_t^{(S_\infty)}$.

Proposition 6.4.2 *For any Borel bounded or positive function f, we have:*

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = f(S_t)\left(1 - \frac{N_t}{S_t}\right) + \int_0^{N_t/S_t} dx f\left(\frac{N_t}{x}\right)$$

PROOF: In the following, U is a random variable, which follows the standard uniform law and which is independent of $\mathcal{F}_t$, and $S^t = \sup s \geq t N_s$. Then, from Lemma 1.1.13,

$$\begin{aligned} \mathbb{E}(f(S_\infty)|\mathcal{F}_t) &= \mathbb{E}\left(f\left(S_t \vee S^t\right)|\mathcal{F}_t\right) \\ &= \mathbb{E}\left(f(S_t)\, 1\!\!1_{\{S_t \geq S^t\}}|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(S^t\right) 1\!\!1_{\{S_t < S^t\}}|\mathcal{F}_t\right) \\ &= f(S_t)\, \mathbb{P}\left(S_t \geq S^t|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(S^t\right) 1\!\!1_{\{S_t < S^t\}}|\mathcal{F}_t\right) \\ &= f(S_t)\, \mathbb{P}\left(U \leq \frac{N_t}{S_t}|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(\frac{N_t}{U}\right) 1\!\!1_{\{U < \frac{N_t}{S_t}\}}|\mathcal{F}_t\right) \\ &= f(S_t)\left(1 - \frac{N_t}{S_t}\right) + \int_0^{N_t/S_t} dx f\left(\frac{N_t}{x}\right). \end{aligned}$$

□

We now show that $\mathbb{E}(f(S_\infty)|\mathcal{F}_t)$ is of the form (6.4.1). A straightforward change of variable in the last integral also gives:

$$\begin{aligned} \mathbb{E}(f(S_\infty)|\mathcal{F}_t) &= f(S_t)\left(1 - \frac{N_t}{S_t}\right) + N_t \int_{S_t}^\infty dy \frac{f(y)}{y^2} \\ &= S_t \int_{S_t}^\infty dy \frac{f(y)}{y^2} - (S_t - N_t)\left(\int_{S_t}^\infty dy \frac{f(y)}{y^2} - \frac{f(S_t)}{S_t}\right). \end{aligned}$$

Hence,

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = H(1) + H(S_t) - h(S_t)(S_t - N_t),$$

with

$$H(x) = x \int_x^\infty dy \frac{f(y)}{y^2},$$

and

$$h(x) = H'(x) \equiv \int_x^\infty dy \frac{f(y)}{y^2} - \frac{f(x)}{x} = \int_x^\infty \frac{dy}{y^2}(f(y) - f(x)).$$

Moreover, from the Azéma-Yor type formula (6.4.1), we have the following representation of $\mathbb{E}(f(S_\infty)|\mathcal{F}_t)$ as a stochastic integral:

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = \mathbb{E}(f(S_\infty)) + \int_0^t h(S_s)\, dN_s.$$

Moreover, there exist two families of random measures $(\lambda_t(dx))_{t\geq 0}$ and $\left(\dot{\lambda}_t(dx)\right)_{t\geq 0}$, with

$$\begin{aligned}
\lambda_t(dx) &= \left(1-\frac{N_t}{S_t}\right)\delta_{S_t}(dx) + N_t \mathbb{1}_{\{x>S_t\}}\frac{dx}{x^2} \\
\dot{\lambda}_t(dx) &= -\frac{1}{S_t}\delta_{S_t}(dx) + \mathbb{1}_{\{x>S_t\}}\frac{dx}{x^2},
\end{aligned}$$

such that

$$\begin{aligned}
\mathbb{E}\left(f(S_\infty)\,|\mathcal{F}_t\right) = \lambda_t(f) &= \int \lambda_t(dx)\, f(x) \\
\dot{\lambda}_t(f) &= \int \dot{\lambda}_t(dx)\, f(x).
\end{aligned}$$

Finally, we notice that there is an absolute continuity relationship between $\lambda_t(dx)$ and $\dot{\lambda}_t(dx)$; more precisely,

$$\dot{\lambda}_t(dx) = \lambda_t(dx)\,\rho(x,t),$$

with

$$\rho(x,t) = \frac{-1}{S_t - N_t}\mathbb{1}_{\{S_t=x\}} + \frac{1}{N_t}\mathbb{1}_{\{S_t<x\}}.$$

Theorem 6.4.3 *Let N be a positive continuous local martingale in the class $\mathcal{C}_0$ with $N_0 = 1$. Then, any $\mathbb{F}$ martingale X is an $\mathbb{F}^{(S_\infty)}$-emimartingale with canonical decomposition:*

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}}\frac{d\langle X,N\rangle_s}{N_{s-}} - \int_0^t \mathbb{1}_{\{g\leq s\}}\frac{d\langle X,N\rangle_s}{S_\infty - N_{s-}},$$

where $\widetilde{X}$ is a $\mathbb{F}^{(S_\infty)}$-local martingale.

PROOF: We can first assume that X is in $\mathbb{H}^1$; the general case follows by localization. Let K_s be an $\mathcal{F}_s$ measurable set, and take $t > s$. Then, for any bounded test function f, $\lambda_t(f)$ is a bounded martingale, hence in BMO, and we have:

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}_{K_s} f(S_\infty)(X_t - X_s)\right) &= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\lambda_t(f)X_t - \lambda_s(f)X_s\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\langle\lambda(f),X\rangle_t - \langle\lambda(f),X\rangle_s\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t \dot{\lambda}_u(f)\, d\langle X,N\rangle_u\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t \int \lambda_u(dx)\,\rho(x,u)\, f(x)\, d\langle X,N\rangle_u\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t d\langle X,N\rangle_u \rho(S_\infty,u)\right)\right).
\end{aligned}$$

But we also have:

$$\rho(S_\infty,t) = \frac{-1}{S_t - N_t}\mathbb{1}_{\{S_t=S_\infty\}} + \frac{1}{N_t}\mathbb{1}_{\{S_t<S_\infty\}}.$$

It now suffices to use the fact that S is constant after g and g is the first time when $S_\infty = S_t$, or in other words:

$$\mathbb{1}_{\{S_\infty>S_t\}} = \mathbb{1}_{\{g>t\}}, \text{ and } \mathbb{1}_{\{S_\infty=S_t\}} = \mathbb{1}_{\{g\leq t\}}.$$

Chapter 7

Filtering

In this chapter, our goal is to show how one can apply the idea of change of probability framework to a filtering problem (due to Kallianpur and Striebel [89]), to obtain the Kallianpur-Striebel formula for the conditional density (see also Meyer [111]). Our results are established in a very simple way, in a filtering model, when the signal is a random variable, and contain, in the simple case, the results of Filipovic et al. [57]. We end the section with the examples of the traditional Gaussian filtering problem and of disorder.

7.1 Change of probability measure

One starts with the elementary model where, on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, an $\mathcal{A}$-measurable random variable X is independent from the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and its law admits a density probability g_0, so that

$$\mathbb{P}(X > \theta|\mathcal{F}_t) = \mathbb{P}(X > \theta) = \int_\theta^\infty g_0(u)du\,.$$

We denote by $\mathbb{F}^{(X)} = \mathbb{F} \vee \sigma(X)$ the filtration generated by $\mathbb{F}$ and X.
Let $(\beta_t(u), t \in I\!\!R_+)$ be a family of positive $(\mathbb{P}, \mathbb{F})$-martingales such that $\beta_0(u) = 1$ for all $u \in I\!\!R$. Note that, due to the assumed independence of X and $\mathbb{F}$, the process $(\beta_t(X), t \geq 0)$ is an $\mathbb{F}^{(X)}$-martingale and one can define a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_t^{(X)})$, by $d\mathbb{Q} = \beta_t(X)d\mathbb{P}$. Since $\mathbb{F}$ is a subfiltration of $\mathbb{F}^{(X)}$, the positive $\mathbb{F}$-martingale

$$m_t^\beta := \mathbb{E}(\beta_t(X)|\mathcal{F}_t) = \int_{-\infty}^\infty \beta_t(u)g_0(u)du$$

is the Radon-Nikodým density of the measure $\mathbb{Q}$, restricted to $\mathcal{F}_t$ with respect to $\mathbb{P}$ (note that $m_0^\beta = 1$). Moreover, the $\mathbb{Q}$-conditional density of X with respect to $\mathcal{F}_t$ can be computed, from the Bayes' formula

$$\mathbb{Q}(X \in B|\mathcal{F}_t) = \frac{1}{\mathbb{E}(\beta_t(X)|\mathcal{F}_t)}\mathbb{E}(1\!\!1_B(X)\beta_t(X)|\mathcal{F}_t) = \frac{1}{m_t^\beta}\int_B \beta_t(u)g_0(u)du$$

where we have used, in the last equality the independence between X and $\mathbb{F}$, under $\mathbb{P}$. Let us summarize this simple but important result:

Proposition 7.1.1 *If X is a r.v. with probability density g_0, independent from $\mathbb{F}$ under $\mathbb{P}$, and if $\mathbb{Q}$ is a probability measure, equivalent to $\mathbb{P}$ on $\mathbb{F} \vee \sigma(X)$ with Radon-Nikodým density $\beta_t(X), t \geq 0$,* *then the* $(\mathbb{Q},\mathbb{F})$ *density process of* X *is*

$$g_t^{\mathbb{Q}}(u)du := \mathbb{Q}(X \in du|\mathcal{F}_t) = \frac{1}{m_t^\beta}\,\beta_t(u)g_0(u)du \tag{7.1.1}$$

where m^β *is the normalizing factor* $m_t^\beta = \int_{-\infty}^{\infty}\beta_t(u)g_0(u)du$. *In particular*

$$\mathbb{Q}(X \in du) = \mathbb{P}(X \in du) = g_0(u)du\,.$$

The right-hand side of (7.1.1) can be understood as the ratio of $\beta_t(u)g_0(u)$ (the change of probability times the $\mathbb{P}$ probability density) and a normalizing coefficient m_t^β. One can say that $(\beta_t(u)g_0(u), t \geq 0)$ is the un-normalized density, obtained by a linear transformation from the initial density. The normalization factor m_t^β introduces a nonlinear dependence of $g_t^{\mathbb{Q}}(u)$ with respect to the initial density.

Remark 7.1.2 We present here some important remarks.
(1) If, for any t, $m_t^\beta = 1$, then the probability measures $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathbb{F}$.
(2) Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the usual right-continuous and complete filtration in the default framework (i.e. when $X = \tau$ is a non negative r.v.) generated by $\mathcal{F}_t \vee \sigma(\tau \wedge t)$. Similar calculation may be made with respect to $\mathcal{G}_t$. The only difference is that the conditional distribution of τ is a Dirac mass on the set $\{t \geq \tau\}$. On the set $\{\tau > t\}$, and under $\mathbb{Q}$, the distribution of τ admits a density given by:

$$\mathbb{Q}(\tau \in du|\mathcal{G}_t) = \beta_t(u)g_0(u)\frac{1}{\int_0^\infty \beta_t(\theta)g_0(\theta)d\theta}du\,.$$

(3) This methodology can be easily extended to a multivariate setting: one starts with an elementary model, where the $\tau_i, i = 1,\dots,d$ are independent from $\mathbb{F}$, with joint density $g(u_1,\dots,u_d)$. With a family of non-negative martingales $\beta(\theta_1,\dots,\theta_d)$, the associated change of probability provides a multidimensional density process.

7.2 Filtering theory

The change of probability approach presented in the previous Section 7.1 is based on the idea that, in order to present modesl with a conditional density, one can restrict our attention to the simple case where the random variable is independent from the filtration and use a change of probability. The same idea is the building block of filtering theory as we present now.

Let W be a Brownian motion on the probability space $(\Omega,\mathcal{A},\mathbb{P})$, and X be a random variable independent of W, with probability density g_0. We denote by

$$dY_t = a(t,Y_t,X)dt + b(t,Y_t)dW_t \tag{7.2.1}$$

the observation process, where a and b are smooth enough to have a solution and where b does not vanish. The goal is to compute the conditional density of X with respect to the filtration $\mathbb{F}^Y$. The way we shall solve the problem is to construct a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that, under $\mathbb{Q}$, the signal X and the observation $\mathbb{F}^Y$ are independent, and to compute the density of X under $\mathbb{P}$ by means of the change of probability approach of the previous section. It is known in nonlinear filtering theory as the Kallianpur-Striebel methodology [89], a way to linearize the problem.
Note that, from the independence assumption between X and W, we see that W is a $\mathbb{F}^{(X)} = \mathbb{F}^W \vee \sigma(X)$-martingale under $\mathbb{P}$.

7.2.1 Simple case

We start with the simple case where the dynamics of the observation is

$$dY_t = a(t,X)dt + dW_t\,.$$

We assume that a is smooth enough so that the solution of

$$d\beta_t(X) = -\beta_t(X)a(t,X)dW_t,\ \beta_0(X) = 1$$

is a $(\mathbb{P}, \mathbb{F}^{(X)})$-martingale, and we define a probability measure $\mathbb{Q}$ on $\mathcal{F}_t^{(X)}$ by $d\mathbb{Q} = \beta_t(X)d\mathbb{P}$. Then, by Girsanov's theorem, the process Y is a $(\mathbb{Q}, \mathbb{F}^{(X)})$-Brownian motion, hence is independent from $\mathbb{F}_0^{(X)} = \sigma(X)$, under $\mathbb{Q}$. Then, we apply our change of probability methodology, writing

$$d\mathbb{P} = \frac{1}{\beta_t(X)}d\mathbb{Q} =: \ell_t(X)d\mathbb{Q}$$

with

$$d\ell_t(X) = \ell_t(X)a(t,X)dY_t,\ \ell_0(X) = 1\,;$$

in other words, $\ell_t(u) = \frac{1}{\beta_t(u)} = \exp\left(\int_0^t a(s,u)dY_s - \frac{1}{2}\int_0^t a^2(s,u)ds\right)$. From Proposition 7.1.1, we obtain that the density of X under $\mathbb{P}$, with respect to $\mathbb{F}^Y$, is $g_t(u)$, given by

$$\mathbb{P}(X \in du|\mathcal{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell}g_0(u)\ell_t(u)du$$

where $m_t^\ell = \mathbb{E}_{\mathbb{Q}}(\ell_t(X)|\mathcal{F}_t^Y) = \int_{-\infty}^{\infty}\ell_t(u)g_0(u)du$. Using the fact that

$$dm_t^\ell = \left(\int_{-\infty}^{\infty}\ell_t(u)a(t,u)g_0(u)du\right)\,dY_t = m_t^\ell\left(\int_{-\infty}^{\infty}g_t(u)a(t,u)du\right)\,dY_t$$

and setting

$$\widehat{a}_t := \mathbb{E}(a(t,X)|\mathcal{F}_t^Y) = \int_{-\infty}^{\infty}g_t(u)a(t,u)du\,,$$

Girsanov's theorem implies that the process B given by

$$dB_t = dY_t - \widehat{a}_t dt = dW_t + (a(t,X) - \widehat{a}_t)\,dt$$

is a $(\mathbb{P}, \mathbb{F}^Y)$ Brownian motion (called the innovation process). From Itô's calculus, it is easy to show that the density process satisfies the nonlinear filtering equation

$$\begin{aligned} dg_t(u) &= g_t(u)\left(a(t,u) - \frac{1}{m_t^\ell}\int_{-\infty}^{\infty}dy\,g_0(y)a(t,y)\ell_t(y)\right)dB_t \\ &= g_t(u)\,(a(t,u) - \widehat{a}_t)\,dB_t\,. \end{aligned} \tag{7.2.2}$$

Remarks 7.2.1 (a) Observe that conversely, given a solution $g_t(u)$ of (7.2.2), and the process μ solution of $d\mu_t = \mu_t\widehat{a}_t dY_t$, then $h_t(u) = \mu_t g_t(u)$ is solution of the linear equation $dh_t(u) = h_t(u)a(t,u)dY_t$.
(b) It is interesting to compare this methodology of change of probability measure with the one used in Chapter 9

7.2.2 Case with drift coefficient

Using the same ideas, we now solve the filtering problem in the case where the observation follows (7.2.1). Let $\beta(X)$ be the $\mathbb{F}^{(X)}$ local martingale, solution of

$$d\beta_t(X) = \beta_t(X)\sigma_t(X)dW_t,\ \beta_0(X) = 1$$

with $\sigma_t(X) = -\frac{a(t,Y_t,X)}{b(t,Y_t)}$. We assume that a and b are smooth enough so that β is a martingale. Let $\mathbb{Q}$ be defined on $\mathcal{F}_t^{(X)}$ by $d\mathbb{Q} = \beta_t(X)d\mathbb{P}$.
From Girsanov's theorem, the process $\widehat{W}$ defined as

$$d\widehat{W}_t = dW_t - \sigma_t(X)dt = \frac{1}{b(t,Y_t)}dY_t$$

is a $(\mathbb{Q}, \mathbb{G}^X)$-Brownian motion, hence $\widehat{W}$ is independent from $\mathcal{G}_0^X = \sigma(X)$. Being $\mathbb{F}^Y$-adapted, the process $\widehat{W}$ is a $(\mathbb{Q}, \mathbb{F}^Y)$-Brownian motion, X is independent from $\mathbb{F}^Y$ under $\mathbb{Q}$, and, as mentioned in Proposition 7.1.1, admits, under $\mathbb{Q}$, the probability density g_0.
We now assume that the natural filtrations of Y and $\widehat{W}$ are the same. To do so, note that it is obvious that $\mathbb{F}^{\widehat{W}} \subseteq \mathbb{F}^Y$. If the SDE $dY_t = b(t, Y_t)d\widehat{W}_t$ has a strong solution (e.g., if b is Lipschitz, with linear growth) then $\mathbb{F}^Y \subseteq \mathbb{F}^{\widehat{W}}$ and the equality between the two filtrations holds.
Then, we apply our change of probability methodology, with $\mathbb{F}^Y$ as the reference filtration, writing $d\mathbb{P} = \ell_t(X)d\mathbb{Q}$ with $d\ell_t(X) = -\ell_t(X)\sigma_t(X)d\widehat{W}_t$ (which follows from $\ell_t(X) = \frac{1}{\beta_t(X)}$) and we get that the density of X under $\mathbb{P}$, with respect to $\mathbb{F}^Y$ is $g_t(u)$ given by

$$g_t(u) = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)$$

with dynamics

$$\begin{aligned} dg_t(u) &= -g_t(u)\left(\sigma_t(u) - \frac{1}{m_t^\ell}\int_{-\infty}^{\infty} dy\, g_0(y)\sigma_t(y)\ell_t(y)\right) dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{1}{b(t,Y_t)}\int_{-\infty}^{\infty} dy\, g_t(y)a(t,Y_t,y)\right) dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right) dB_t\,. \end{aligned} \tag{7.2.3}$$

Here B is a $(\mathbb{P}, \mathbb{F}^Y)$ Brownian motion (the innovation process) given by

$$dB_t = dW_t + \left(\frac{a(t,Y_t,X)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right) dt\,,$$

where $\widehat{a}_t = \mathbb{E}(a(t, Y_t, X)|\mathcal{F}_t^Y)$.

Proposition 7.2.2 *If the signal X has probability density $g_0(u)$ and is independent from the Brownian motion W, and if the observation process Y follows*

$$dY_t = a(t, Y_t, X)dt + b(t, Y_t)dW_t\,,$$

then, the conditional density of X given $\mathcal{F}_t^Y$ is

$$\mathbb{P}(X \in du|\mathcal{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)du \tag{7.2.4}$$

where $\ell_t(u) = \exp\left(\int_0^t \frac{a(s,Y_s,u)}{b^2(s,Y_s)}dY_s - \frac{1}{2}\int_0^t \frac{a^2(s,Y_s,u)}{b^2(s,Y_s)}ds\right)$, $m_t^\ell = \int_{-\infty}^{\infty} \ell_t(u)g_0(u)du$, *and its dynamics is given in (7.2.3).*

7.2.3 Case where X has a Conditional Law

Assume now that X has a non trivial conditional law w.r.t. the Brownian motion driving the observation process. We assume that

$$\mathbb{P}(X > u|\mathcal{F}_t^W) = \int_u^\infty p_t(v)dv$$

and that the observation is

$$dY_t = a(t, Y_t, X)dt + b(t, Y_t)dW_t$$

Then, the process

$$B_t := W_t + \int_0^t \frac{d\langle p.(\theta), W\rangle_s|_{\theta=X}}{p_s(X)}$$

is a $\mathbb{F}^W \vee \sigma(X)$ Brownian motion, independent of X. It follows that

$$dY_t = \left(a(t, Y_t, X)dt - b(t, Y_t)\frac{d\langle p.(\theta), W\rangle_t|_{\theta=X}}{p_t(X)}\right) + b(t, Y_t)dB_t$$

and we can apply the previous results.

7.2.4 Gaussian filter

We apply our results to the well known case of Gaussian filter. Let W be a Brownian motion, X a random variable (the signal) with density probability g_0 a Gaussian law with mean m_0 and variance γ_0, independent of the Brownian motion W and let Y (the observation) be the solution of

$$dY_t = (a_0(t, Y_t) + a_1(t, Y_t)X)dt + b(t, Y_t)dW_t,$$

Then, from the previous results, the density process $g_t(u)$ is of the form

$$\frac{1}{m_t^\ell}\exp\left(\int_0^t \frac{a_0(s, Y_s) + a_1(s, Y_s)u}{b^2(s, Y_s)}\,dY_t - \frac{1}{2}\int_0^t \left(\frac{a_0(s, Y_s) + a_1(s, Y_s)u}{b(s, Y_s)}\right)^2 ds\right) g_0(u)$$

The logarithm of $g_t(u)$ is a quadratic form in u with stochastic coefficient, so that $g_t(u)$ is a Gaussian density, with mean m_t and variance γ_t (as proved already by Liptser and Shiryaev [104]). A tedious computation, purely algebraic, shows that

$$\gamma_t = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \frac{a_1^2(s,Y_s)}{b^2(s,Y_s)}ds}, \quad m_t = m_0 + \int_0^t \gamma_s \frac{a_1(s, Y_s)}{b(s, Y_s)}\,dB_s$$

with $dB_t = dW_t + \frac{a_1(t,Y_t)}{b(t,Y_t)}(X - \mathbb{E}(X|\mathcal{F}_t^Y))dt$.

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In the case where the coefficients of the process Y are deterministic functions of time, i.e.,

$$dY_t = (a_0(t) + a_1(t)X)dt + b(t)dW_t$$

the variance $\gamma(t)$ is deterministic and the mean m is an $\mathbb{F}^Y$-Gaussian martingale

$$\gamma(t) = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \alpha^2(s)ds}, \quad m_t = m_0 + \int_0^t \gamma(s)\alpha(s)dB_s$$

where $\alpha = a_1/b$. Furthermore, $\mathbb{F}^Y = \mathbb{F}^B$.

Filtering versus enlargement: Choosing $f(s) = \frac{\gamma(s)a_1(s)}{b(s)}$ in the example of Section 6.4.4 leads to the same conditional law (with $m_0 = 0$); indeed, it is not difficult to check that this choice of parameter leads to $\int_t^\infty f^2(s)ds = \sigma^2(t) = \gamma(t)$ so that the two variances are equal.
The similarity between filtering and the example of Section 6.4.4 can be also explained as follows. Let us start from the setting of Section 6.4.4 where $X = \int_0^\infty f(s)dB_s$ and introduce $\mathbb{F}^{(X)} = \mathbb{F}^B \vee \sigma(X)$, where B is the given Brownian motion. We have seen that

$$W_t := B_t + \int_0^t \frac{X - m_s}{\sigma^2(s)} f(s)ds$$

is an $\mathbb{F}^{(X)}$-BM, hence is a $\mathbb{G}^W$-BM independent of X. So, the example presented in Section 6.4.4 is equivalent to the following filtering problem: the signal X is a Gaussian variable, centered, with variance $\gamma(0) = \int_0^\infty f^2(s)ds$ and the observation

$$dY_t = f(t)Xdt + \left(\int_t^\infty f^2(s)ds\right) dW_t = f(t)Xdt + \sigma^2(t)dW_t\,.$$

7.2.5 Disorder

Classical case: the signal is independent of the driving Brownian motion

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and τ be a random time, independent of W and such that $\mathbb{P}(\tau > t) = e^{-\lambda t}$, for all $t \geq 0$ and some $\lambda > 0$ fixed. We define $Y = (Y_t)_{t\geq 0}$ as the solution of the stochastic differential equation

$$dY_t = \big(a + b\,1\!\!1_{\{t>\tau\}}\big)\,dt + Y_t\,\sigma\,dW_t\,.$$

Let $\mathbb{F}^Y = (\mathcal{F}^Y_t, t\geq 0)$ be the natural filtration of the process Y (note that $\mathbb{F}^Y$ is smaller than $\mathbb{F}^W \vee \sigma(\tau)$). From $Y_t = x + \int_0^t (a + b\,1\!\!1_{\{s>\tau\}})ds + \int_0^t \sigma\,dW_s$, it follows that (from Exercise 1.6.4)

$$dY_t = \big(a + b\,(1 - G_t)\big)\,dt + d\text{mart}$$

Here, $G = (G_t)_{t\geq 0}$ is the Azéma supermartingale given by $G_t = \mathbb{P}(\tau > t\,|\,\mathcal{F}_t)$. Identifying the brackets, one has $d\text{mart} = \sigma d\bar{W}_t$ where $\bar{W}$ is a martingale with bracket t, hence is a BM. It follows that the process Y admits the following representation in its own filtration

$$dY_t = \big(a + b\,(1 - G_t)\big)\,dt + \sigma\,d\bar{W}_t\,.$$

Here $\bar{W} = (\bar{W}_t)_{t\geq 0}$ is the innovation process defined by

$$\bar{W}_t = W_t + \frac{b}{\sigma}\int_0^t \big(1\!\!1_{\{s>\tau\}} - (1 - G_s)\big)\,ds = W_t - \frac{b}{\sigma}\int_0^t \big(1\!\!1_{\{\tau>s\}} - G_s\big)\,ds$$

and is a standard $\mathbb{F}$-Brownian motion. Using the previous results with $a(t, Y_t, \tau) = a + b1\!\!1_{t>\tau}$, one obtains easily

$$\begin{aligned}
\ell_t(u) &= \exp\left(\frac{a}{\sigma^2}Y_t - \frac{1}{2}\frac{a^2}{\sigma^2}t\right) =: Z_t \quad u > t \\
&= \exp\left(\frac{a+b}{\sigma^2}Y_t - \frac{1}{2}\frac{(a+b)^2}{\sigma^2}t - \frac{b}{\sigma^2}Y_u + \frac{1}{2}(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2})u\right) \\
&= \frac{Z_t}{U_t}U_u \quad u \leq t
\end{aligned}$$

where $U_u = e^{-\frac{b}{\sigma^2}Y_u + \frac{1}{2}(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2})u}$ and $G_t = \frac{1}{m^\ell_t}e^{-\lambda t}Z_t$ where

$$\begin{aligned}
m^\ell_t &= \lambda\,e^{\frac{a+b}{\sigma^2}Y_t - \frac{1}{2}\frac{(a+b)^2}{\sigma^2}t}\int_0^t e^{-\lambda u}e^{-\frac{b}{\sigma^2}Y_u + \frac{1}{2}(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2})u}du + e^{-\lambda t}Z_t \\
&= \lambda\frac{Z_t}{U_t}\int_0^t e^{-\lambda u}U_u du + e^{-\lambda t}Z_t
\end{aligned}$$

Moreover

$$\begin{aligned}
g_t(u) &= \frac{U_t}{e^{-\lambda t}U_t + \lambda\int_0^t e^{-\lambda u}U_u du}\big(1\!\!1_{u>t}e^{-\lambda u}U_t + 1\!\!1_{t>u}U_u\big) \\
G_t(u) &= \frac{Z_t}{m_t}\left(e^{-\lambda t} + 1\!\!1_{t>u}\frac{1}{U_t}\int_u^t \lambda e^{-\lambda s}U_s ds\right)
\end{aligned}$$

After some computation, we recover that the process G solves the stochastic differential equation

$$dG_t = -\lambda G_t\,dt + \frac{b}{\sigma}\,G_t(1 - G_t)\,d\bar{W}_t\,. \tag{7.2.5}$$

Observe that the process $n = (n_t)_{t \geq 0}$ with $n_t = e^{\lambda t} G_t$ admits the representation

$$dn_t = d(e^{\lambda t} G_t) = \frac{b}{\sigma} e^{\lambda t} G_t(1 - G_t)\, d\bar{W}_t$$

and thus, n is an $\mathbb{F}$-martingale (to establish the true martingale property, note that the process $(G_t(1-G_t))_{t\geq 0}$ is bounded). The equality (7.2.5) provides the (additive) Doob-Meyer decomposition of the supermartingale G, while $G_t = (G_t\, e^{\lambda t})\, e^{-\lambda t}$ gives its multiplicative decomposition. It follows from these decompositions that the $\mathbb{F}$-intensity rate of τ is λ, so that, the process $M = (M_t)_{t\geq 0}$ with $M_t = 1\!\!1_{\tau \leq t} - \lambda(t \wedge \tau)$ is a $\mathbb{G}$-martingale.

It follows from the definition of the conditional survival probability process G and the fact that $(G_t\, e^{\lambda t})_{t\geq 0}$ is a martingale that the expression

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \mathbb{E}[\mathbb{P}(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t] = \mathbb{E}[G_u\, e^{\lambda u} \,|\, \mathcal{F}_t]\, e^{-\lambda u} = G_t\, e^{\lambda(t-u)}$$

holds for $0 \leq t < u$. One can easily extend the results to the case

$$dY_t = \big(a(t, Y_t) + b(t, Y_t)\, 1\!\!1_{t>\tau}\big)\, dt + \sigma(t, Y_t)\, dW_t\,.$$

Using the previous results with $a(t, Y_t, \tau) = a(t, Y_t) + b(t, Y_t) 1\!\!1_{t>\tau} := a_t + b_t 1\!\!1_{t>\tau}$, one obtains easily

$$\begin{aligned}
\ell_t(u) &= \exp\left(\int_0^t \frac{a_s}{\sigma_s^2} dY_s - \frac{1}{2}\frac{a_s^2}{\sigma_s^2} ds\right) =: Z_t \quad u > t \\
&= \exp\left(\int_0^u \frac{a_s}{\sigma_s^2} dY_s - \int_0^u \frac{1}{2}\frac{a_s^2}{\sigma_s^2} ds + \int_u^t \frac{a_s + b_s}{\sigma_s^2} dY_s - \int_u^t \frac{1}{2}\frac{(a_s + b_s)^2}{\sigma_s^2} ds\right) \quad u \leq t
\end{aligned}$$

and $G_t = \frac{1}{m_t^\ell} e^{-\lambda t} Z_t$ where

$$\begin{aligned}
m_t^\ell &= \lambda \exp\left(\int_0^t \frac{a_s + b_s}{\sigma_s^2} dY_s - \int_0^t \frac{1}{2}\frac{(a_s + b_s)^2}{\sigma_s^2} ds\right) \int_0^t e^{-\lambda u} U_u du + e^{-\lambda t} Z_t \\
&= \lambda \frac{Z_t}{U_t} \int_0^t e^{-\lambda u} U_u du + e^{-\lambda t} Z_t
\end{aligned}$$

with $U_u = \exp\left(-\int_0^u \frac{b_s}{\sigma_s^2} dY_s + \frac{1}{2}\int_0^u \frac{b_s}{\sigma_s^2} + 2\frac{a_s b_s}{\sigma_s^2} ds\right)$

Chapter 8

Progressive Enlargement

In this chapter, we study the case of progressive enlargements of the form $\mathcal{F}_t \vee \sigma(\tau \wedge t)$ for a non negative random variable τ. More precisely, we assume that τ is a finite random time, i.e., a finite non-negative random variable constructed on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, and we denote by $\mathbb{G}$ the right-continuous filtration

$$\mathcal{G}_t := \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))\} .$$

We define, as before, the right-continuous process H, called the default indicator as

$$H_t = 1\!\!1_{\{\tau \le t\}} .$$

We denote by $\mathbb{H} = (\mathcal{H}_t, t \ge 0)$ its natural filtration (after regularization). With the usual abuse of notation, we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ for the right-continuous progressively enlarged filtration. Note that τ is an $\mathbb{H}$-stopping time, hence a $\mathbb{G}$-stopping time. (In fact, $\mathbb{H}$ is the smallest right-continuous filtration making τ a stopping time, and $\mathbb{G}$ is the smallest right-continuous filtration containing $\mathbb{F}$ and making τ a stopping time).

We recall the result obtained in Subsection 2.2.1: if Y is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process $Y^{\mathbb{F}}$, called the predefault-value of Y, such that $1\!\!1_{\{t<\tau\}} Y_t = 1\!\!1_{\{t<\tau\}} Y^{\mathbb{F}}_t$.

For a general random time τ, it is not true that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales. Here is an example: due to the separability of the Brownian filtration, there exists a bounded random variable τ such that $\mathcal{F}_\infty = \sigma(\tau)$. Hence, $\mathcal{F}^\tau_{\tau+t} = \mathcal{F}_\infty, \forall t$ so that the $\mathbb{G}$-martingales are constant after τ. Consequently, $\mathbb{F}$-martingales are not $\mathbb{G}$-semi-martingales.

In this chapter, we study

- the $\mathbb{G}$ semi-martingale decomposition of $\mathbb{F}$ martingales stopped at τ
- pseudo honest times
- Honest times and the $\mathbb{G}$ semi-martingale decomposition of $\mathbb{F}$ martingales
- Arbitrage opportunities

The study of initial and equivalent times is deferred to the following chapters. The study of the particular and important case of last passage times is presented in Chapter 8.10.

We recall the two important conditions that we shall sometimes assume (see Lemma1.4.9)
(C) All $\mathbb{F}$-martingales are continuous
(A) τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for any $\mathbb{F}$-stopping time ϑ.

We recall our notation

$$\boxed{\mathbb{F} \subset \mathbb{G} = \mathbb{F} \vee \mathbb{H} \subset \mathbb{F}^{(\tau)} = \mathbb{F} \vee \sigma(\tau)}$$

8.1 Two Important Supermartingales

We introduce the Azéma supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ and call it sometimes the conditional survival process. The process Z is a super-martingale of class (D). Therefore, it admits a Doob-Meyer decomposition. We recall that the process $A^p = A^{p,\mathbb{F}}$ is the $\mathbb{F}$-predictable compensator of H, see Definition 1.4.6.

Lemma 8.1.1 *Let τ be a positive random time and*

$$Z_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = \mu_t - A_t^p$$

the Doob-Meyer decomposition of the super-martingale Z. Then, for any $\mathbb{F}$-predictable positive process Y,

$$\begin{aligned} \mathbb{E}(Y_\tau) &= \mathbb{E}\left(\int_0^\infty Y_u dA_u^p\right) \\ \mathbb{E}(Y_\tau \mathbb{1}_{t<\tau\leq T}|\mathcal{F}_t) &= \mathbb{E}\left(\int_t^T Y_u dA_u^p|\mathcal{F}_t\right) = -\mathbb{E}\left(\int_t^T Y_u dZ_u|\mathcal{F}_t\right) \end{aligned}$$

Proof: The first equality is a consequence of the definition of dual projection (see Proposition 1.4.7).
For any càglàd process Y of the form $Y_u = y_s \mathbb{1}_{]s,t]}(u)$ with $y_s \in b\mathcal{F}_s$, one has

$$\mathbb{E}(Y_\tau) = \mathbb{E}(y_s \mathbb{1}_{]s,t]}(\tau)) = \mathbb{E}(y_s(A_t - A_s)).$$

The result follows from MCT. □

Another important $\mathbb{F}$-supermartingale is

$$\widetilde{Z}_t := P\left(\tau \geq t \,\middle|\, \mathcal{F}_t\right). \tag{8.1.1}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $\mathbb{1}_{]\!]0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $\mathbb{1}_{]\!]0,\tau]\!]}$. An optional decomposition of Z leads to an important $\mathbb{F}$-martingale m, given by

$$\boxed{m := Z + A^{o,\mathbb{F}}} \tag{8.1.2}$$

where $A^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of H. The supermartingales Z and $\widetilde{Z}$ are related through $\tilde{Z} = Z + \Delta A^{o,\mathbb{F}}$ and $\widetilde{Z} = Z_- + \Delta m$.

The following results (see Lemma 1.4.9 and [113]) will be important

- If assumption (C) or (A) is satisfied, then $Z = \widetilde{Z}$.
- Under assumptions (C) and (A), the supermartingale $Z = \widetilde{Z}$ is a continuous process.
- Under (C), $A^p = A^o$

Note that $m_t = \mathbb{E}(A_\infty^{o,\mathbb{F}}|\mathcal{F}_t)$ and, for any $\mathbb{F}$ uniformly integrable martingale n, $E(n_\tau) = \mathbb{E}(n_\infty m_\infty)$. Indeed, one has

$$\mathbb{E}(n_\tau) = \mathbb{E}\left(\int_0^\infty n_s dA_s^{o,\mathbb{F}}\right) = \mathbb{E}(n_\infty (A_\infty^{o,\mathbb{F}}) = \mathbb{E}(n_\infty m_\infty)$$

where the second equality comes from Yoeurp's lemma 1.2.11.

If $R := \inf\{t : Z_t = 0\}$, then $R = \inf\{t : \widetilde{Z}_t = 0\} = \inf\{t : Z_{t-} = 0\}$ and $\tau \le R$.

Comment 8.1.2 The process μ is a square integrable martingale. Indeed, from Doob-Meyer decomposition, since Z is bounded, μ is a square integrable martingale.

8.2 General Facts

For what concerns the progressive enlargement setting, the following result is analogous to Proposition 6.1.1. This results can be found in Jeulin [82, Lemma 4.4].

Proposition 8.2.1 *One has*

(i) A random variable Y_t is $\mathcal{G}_t$-measurable if and only if it is of the form

$$Y_t(\omega) = \tilde{y}_t(\omega)\mathbb{1}_{t<\tau(\omega)} + \hat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)\le t}$$

for some $\mathcal{F}_t$-measurable random variable $\tilde{y}_t$ and some family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variables $\hat{y}_t(\cdot, u), t \ge u$.

(ii) A process Y is $\mathbb{G}$-predictable if and only if it is of the form

$$Y_t(\omega) = \tilde{y}_t(\omega)\mathbb{1}_{t\le\tau(\omega)} + \hat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)<t}, t \ge 0,$$

where $\tilde{y}$ is $\mathbb{F}$-predictable and $(t, \omega, u) \mapsto \hat{y}_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.

PROOF: For part (i), it suffices to recall that $\mathcal{G}_t$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega)f\big(t \wedge \tau(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f a bounded Borel function on $\mathbb{R}^+$.

(ii) It suffices to notice that $\mathbb{G}$-predictable processes are generated by processes of the form $X_t = x_t\mathbb{1}_{t\le\tau} + \hat{x}_t f(\tau)\mathbb{1}_{\tau<t}$, $t \ge 0$, where $x, \hat{x}$ are $\mathbb{F}$-predictable and f is a bounded Borel function, defined on $\mathbb{R}^+$. □

Such a characterization result does *not* hold for optional processes, in general. We refer to Barlow [18, Remark on pages 318 and 319], for a counterexample (see also Example 8.8.10). See Song [125] for a general study.

Proposition 8.2.2 *For any $\mathbb{G}$-predictable process Y, there exists an $\mathbb{F}$-predictable process y such that $Y_t\mathbb{1}_{\{t\le\tau\}} = y_t\mathbb{1}_{\{t\le\tau\}}$. Under the condition $\forall t, \mathbb{P}(\tau \le t|\mathcal{F}_t) < 1$, the process $(y_t, t \ge 0)$ is unique.*

PROOF: We refer to Dellacherie [45] and Dellacherie et al. [41, p.186]. The process y may be recovered as the ratio of the $\mathbb{F}$-predictable projections of $Y_t\mathbb{1}_{\{t\le\tau\}}$ and $\mathbb{1}_{\{t\le\tau\}}$. □

Lemma 8.2.3 ***Key Lemma:*** *Let $X \in \mathcal{F}_T$ be an integrable r.v. Then, for any $t \le T$,*

$$\boxed{\mathbb{E}(X\mathbb{1}_{\{\tau<T\}}|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}(XZ_T|\mathcal{F}_t)}{Z_t}}$$

PROOF: On the set $\{t < \tau\}$, any $\mathcal{G}_t$ measurable random variable is equal to an $\mathcal{F}_t$-measurable random variable, therefore

$$\mathbb{E}(X 1\!\!1_{\{\tau<T\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} y_t$$

where y_t is $\mathcal{F}_t$-measurable. Taking conditional expectation w.r.t. $\mathcal{F}_t$, we get $y_t = \dfrac{\mathbb{E}(Y_t 1\!\!1_{\{t<\tau\}}|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}$. (it can be proved that $\mathbb{P}(t < \tau|\mathcal{F}_t)$ does not vanish on the set $\{t < \tau\}$, see the following Exercise 8.2.5.) □

Exercise 8.2.4 Prove that, if τ is an $\mathbb{F}$ stopping time, $\mathbb{G} = \mathbb{F}$. ◁

Exercise 8.2.5 Prove that

$$\{\tau > t\} \subset \{Z_t > 0\} \tag{8.2.1}$$

(where the inclusion is up to a negligible set). ◁

Proposition 8.2.6 *The $\mathbb{G}$-adapted process Y*

$$Y_t = y_t 1\!\!1_{t<\tau} + y_t(\tau) 1\!\!1_{\tau\leq t}$$

is a martingale if for any u, $(y_t(u), t \geq u)$ is a martingale and if $\mathbb{E}(y_t|\mathcal{F}_t)$ is a $\mathbb{F}$-martingale.

See [8].

8.3 Before τ

It is proved in Yor [133] that, if X is an $\mathbb{F}$-martingale, then the processes $(X_{t\wedge\tau}, t \geq 0)$ and $(X_t(1 - H_t), t \geq 0)$ are $\mathbb{G}$ semi-martingales. Furthermore, the decompositions of the $\mathbb{F}$-martingales in the filtration $\mathbb{G}$ are known up to time τ (Jeulin and Yor [83]).

Proposition 8.3.1 *Under (CA), every $\mathbb{F}$-martingale X stopped at time τ is a $\mathbb{G}$-semi-martingale with canonical decomposition*

$$\boxed{X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X, \mu\rangle_s}{Z_s}}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

PROOF: Let Y_s be an $\mathcal{G}_s$-measurable random variable. There exists an $\mathcal{F}_s$-measurable random variable y_s such that $Y_s 1\!\!1_{\{s<\tau\}} = y_s 1\!\!1_{\{s<\tau\}}$, hence, if X is an $\mathbb{F}$-martingale, for $s < t$,

$$\begin{aligned}
&\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) = \mathbb{E}(Y_s 1\!\!1_{\{s<\tau\}}(X_{t\wedge\tau} - X_{s\wedge\tau})) \\
&= \mathbb{E}(y_s 1\!\!1_{\{s<\tau\}}(X_{t\wedge\tau} - X_{s\wedge\tau})) \\
&= \mathbb{E}\left(y_s(1\!\!1_{\{s<\tau\leq t\}}(X_\tau - X_s) + 1\!\!1_{\{t<\tau\}}(X_t - X_s))\right)
\end{aligned}$$

From the definition of Z (see also Definition 1.4.6 and Lemma 8.1.1),

$$\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\leq t\}} X_\tau\right) = -\mathbb{E}\left(y_s \int_s^t X_u dZ_u\right).$$

From integration by parts formula (taking into account the continuity of Z and X)

$$\int_s^t X_u dZ_u = -X_s Z_s + Z_t X_t - \int_s^t Z_u dX_u - \langle X, Z\rangle_t + \langle X, Z\rangle_s$$

We have also

$$\begin{aligned}\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\le t\}} X_s\right) &= \mathbb{E}\left(y_s X_s (Z_s - Z_t)\right)\\ \mathbb{E}\left(y_s 1\!\!1_{\{t<\tau\}} (X_t - X_s)\right) &= \mathbb{E}\left(y_s Z_t (X_t - X_s)\right)\end{aligned}$$

hence, from the martingale property of X

$$\begin{aligned}&\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) = \mathbb{E}(y_s(\langle X,\mu\rangle_t - \langle X,\mu\rangle_s))\\ &= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X,\mu\rangle_u}{Z_u} Z_u\right) = \mathbb{E}\left(y_s \int_s^t \frac{d\langle X,\mu\rangle_u}{Z_u} \mathbb{E}(1\!\!1_{\{u<\tau\}}|\mathcal{F}_u)\right)\\ &= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X,\mu\rangle_u}{Z_u^\tau} 1\!\!1_{\{u<\tau\}}\right) = \mathbb{E}\left(y_s \int_{s\wedge\tau}^{t\wedge\tau} \frac{d\langle X,\mu\rangle_u}{Z_u}\right).\end{aligned}$$

The result follows. □

The general result is more delicate:

Proposition 8.3.2 *Every $\mathbb{F}$-local martingale X stopped at time τ is a $\mathbb{G}$-semi-martingale with canonical decomposition*

$$\boxed{X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X,m\rangle_s}{Z_{s-}}}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

In other terms,

$$X_{t\wedge\tau} = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X,\mu\rangle_s + dJ_s}{Z_{s-}},$$

where J is the $\mathbb{F}$-dual predictable projection of the process $\Delta X_\tau 1\!\!1_{[\![\tau,\infty[\![}$. Another interesting decomposition is (see Aksamit [2]). Let us introduce the $\mathbb{F}$-stopping time R $R := \inf\{t : Z_t = 0\}$ and $\widetilde{R} = R_{\{\widetilde{Z}_R = 0 < Z_{R-}\}}$, where $R_A = R 1\!\!1_A + \infty 1\!\!1_{A^c}$. Then, if X is an $\mathbb{F}$-local martingale, the process

$$X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[m,X]_s + (\Delta X_{\widetilde{R}} 1\!\!1_{[\![\widetilde{R},\infty[\![})^{p,\mathbb{F}}_{t\wedge\tau}, \quad t \ge 0$$

is a $\mathbb{G}$-local martingale.

This result remains valid for any filtration $\mathbb{G}$ that coincide with $\mathbb{F}$ before τ.

8.4 Basic Results

We recall the results obtained in Proposition 2.2.7:

Proposition 8.4.1 *a) The process*

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_u^p}{Z_{u-}}, \quad t \ge 0$$

is a $\mathbb{G}$-martingale.
b) For any bounded $\mathbb{G}$-predictable process Y, the process

$$Y_\tau 1\!\!1_{\tau\le t} - \int_0^{t\wedge\tau} \frac{Y_s}{Z_{s-}} dA_s^p, \quad t \ge 0$$

is a $\mathbb{G}$-martingale.
c) The process $L_t := (1-H_t)/Z_t, \quad t \ge 0$ is a $\mathbb{G}$ -martingale.

Definition 8.4.2 *In the case where the process A^p is absolutely continuous w.r.t. Lebesgue's measure, i.e., $dA^p_t = a_t dt$, the process $\lambda_t = \frac{a_t}{Z_{t-}}$ is called the $\mathbb{F}$-intensity of τ, the process $\lambda^{\mathbb{G}}_t = 1\!\!1_{t<\tau}\lambda_t$ is the $\mathbb{G}$-intensity, and the process*

$$H_t - \int_0^{t\wedge\tau} \lambda_s ds = H_t - \int_0^t (1-H_s)\lambda_s ds = H_t - \int_0^t \lambda^{\mathbb{G}}_s ds, \quad t \geq 0$$

is a $\mathbb{G}$-martingale.

We also recall

Lemma 8.4.3 *The process λ satisfies*

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h|\mathcal{F}_t)}{\mathbb{P}(t < \tau|\mathcal{F}_t)} .$$

The converse is known as Aven's lemma [15].

Lemma 8.4.4 *Let $(\Omega, \mathbb{G}, \mathbb{P})$ be a filtered probability space and N be a counting process. Assume that $E(N_t) < \infty$ for any t. Let $(h_n, n \geq 1)$ be a sequence of real numbers converging to 0, and*

$$Y^{(n)}_t = \frac{1}{h_n} E(N_{t+h_n} - N_t|\mathcal{G}_t)$$

Assume that there exists λ and y non-negative $\mathbb{G}$-adapted processes such that

(i) For any t, $\lim Y^{(n)}_t = \lambda_t$

(ii) For any t, there exists for almost all ω an $n_0 = n_0(t,\omega)$ such that

$$|Y^{(n)}_s(\omega) - \lambda_s(\omega)| \leq y_s(\omega)\,, \ s \leq t, \, n \geq n_0(t,\omega)$$

(iii) $\int_0^t y_s ds < \infty, \forall t, a.s.$

Then, $N_t - \int_0^t \lambda_s ds$ is a $\mathbb{G}$-martingale.

Suppose from now on that a second filtration $\widetilde{\mathbb{F}}$ is given, with $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$ and define the associated σ-algebra $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ and the $\widetilde{\mathbb{F}}$ Azéma super-martingale

$$\widetilde{Z}_t = \mathbb{P}(t < \tau|\widetilde{\mathcal{F}}_t) = \mathbb{E}(Z_t|\widetilde{\mathcal{F}}_t)\,.$$

Let $Z_t = \mu_t - A^p_t$ be the $\mathbb{F}$-Doob-Meyer decomposition of the $\mathbb{F}$-supermartingale Z and assume that A^p is absolutely continuous with respect to Lebesgue's measure: $A^p_t = \int_0^t a_s ds$. The process $\widetilde{A}$ defined as $\widetilde{A}_t := \mathbb{E}(A_t|\widetilde{\mathcal{F}}_t)$ is an $\widetilde{\mathbb{F}}$-submartingale and its $\widetilde{\mathbb{F}}$-Doob-Meyer decomposition is denoted

$$\widetilde{A}_t = \widetilde{n}_t + \widetilde{\alpha}_t\,.$$

where $\widetilde{n}$ is the $\widetilde{\mathbb{F}}$-martingale part and, from Exercise 1.6.4, $\widetilde{\alpha}_t = \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$. Hence, setting $\widetilde{\mu}_t = \mathbb{E}(\mu_t|\widetilde{\mathcal{F}}_t)$, the super-martingale $\widetilde{Z}$ admits a $\widetilde{\mathbb{F}}$-Doob-Meyer decomposition as

$$\widetilde{Z}_t = \widetilde{\mu}_t - \widetilde{n}_t - \widetilde{\alpha}_t$$

where $\widetilde{\mu} - \widetilde{n}$ is the $\widetilde{\mathbb{F}}$-martingale part. It follows that

$$H_t - \int_0^{t\wedge\tau} \frac{d\widetilde{\alpha}_s}{\widetilde{Z}_s} ds = H_t - \int_0^{t\wedge\tau} \frac{\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)}{\widetilde{Z}_s} ds, \quad t \geq 0$$

is a $\widetilde{\mathbb{G}}$-martingale and that the $\widetilde{\mathbb{F}}$-intensity of τ is equal to $\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{Z}_s$, and not "as one could think" to $\mathbb{E}(a_s/Z_s|\widetilde{\mathcal{F}}_s)$.

This result can be proved directly thanks to Brémaud's following result (a consequence of Exercise 1.6.4): if $H_t - \int_0^t \widetilde{\lambda}_s^{\mathbb{G}} ds$ is a $\mathbb{G}$-martingale, then $H_t - \int_0^t \mathbb{E}(\widetilde{\lambda}_s^{\mathbb{G}}|\widetilde{\mathcal{G}}_s)ds$ is a $\widetilde{\mathbb{G}}$-martingale. Since

$$\begin{aligned}
\mathbb{E}(\widetilde{\lambda}_s^{\mathbb{G}}|\mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s^{\mathbb{F}}|\widetilde{\mathcal{G}}_s) &= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{Z}_s}\mathbb{E}(\mathbb{1}_{\{s\leq\tau\}}\lambda_s^{\mathbb{F}}|\widetilde{\mathcal{F}}_s) \\
&= \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{Z}_s}\mathbb{E}(Z_s\lambda_s^{\mathbb{F}}|\widetilde{\mathcal{F}}_s) = \frac{\mathbb{1}_{\{s\leq\tau\}}}{\widetilde{Z}_s}\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)
\end{aligned}$$

it follows that $H_t - \int_0^{t\wedge\tau} \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{Z}_s ds$ is a $\widetilde{\mathbb{G}}$-martingale, and we are done.

Exercise 8.4.5 Prove that if X is a (square-integrable) $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale, where L is defined in Proposition 8.4.1. $\lhd$

Exercise 8.4.6 We consider, as in the paper of Biagini et al. [23] a mortality bond, a financial instrument with payoff $Y = \int_0^{\tau\wedge T} Z_s ds$, where $Z_s = \mathbb{P}(\tau > s|\mathcal{F}_s)$ where $\mathbb{F}$ is a continuous filtration. We assume that Z is continuous, admits a Doob-Meyer decomposition as $Z = \mu - A$ and does not vanish.

1. Compute, in the case $r = 0$, the price Y_t of the mortality bond. It will be convenient to introduce $N_t = \mathbb{E}(\int_0^T Z_s^2 ds|\mathcal{F}_t)$. Is the process N a $(\mathbb{P},\mathbb{F})$ martingale? a $(\mathbb{P},\mathbb{G})$-martingale?
2. Determine the processes α, β and γ so that
$$dY_t = \alpha_t dM_t + \beta_t(dN_t - \frac{1}{Z_t}d\langle N, Z\rangle_t) + \gamma_t(dZ_t - \frac{1}{Z_t}d\langle Z\rangle_t)$$
3. Determine the price $D(t,T)$ of a defaultable zero-coupon bond with maturity T, i.e., a financial asset with terminal payoff $\mathbb{1}_{T<\tau}$. Give the dynamics of this price.
4. We now assume that $\mathbb{F}$ is a Brownian filtration, and that a risky asset with dynamics
$$dS_t = S_t(bdt + \sigma dW_t)$$
is traded. Explain how one can hedge the mortality bond.

$\lhd$

8.5 Multiplicative Decomposition of the Azéma supermartingale

Lemma 8.5.1 *Assume that (CA) holds and that the super-martingale Z does not vanish. Then, Z admits a multiplicative decomposition as $Z_t = N_t e^{\Gamma_t}$ where Γ is an increasing $\mathbb{F}$-predictable process and N a local $\mathbb{F}$-martingale. Moreover $(H_t - \Gamma_{t\wedge\tau},\, t \geq 0)$ is a $\mathbb{G}$-martingale.*

PROOF: The proof was done in Lemma 1.1.17 $\square$

Lemma 8.5.2 *Assume that the super-martingale Z does not vanish and let $Z_t = N_t D_t$ its multiplicative decomposition. Then, $H_t - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale, where $\Lambda_t = \int_0^t \frac{1}{D_{t-}} dD_t$.*

PROOF: We start with the result in the proof of Lemma 1.1.17, and deduce from (1.1.3) that $\frac{dA^p_t}{Z_{t-}} = \frac{1}{D_{t-}} dD_t$. Note that, if we write $D = e^{-\Gamma}$, then

$$dD_t = -D_{t-}\left(-d\Gamma_t + (e^{-\Delta\Gamma_t} - 1 - \Delta\Gamma_t)\right) .$$

□

See Kardaras for a useful optional decomposition of Z.

8.6 Construction of Random Time with Given Intensity

In this section, we are interested with the following problem: let Λ be a given continuous increasing process. The Cox process modeling provides a construction of τ such that Λ is the compensator of H. Is it possible to have a different construction with the same property? We can do that, as soon as one can construct a random time τ such that the multiplicative decomposition of the Azéma supermartingale is $N_t e^{-\Lambda_t}$. We shall give some constructions and we refer the reader to [80] where there are infinitely many possibilities enjoying the property that $\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t}$. The same problem is studied in Li and Rutkowski [103]. The case where Λ is not continuous is studied in Song [124].

In a first step, using a change of probability measure framework, a local martingale N and an absolutely continuous increasing process Λ being given (such that $0 < N_t e^{-\Lambda_t} < 1$ for $t > 0$ and $N_0 = 1$), and τ being constructed as in the Cox process model with intensity λ, we construct a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$ such that $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}(\tau > t|\mathcal{F}_t) = Z_t = N_t e^{-\Lambda_t}$. This will imply that the $\mathbb{Q}$ intensity of τ remains λ, but immersion fails to hold under $\mathbb{Q}$.

Proposition 8.6.1 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a given filtered probability space, where $\mathbb{F}$ is a Brownian filtration. Assume that N is a continuous $(\mathbb{P}, \mathbb{F})$-local martingale and Λ an absolutely continuous $\mathbb{F}$-adapted increasing process such that $0 < N_t e^{-\Lambda_t} < 1$ for $t > 0$, $N_0 = 1$. Let $\tau := \inf\{t : \Lambda_t > \Theta\}$ where Θ is a unit exponential r.v. independent form $\mathbb{F}$. Then, there exists a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$, which satisfies $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$) and $\mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t}$.*

PROOF: Let $\Lambda_t = \int_0^t \lambda_u du$. We are looking for conditional probabilities with a particular form (the idea is linked with the results obtained in Subsection 7.2.5). From the Cox construction, $\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t}$.
We shall prove that there exists a $\mathbb{G}$-martingale L of the form

$$L_t = \ell_t 1\!\!1_{t<\tau} + \ell_t(\tau) 1\!\!1_{\tau \le t}$$

and satisfying the condition of Proposition 8.2.6 such that, setting $d\mathbb{Q} = L d\mathbb{P}$
(i) $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$
(ii) $\mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t}$

It is not difficult to check that L is a $\mathbb{G}$-martingale if $m^Y_t := \mathbb{E}(L_t|\mathcal{F}_t)$ is an $\mathbb{F}$ martingale and, for any u, $\ell(u)$ is a family of $\mathbb{F}$ martingales: indeed, in that case, for $s < t$

$$\mathbb{E}(L_t|\mathcal{G}_s) = \mathbb{E}(\ell_t 1\!\!1_{\{\tau>t\}}|\mathcal{G}_s) + \mathbb{E}(\ell_t(\tau) 1\!\!1_{\{s<\tau\le t\}}|\mathcal{G}_s) + \mathbb{E}(\ell_t(\tau) 1\!\!1_{\tau \le s}|\mathcal{G}_s) = I_1 + I_2 + I_3.$$

For I_1 and I_2, we apply the Key Lemma, and we set $Z_t = e^{-\Lambda_t}$

$$I_1 + I_2 = 1\!\!1_{\tau>s} \frac{1}{Z_s} \mathbb{E}(\ell_t Z_t|\mathcal{F}_s) + 1\!\!1_{\tau>s} \frac{1}{Z_s} \mathbb{E}(\ell_t(\tau) 1\!\!1_{s<\tau\le t}|\mathcal{F}_s),$$

whereas for I_3, we obtain

$$I_3 = \mathbb{E}(\ell_t(\tau) 1\!\!1_{\tau \le s}|\mathcal{G}_s) = 1\!\!1_{\tau \le s}\mathbb{E}(\ell_t(u)|\mathcal{F}_s)_{u=\tau} = 1\!\!1_{\tau \le s}\mathbb{E}(\ell_s(u)|\mathcal{F}_s)_{u=\tau} = 1\!\!1_{\tau \le s}\ell_s(\tau),$$

where the first equality holds under the H-hypothesis and the second follows from the martingale property of $\ell(u)$.
It remains to prove that $I_1 + I_2 = \ell_s \mathbb{1}_{\tau>s}$. Since

$$\mathbb{E}(L_t|\mathcal{F}_t) = \mathbb{E}(\ell_t Z_t + \ell_t(\tau)\mathbb{1}_{\tau\leq t}|\mathcal{F}_t) = \ell_t Z_t + \int_0^t \ell_t(u)\lambda_u e^{-\Lambda_u} du$$

is a martingale, we see that

$$\mathbb{E}(\ell_t Z_t|\mathcal{F}_s) + \mathbb{E}(\ell_t(\tau)\mathbb{1}_{\tau\leq t}|\mathcal{F}_s) - \mathbb{E}(\ell_s(\tau)\mathbb{1}_{\tau\leq s}|\mathcal{F}_s) = \ell_s Z_s.$$

Therefore,

$$I_1 + I_2 = \mathbb{1}_{\tau>s}\frac{1}{Z_s}\Big(\ell_s Z_s + \mathbb{E}((\ell_s(\tau) - \ell_t(\tau))\mathbb{1}_{\tau\leq s}|\mathcal{F}_s)\Big) = \ell_s \mathbb{1}_{\tau>s},$$

where the last equality holds since

$$\mathbb{E}((\ell_s(\tau) - \ell_t(\tau))\mathbb{1}_{\tau\leq s}|\mathcal{F}_s) = \mathbb{1}_{\tau\leq s}\mathbb{E}((\ell_s(u) - \ell_t(u))|\mathcal{F}_s)_{u=\tau} = 0.$$

For the last equality in the formula above, we have again used the martingale property of $\ell(u)$.

(This result is a particular case of Proposition 9.3.2)

The condition (i) is satisfied if $1 = \mathbb{E}(L_t|\mathcal{F}_t)$. Then

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbb{E}(\mathbb{1}_{\tau>t} L_t|\mathcal{F}_t) = \mathbb{E}(\mathbb{1}_{\tau>t}\ell_t|\mathcal{F}_t) = \ell_t Z_t$$

is equal to NZ if (and only if) $\ell = N$. We chose $\ell_t(t) = \ell_t$ (this is a particular choice). We are now reduced to find a family of martingales $\ell_t(u), t \geq u$ such that

$$\ell_u(u) = N_u,\; 1 = N_t e^{-\Lambda_t} + \int_0^t \ell_t(u)\lambda_u e^{-\Lambda_u} du$$

We restrict our attention to families ℓ of the form

$$\ell_t(u) = X_t Y_u,\; t \geq u$$

where X is an $\mathbb{F}$ martingale such that

$$X_t Y_t = N_t,\;\; 1 = N_t e^{-\Lambda_t} + X_t \int_0^t Y_u \lambda_u e^{-\Lambda_u} du\,.$$

It is easy to show that

$$Y_t = Y_0 + \int_0^t e^{\Lambda_u} d(\frac{1}{X_u})$$

In a Brownian filtration case, there exists a process ν such that $dN_t = \nu_t N_t dW_t$ and the positive martingale X is of the form $dX_t = x_t X_t dW_t$. Then, using the fact that integration by parts implies

$$d(X_t Y_t) = Y_t dX_t - e^{\Lambda_t}\frac{1}{X_t} dX_t = x_t(X_t Y_t - e^{\Lambda_t})dW_t = dN_t\,,$$

we are lead to choose

$$x_t = \frac{\nu_t Z_t}{Z_t - 1}$$

We now present a more general methodology presented in [80]. We construct a family of martingales $G_t(u)$, valued in $]0,1[$, such that $G_t(t) = Z_t = N_t e^{-\Lambda_t}$ and $G_t(\cdot)$ is decreasing. Then, one can construct a probability $\mathbb{Q}$ on a product space such that $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}(\tau > u|\mathcal{F}_t) = G_t(u)$. From the conditional probability, one can deduce a density process, hence one can construct a random time admitting $G_t(u)$ as conditional probability. See also [103] for related results.

Proposition 8.6.2 *Let $0 < \theta < \infty$ be fixed and consider the process defined for $\theta \le t \le \infty$*

$$G_t(\theta) := (1 - Z_t) \exp \left\{ - \int_\theta^t \frac{Z_s}{1 - Z_s} d\Lambda_s \right\}$$

Then, the process $(G_t(\theta), \theta \le t \le \infty)$ is a $(\mathbb{P}, \mathbb{F})$ uniformly integrable martingale.

PROOF: Applying the integration by parts formula on $G(\theta)$, for $\theta \le t < \infty$, one gets

$$dG_t(\theta) \quad = \quad -\exp \left\{ - \int_\theta^t \frac{Z_s \, d\Lambda_s}{1 - Z_s} \right\} e^{-\Lambda_t} dN_t$$

hence, $G(\theta)$ is a $(\mathbb{P}, \mathbb{F})$ local martingale on $[\theta, \infty)$. Being clearly positive and bounded by 1, it is a uniformly integrable martingale on $[\theta, \infty]$. □

It is then possible to construct a random time τ admitting $G_t(u)$ as conditional probability.

We illustrate this construction in the Gaussian example presented in Section 6.4.4 where we set $Y_t = \frac{m_t - h(t)}{\sigma(t)}$. The multiplicative decomposition of the supermartingale $Z_t = P(\tau > t | \mathcal{F}_t^B)$ is $Z_t = N_t \exp \left(- \int_0^t \lambda_s ds \right)$ where

$$dN_t = N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h'(t) \, \varphi(Y_t)}{\sigma(t) \, \Phi(Y_t)}.$$

Using the fact that $Z_t = \Phi(Y_t)$, one checks that the basic martingale survival process satisfies

$$dG_t(\theta) = (1 - G_t(\theta)) \frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t, \, t \ge \theta, \quad G_\theta(\theta) = \Phi(Y_\theta)$$

which provides a new example of martingale survival processes, with density process

$$g_t(\theta) = (1 - Z_t) e^{- \int_\theta^t \frac{Z_s}{1 - Z_s} \lambda_s ds} \frac{Z_\theta \lambda_\theta}{1 - Z_\theta}, \, \theta \le t.$$

Other constructions of martingale survival processes having a given survival process can be found in [80], as well as constructions of local-martingales N such that $Ne^{-\Lambda}$ is valued in $[0, 1]$ for a given increasing continuous process Λ.

8.7 Pseudo-stopping Times

As we have mentioned, if $\mathbb{F}$ is immersed in $\mathbb{G}$, the process $(Z_t, t \ge 0)$ is a decreasing process. The converse is not true. The decreasing property of Z is closely related with the definition of pseudo-stopping timnotion developed by Nikeghbali and Yor [114], from D. Williams example (see Example 8.7.3 below).

Definition 8.7.1 *A random time τ is a pseudo-stopping time if, for any bounded $\mathbb{F}$-martingale M, $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.*

Proposition 8.7.2 *The random time τ is a pseudo-stopping time if and only if one of the following equivalent properties holds:*
(i) For any local $\mathbb{F}$-martingale M, the process $(M_{t \wedge \tau}, t \ge 0)$ is a local $\mathbb{G}$-martingale
(ii)$A^p_\infty = 1$,
(iii) $\mu_t = 1$, $\forall t \ge 0$,
(iv) The process Z is a decreasing $\mathbb{F}$-predictable process.

PROOF: The implication $(iv) \Rightarrow (i)$ is a consequence of Jeulin result established in Theorem 8.3.1. The implication $(i) \Rightarrow (ii)$ follows from the properties of the compensator A^τ: indeed

$$\mathbb{E}(M_\tau) = \mathbb{E}(\int_0^\infty M_u dA^p_u) = \mathbb{E}(M_\infty A^p_\infty) = m_0$$

implies that $A^p_\infty = 1$. We refer to Nikeghbali and Yor [114]. □

Example 8.7.3 The first example of a pseudo-stopping time was given by Williams [129]. Let B be a Brownian motion and define the stopping time $T_1 = \inf\{t : B_t = 1\}$ and the random time $\vartheta = \sup\{t < T_1 : B_t = 0\}$. Set

$$\tau = \sup\{s < \theta : B_s = B^*_s\}$$

where B^* is the running maximum of the Brownian motion. Then, τ is a pseudo-stopping time. Note that $\mathbb{E}(B_\tau)$ is not equal to 0; this illustrates the fact we cannot take any martingale in Definition 8.7.1. The martingale $(B_{t\wedge T_1}, t \geq 0)$ is neither bounded, nor uniformly integrable. In fact, since the maximum B^*_θ $(=B_\tau)$ is uniformly distributed on $[0,1]$, one has $\mathbb{E}(B_\tau) = 1/2$.

Pseudo stopping times are not stable by change of probability. See Aksamit [2] and Kreher [97] for a related study.

Example Let W be a Brownian motion and let $\tau = \sup\{t \leq 1 : W_1 - 2W_t = 0\}$, that is the last time before 1 when the Brownian motion is equal to half of its terminal value at time 1. Then,

$$\{\tau \leq t\} = \left\{\inf_{t\leq s\leq 1} 2W_s \geq W_1 \geq 0\right\} \cup \left\{\sup_{t\leq s\leq 1} 2W_s \leq W_1 \leq 0\right\}.$$

► The quantity

$$\mathbb{P}(\tau \leq t, W_1 \geq 0|\mathcal{F}_t) = \mathbb{P}\left(\inf_{t\leq s\leq 1} 2W_s \geq W_1 \geq 0|\mathcal{F}_t\right)$$

can be evaluated using the equalities

$$\begin{aligned}\left\{\inf_{t\leq s\leq 1} W_s \geq \frac{W_1}{2} \geq 0\right\} &= \left\{\inf_{t\leq s\leq 1}(W_s - W_t) \geq \frac{W_1}{2} - W_t \geq -W_t\right\}\\ &= \left\{\inf_{0\leq u\leq 1-t}(\widetilde{W}_u) \geq \frac{\widetilde{W}_{1-t}}{2} - \frac{W_t}{2} \geq -W_t\right\},\end{aligned}$$

where $(\widetilde{W}_u = W_{t+u} - W_t, u \geq 0)$ is a Brownian motion independent of $\mathcal{F}_t$. It follows that

$$\mathbb{P}\left(\inf_{t\leq s\leq 1} W_s \geq \frac{W_1}{2} \geq 0|\mathcal{F}_t\right) = \Psi(1-t, W_t),$$

where

$$\begin{aligned}\Psi(s,x) &= \mathbb{P}\left(\inf_{0\leq u\leq s} \widetilde{W}_u \geq \frac{\widetilde{W}_s}{2} - \frac{x}{2} \geq -x\right) = \mathbb{P}\left(2M_s - W_s \leq \frac{x}{2}, W_s \leq \frac{x}{2}\right)\\ &= \mathbb{P}\left(2M_1 - W_1 \leq \frac{x}{2\sqrt{s}}, W_1 \leq \frac{x}{2\sqrt{s}}\right).\end{aligned}$$

► The same kind of computation leads to

$$\mathbb{P}\left(\sup_{t\leq s\leq 1} 2W_s \leq W_1 \leq 0|\mathcal{F}_t\right) = \Psi(1-t, -W_t).$$

► The quantity $\Psi(s,x)$ can now be computed from the joint law of the maximum and of the process at time 1; however, we prefer to use Pitman's theorem (see [3M]): let $\widetilde{U}$ be a r.v. uniformly distributed on $[-1,+1]$ independent of $R_1 := 2M_1 - W_1$, then

$$\begin{aligned}\mathbb{P}(2M_1 - W_1 \le y, W_1 \le y) &= \mathbb{P}(R_1 \le y, \widetilde{U}R_1 \le y)\\ &= \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \le y, uR_1 \le y)du\,.\end{aligned}$$

For $y > 0$,

$$\frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \le y, uR_1 \le y)du = \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \le y)du = \mathbb{P}(R_1 \le y)\,.$$

For $y < 0$

$$\int_{-1}^{1} \mathbb{P}(R_1 \le y, uR_1 \le y)du = 0\,.$$

Therefore

$$\mathbb{P}(\tau \le t|\mathcal{F}_t) = \Psi(1-t, W_t) + \Psi(1-t, -W_t) = \rho\left(\frac{|W_t|}{\sqrt{1-t}}\right)$$

where

$$\rho(y) = \mathbb{P}(R_1 \le y) = \sqrt{\frac{2}{\pi}}\int_0^y x^2 e^{-x^2/2}dx\,.$$

Then $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = 1 - \rho(\frac{|W_t|}{\sqrt{1-t}})$. We can now apply Tanaka's formula to the function ρ. Noting that $\rho'(0) = 0$, the contribution to the Doob-Meyer decomposition of Z of the local time of W at level 0 is 0. Furthermore, the increasing process A of the Doob-Meyer decomposition of Z is given by

$$\begin{aligned}dA_t &= \left(\frac{1}{2}\rho''\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{1}{1-t} + \frac{1}{2}\rho'\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{|W_t|}{\sqrt{(1-t)^3}}\right)dt\\ &= \frac{1}{1-t}\frac{|W_t|}{\sqrt{1-t}}e^{-W_t^2/2(1-t)}dt\,.\end{aligned}$$

We note that A may be obtained as the dual predictable projection on the Brownian filtration of the process $A_s^{(W_1)}, s \le 1$, where $(A_s^{(x)}, s \le 1)$ is the compensator of τ under the law of the Brownian bridge $\mathbb{P}^{(1)}_{0\to x}$.

Comment 8.7.4 Note that the random time τ presented in this subsection is not the end of a predictable set, hence, is not honest. However, $\mathbb{F}$-martingales are semi-martingales in the progressive enlarged filtration: it suffices to note that $\mathbb{F}$-martingales are semi-martingales in the filtration initially enlarged with W_1.

We now follow the same idea and define another random time, more appropriate to Finance. Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion and σ a constant.

Let $\tau = \sup\{t \le 1 \,:\, S_1 - 2S_t = 0\}$, that is the last time before 1 when the price is equal to half of its terminal value at time 1.

Proposition 8.7.5 *In the above model NA holds before τ. There are classical arbitrages after τ.*

PROOF: Note that

$$\{\tau \le t\} = \{\inf_{t\le s\le 1} 2S_s \ge S_1\} = \{\inf_{t\le s\le 1} 2\frac{S_s}{S_t} \ge \frac{S_1}{S_t}\}$$

Since $\frac{S_s}{S_t}, s \geq t$ and $\frac{S_1}{S_t}$ are independent from $\mathcal{F}_t$, therefore

$$\mathbb{P}(\inf_{t \leq s \leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}|\mathcal{F}_t) = \mathbb{P}(\inf_{t \leq s \leq 1} 2S_{s-t} \geq S_{1-t}) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s \leq u} 2S_s \geq S_u)$. It follows that the Azéma super-martingale is a deterministic decreasing function, hence, τ is a pseudo-stopping time and S is a $\mathbb{G}$ martingale up to time τ and there are no arbitrages up to τ.

There are obviously arbitrages after τ, since, at time τ, one knows the value of S_1 and $S_1 > S_\tau$. In fact, for $t > \tau$, one has $S_t > S_\tau$, and the arbitrage occurs at any time before 1. □

Remark 8.7.6 It is not difficult to prove that $(\mathcal{H}')$ hypothesis holds for that example, even if τ is neither honest(see Section 8.8), does not admit a positive density (see Hypothesis **??**) and immersion is not satisfied.

8.8 Honest Times

There exists an interesting class of random times τ such that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales, called honest times, introduced by Meyer [112] and studied by Barlow [18] and Jeulin [82] among others.

8.8.1 Definition

Definition 8.8.1 *A random time τ is honest if for $s \leq t$*

$$\{\tau \leq s\} = F_{s,t} \cap \{\tau \leq t\}, \textit{ for some } F_{s,t} \in \mathcal{F}_t$$

or equivalently, if τ is equal to an $\mathcal{F}_t$-measurable random variable on $\tau < t$.

Examples 8.8.2 (i) Let B a Brownian motion and set $\tau = g_1$ where $g_t = \sup\{s < t \,:\, B_s = 0\}$. Then, for $t < 1$, $g_1 = g_t$ on $\{g_1 < t\}$, and g_t is $\mathcal{F}_t$-measurable.
(ii) Let X be an adapted continuous process and $X^* = \sup X_s, X^*_t = \sup_{s \leq t} X_s$. The random time

$$\tau = \sup\{s \,:\, X_s = X^*\}$$

is honest. Indeed, on the set $\{\tau < t\}$, one has $\tau = \sup\{s \,:\, X_s = X^*_s\}$.
(iii) An $\mathbb{F}$-stopping time is honest: indeed $\tau = \tau \wedge t$ on $\tau < t$.

If τ is honest,

$$\mathcal{G}_t = \{A \in \mathcal{F}_\infty, : \, A = (\widetilde{A}_t \cap \{\tau \leq t\}) \cup (\widehat{A}_t \cap \{\tau > t\}) \text{ for some } \widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$$

This filtration is continuous on right.

Exercise 8.8.3 Let τ be an honest time. Prove that

$$\mathbb{E}(f(\tau)|\mathcal{F}_t) = f(\tau)(1-Z_t) + \mathbb{E}(\int_t^\infty f(s) dA^p_s|\mathcal{F}_t)$$

◁

Exercise 8.8.4 Prove that $\mathcal{G}^*_t := \{A \in \mathcal{F}_\infty \,:\, A = (\widetilde{A}_t \cap \{\tau \leq t\}) \cup (\widehat{A}_t \cap \{\tau > t\}) \text{ for some } \widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$ defines indeed a filtration (i.e., the increasing property holds). ◁

8.8.2 Martingales

Proposition 8.8.5 *Let X be a càdlàg $\mathbb{G}$-adapted integrable process. Then X is a $\mathbb{G}$ martingale if and only if*
(i) $(\mathbb{E}(X_t|\mathcal{F}_t),\, t \geq 0)$ is an $\mathbb{F}$-martingale
(ii) For $s < t$, $\mathbb{E}(1\!\!1_{\tau \leq s} X_t|\mathcal{F}_s) = \mathbb{E}(1\!\!1_{\tau \leq s} X_s|\mathcal{F}_s)$.

PROOF: This easy proof is left to the reader. □

8.8.3 Stability

Let τ and τ^* be two honest times. We show in the following lemma that $\tau \vee \tau^*$ is an honest time.

Lemma 8.8.6 *Let τ and τ^* be two honest times, then $\tau \vee \tau^*$ times. We show in the following lemma that $\tau \vee \tau^*$ is an honest time.*

PROOF: The random time τ and τ^* are honest, this implies that for every $t \geq 0$ there exist $\mathcal{F}_t$ measurable random variables τ_t and τ_t^* such that

$$\tau 1\!\!1_{\tau < t} = \tau_t 1\!\!1_{\tau < t} \quad \text{and} \quad \tau^* 1\!\!1_{\tau^* < t} = \tau_t^* 1\!\!1_{\tau^* < t}$$

holds. Let us consider the random time $\tau \vee \tau^*$.

$$\tau \vee \tau^* 1\!\!1_{\tau \vee \tau^* < t} = \tau \vee \tau^* 1\!\!1_{\tau < t, \tau^* < t} = \tau_t \vee \tau_t^* 1\!\!1_{\tau < t, \tau^* < t} = \tau_t \vee \tau_t^* 1\!\!1_{\tau \vee \tau^* < t},$$

which proves that it is in fact honest time. □

8.8.4 Properties

Lemma 8.8.7 *(Azéma) Let τ be an honest time which avoids $\mathbb{F}$-stopping times. Then:*
(i) A^p_∞ has an exponential law with parameter 1.
(ii) The measure dA^p_t is carried by $\{t \,:\, Z_t = 1\}$
(iii) $\tau = sup\{t \,:\, 1 - Z_t = 1\}$
(iv)$A^p_\infty = A^p_\tau$

In particular, under (CA), $\Lambda_t = \int_0^t \frac{dA^p_s}{Z_s} = A^p_t$ (we have used (ii) above) and Λ_τ has an exponential law.

Proposition 8.8.8 *(Jeulin [82]) A random time τ is honest if and only if one of the equivalent assertions hold*
(a) There exists an optional set Γ such that $\tau(\omega) = \sup\{t \,:\, (t, \omega) \in \Gamma\}$ (it is the end of an optional set) on $\{\tau < \infty\}$
(b) $\widetilde{Z}_\tau = 1$ on $\{\tau < \infty\}$
(c) $\tau = \sup\{t \,:\, \widetilde{Z}_t = 1\}$ on $\{\tau < \infty\}$
(d) $A^o_t = A^o_{t \wedge \tau}$.

In particular, an honest time is $\mathcal{F}_\infty$-measurable. If X is a transient diffusion, the last passage time Λ_a (see Proposition 8.10.1) is honest.

Lemma 8.8.9 *The process Y is $\mathbb{G}$-predictable if and only if there exist two $\mathbb{F}$ predictable processes y and $\widetilde{y}$ such that*

$$Y_t = y_t 1\!\!1_{t \leq \tau} + \widetilde{y}_t 1\!\!1_{t > \tau}.$$

Let $X \in L^1$. Then a càdlàg version of the martingale $X_t = \mathbb{E}[X|\mathcal{G}_t]$ is given by:

$$X_t = \frac{1}{Z_t}\mathbb{E}[\xi \mathbf{1}_{t<\tau}|\mathcal{F}_t]\,\mathbf{1}_{t<\tau} + \frac{1}{1-Z_t}\mathbb{E}[\xi \mathbf{1}_{t\geq\tau}|\mathcal{F}_t]\,\mathbf{1}_{t\geq\tau}.$$

Every $\mathbb{G}$ optional process decomposes as

$$L1\!\!1_{[0,\tau[} + J1\!\!1_{[\tau]} + K1\!\!1_{]\tau,\infty[},$$

where L and K are $\mathbb{F}$-optional processes and where J is a $\mathbb{F}$ progressively measurable process.

See Jeulin [82] for a proof.

Example 8.8.10 We give Barlow's counterexample to prove that an $\mathbb{G}$ optional process can not be decomposed as

$$L1\!\!1_{[0,\tau[} + K1\!\!1_{]\tau,\infty[},$$

where L and K are $\mathbb{F}$-optional processes. Let B be a Brownian motion and $\vartheta = \inf\{t : |B_t| = 1\}$ and $\tau = \sup\{t : B^\vartheta = 0\}$. The process X defined as $X_t = 1\!\!1_{t\geq\tau}\mathrm{sgn}(B_\tau)$ is a $\mathbb{G}$-martingale and is an optional process. Obviously, if $(H, K$ exist, then $H = 0$ and one can choose K predictable. Then $\Delta X_\tau = K_\tau$ would be $\mathcal{G}_{\tau-}$ measurable, which contradicts the martingale property of X.

8.8.5 Progressive versus initial enlargement

Proposition 8.8.11 *If τ is honest, any $\mathbb{F}$ martingale is a $\mathbb{F}^{(\tau)}$ (and a $\mathbb{G}$)-semi-martingale*

8.8.6 Decomposition

Proposition 8.8.12 *Let τ be honest. We assume (CA). Then, any $\mathbb{F}$-local martingale M is a $\mathbb{G}$ semi-martingale with decomposition*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge\tau}\frac{d\langle M,\mu\rangle_s}{Z_s} - \int_\tau^{\tau\vee t}\frac{d\langle M,\mu\rangle_s}{1-Z_s},$$

where $\widetilde{M}$ is a $\mathbb{G}$-local martingale

PROOF: Let M be an $\mathbb{F}$-martingale which belongs to $\mathbb{H}^1$ and $G_s \in \mathcal{G}_s$. We define a $\mathbb{G}$-predictable process Y as $Y_u = 1\!\!1_{G_s}1\!\!1_{]s,t]}(u)$. For $s < t$, one has, using the decomposition of $\mathbb{G}$-predictable processes:

$$\begin{aligned}
\mathbb{E}(1\!\!1_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\infty Y_u dM_u\right)\\
&= \mathbb{E}\left(\int_0^\tau y_u dM_u\right) + \mathbb{E}\left(\int_\tau^\infty \widetilde{y}_u dM_u\right).
\end{aligned}$$

Noting that $\int_0^t \widetilde{y}_u dM_u$ is a martingale yields $\mathbb{E}\left(\int_0^\infty \widetilde{y}_u dM_u\right) = 0$,

$$\begin{aligned}
\mathbb{E}(1\!\!1_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\tau (y_u - \widetilde{y}_u) dM_u\right)\\
&= \mathbb{E}\left(\int_0^\infty dA^p_v \int_0^v (y_u - \widetilde{y}_u) dM_u\right).
\end{aligned}$$

By integration by parts, setting $N_t = \int_0^t (y_u - \widetilde{y}_u) dM_u$, we get

$$\mathbb{E}(1\!\!1_{G_s}(M_t - M_s)) = \mathbb{E}(N_\infty A^p_\infty) = \mathbb{E}(N_\infty \mu_\infty) = \mathbb{E}\left(\int_0^\infty (y_u - \widetilde{y}_u) d\langle M,\mu\rangle_u\right).$$

Now, it remains to note that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^\infty Y_u\left(\frac{d\langle M,\mu\rangle_u}{Z_{u-}}1\!\!1_{\{u\le\tau\}}-\frac{d\langle M,\mu\rangle_u}{1-Z_{u-}}1\!\!1_{\{u>\tau\}}\right)\right)\\
&\quad= \mathbb{E}\left(\int_0^\infty \left(y_u\frac{d\langle M,\mu\rangle_u}{Z_{u-}}1\!\!1_{\{u\le\tau\}}-\widetilde{y}_u\frac{d\langle M,\mu\rangle_u}{1-Z_{u-}}1\!\!1_{\{u>\tau\}}\right)\right)\\
&\quad= \mathbb{E}\left(\int_0^\infty (y_u d\langle M,\mu\rangle_u - \widetilde{y}_u d\langle M,\mu\rangle_u)\right)\\
&\quad= \mathbb{E}\left(\int_0^\infty (y_u-\widetilde{y}_u)\, d\langle M,\mu\rangle_u\right)
\end{aligned}$$

to conclude the result in the case $M\in\mathbb{H}^1$. The general result follows by localization. □

The general version is given in Jeulin [82, Chapitre 5]

Proposition 8.8.13 *Let τ be honest. Then, any $\mathbb{F}$-local martingale M is a $\mathbb{G}$ semi-martingale with decomposition*

$$M_t=\widetilde{M}_t+\int_0^{t\wedge\tau}\frac{d\langle M,m\rangle_s}{Z_{s-}}-\int_\tau^{\tau\vee t}\frac{d\langle M,m\rangle_s}{1-Z_{s-}},$$

where $\widetilde{M}$ is a $\mathbb{G}$-local martingale

Example 8.8.14 Let W be a Brownian motion, and $\tau=g_1$, the last time when the BM reaches 0 before time 1, i.e., $\tau=\sup\{t\le 1: W_t=0\}$. Using the computation of $Z_t^{g_1}=\mathbb{P}(g_1>t|\mathcal{F}_t)$ (see the following Subsection 8.10.3) and applying Proposition 8.8.12, we obtain the decomposition of the Brownian motion in the enlarged filtration

$$\begin{aligned}
W_t &= \widetilde{W}_t-\int_0^t 1\!\!1_{[0,\tau]}(s)\,\frac{\Phi'}{1-\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right)\frac{\mathrm{sgn}(W_s)}{\sqrt{1-s}}\,ds\\
&\quad +1\!\!1_{\{\tau\le t\}}\,\mathrm{sgn}(W_1)\int_\tau^t\frac{\Phi'}{\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right)ds
\end{aligned}$$

where $\Phi(x)=\sqrt{\frac{2}{\pi}}\int_0^x\exp(-u^2/2)du$.

Exercise 8.8.15 Prove that any $\mathbb{F}$-stopping time is honest ◁

Exercise 8.8.16 Prove that, under (CA)

$$\mathbb{E}(\int_0^{t\wedge\tau}\frac{d\langle M,\mu\rangle_s}{Z_{s-}}-\int_\tau^{\tau\vee t}\frac{d\langle M,\mu\rangle_s}{1-Z_{s-}}|\mathcal{F}_t)$$

is an $\mathbb{F}$-local martingale, without using the previous Proposition 8.8.12. ◁

8.8.7 Predictable Representation Theorem

Theorem 8.8.17 *If there exists a family of continuous $\mathbb{F}$ martingales M^i which enjoys the PRT in $\mathbb{F}$, then any continuous $\mathbb{G}$-martingale is a sum of stochastic integrals w.r.t. $\widetilde{M}^i$.*

8.8.8 Multiplicative Decomposition

This section is a part of [115]. For N be a local martingale which belongs to the class $(\mathcal{C}_0)$, with $N_0=x$, we set $S_t=\sup_{s\le t}N_s$. We consider the last time where N reaches its maximum over $[0,\infty]$, i.e., the last time where N equal S:

$$g = \sup\{t\ge 0:\quad N_t=S_\infty\}=\sup\{t\ge 0:\quad S_t-N_t=0\}\,. \tag{8.8.1}$$

Without loss of generality, we restrict our attention to the case $x=1$.

Proposition 8.8.18 *The supermartingale $Z_t = \mathbb{P}(g > t \mid \mathcal{F}_t)$ admits the multiplicative decomposition $Z_t = \frac{N_t}{S_t}$, $t \geq 0$.*

PROOF: We have the following equalities

$$\begin{aligned}\{g > t\} &= \{\exists\, u > t : \; S_u = N_u\} = \{\exists\, u > t : \; S_t \leq N_u\} \\ &= \left\{\sup_{u \geq t} N_u \geq S_t\right\} = \{S^t \geq S_t\}.\end{aligned}$$

Hence, from (1.1.2), we get: $\mathbb{P}(g > t \mid \mathcal{F}_t) = \frac{N_t}{S_t}$. □

Lemma 8.8.19 *Any $\mathbb{F}$-local martingale X is a $\mathbb{F}^g$ semi martingale X with decomposition*

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X, N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\leq s\}} \frac{d\langle X, N\rangle_s}{S_\infty - N_s},$$

where $\widetilde{X}$ is an $\mathbb{F}^g$-local martingale.

PROOF: Let X be an $\mathbb{F}$-martingale which is in $\mathbb{H}^1$; the general case follows by localization. From results given in Section 6.4.5

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X, N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\leq s\}} \frac{d\langle X, N\rangle_s}{S_\infty - N_s},$$

where $\widetilde{X}$ denotes an $\mathbb{F}^{(S_\infty)}$ martingale. Thus, $\widetilde{X}$, which is equal to:

$$X_t - \left(\int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X, N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\leq s\}} \frac{d\langle X, N\rangle_s}{S_\infty - N_s},\right),$$

is $\mathbb{F}^g$ adapted (recall that $\mathcal{F}^g_t \subset \mathcal{F}^{(S_\infty)}_t$), and hence it is an $\mathbb{F}^g$-martingale.

These results extend to honest times:

Theorem 8.8.20 *Let τ be an honest time. Then, under the conditions* ***(CA)****, the supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ admits the following additive and multiplicative representations: there exists a continuous and nonnegative local martingale N, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$\begin{aligned}Z_t &= \mathbb{P}(\tau > t|\mathcal{F}_t) = \frac{N_t}{S_t} \\ Z_t &= \mu_t - A^p_t.\end{aligned}$$

where these two representations are related as follows:

$$\begin{aligned}N_t &= \exp\left(\int_0^t \frac{d\mu_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle\mu\rangle_s}{Z_s^2}\right), \quad S_t = \exp(A^p_t); \\ \mu_t &= 1 + \int_0^t \frac{dN_s}{S_s} = \mathbb{E}(\log S_\infty \mid \mathcal{F}_t), \quad A_t = \log S_t.\end{aligned}$$

8.9 Classical Arbitrages for Honest Times

Throughout this section, we consider a finite honest time (see Aksamit [2] for a generalization) and we consider a financial market, with a savings account with null interest rate and a risky price process S. We assume that $(S, \mathbb{F})$ is a complete market.

The first papers dealing with arbitrages related to honest times are Imkeller [72] and Zwierz [136]. They consider the case of arbitrages occurring after τ under (C). More recent papers are from Fontana et al. [64] and Aksamit et al. [4] and Acciaio et al. [1].

Note that, in Dellacherie et al. [41], the authors had the intuition that there are arbitrages. Studying a case similar to $\tau = \inf\{s : S_s = S^*\}$ where S was a geometric Brownian motion, they wrote: Tous les spéculateurs cherchent à connaitre τ sans jamais y parvenir, d'où son nom de v.a. honnête [1]

We make use of the standard definitions on classical arbitrages recalled in Section 1.5.1.

8.9.1 Existence of Classical Arbitrages

In a first step, we consider the case where conditions (CA) hold, and we assume that $\mathbb{F}$ is a Brownian filtration, as in [64].

Theorem 8.9.1 *Assuming that $\mathbb{F}$ is a Brownian filtration, that (A) is satisfied, and that $(S,\mathbb{F})$ is a complete market, then there are classical arbitrages on the time interval $[0,\tau]$ and on the time interval $]\tau,\infty[$*

PROOF: From the multiplicative decomposition of $Z = N/S$ we see that $Z_\tau = 1$, so that $N_\tau \geq 1$. It remains to use Proposition 1.5.1. □

The following theorem represents our principal result in the general framework.

Theorem 8.9.2 *If τ is a finite honest time which is not an $\mathbb{F}$ stopping time there are classical arbitrages before τ for $(S,\mathbb{G})$ and classical arbitrages after τ for $(S,\mathbb{G})$.*

PROOF: (a) From $m = \widetilde{Z} + A^o_-$ and $\widetilde{Z}_\tau = 1$, we deduce that $m_\tau \geq 1$. Since τ is not an $\mathbb{F}$ stopping time, one has $\mathbb{P}(A^o_{\tau-} > 0) > 0$. The result follows from Proposition 1.5.1.
(b) From $m = Z + A^o$ and the fact that A^o does not increase after τ, one obtains that, for $t > \tau$, $m_t - m_\tau = Z_t - Z_\tau \geq -2$. On the other hand, using $m = \widetilde{Z} + A^o_-$, one obtains that, for $t > \tau$, $m_t - m_\tau = \widetilde{Z}_t - 1 + \Delta A^o_\tau$. Consider the following $\mathbb{G}$-stopping time

$$\nu := \inf\{t > \tau : \widetilde{Z}_t \leq \frac{1 - \Delta A^o_\tau}{2}\}. \tag{8.9.1}$$

Then,

$$m_\nu - m_\tau = \widetilde{Z}_\nu - 1 + \Delta A^o_\tau \leq \frac{\Delta A^o_\tau - 1}{2} \leq 0,$$

and, as τ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) = \mathbb{P}(\Delta A^o_\tau < 1) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of an admissible self-financing strategy with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$. This ends the proof of the theorem.

□

We now reproduce some examples, given in [3].

[1] We provide an English translation for the convenience of the reader: " *Every speculator strives to know when τ will occur, without ever achieving this goal. Hence, the name of honest random variable*".

8.9.2 Classical arbitrage opportunities in a Brownian filtration

Throughout this subsection, we assume given a one-dimensional Brownian motion W and $\mathbb{F}$ is its augmented natural filtration. The market model is represented by the savings account whose process is the constant one and one stock whose price process is given by

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \qquad \sigma > 0 \text{ given.}$$

It is worth mentioning that in this context of Brownian filtration, for any process V with locally integrable variation, we have $V^{o,\mathbb{F}} = V^{p,\mathbb{F}}$.

For some honest times τ, we compute explicitly the arbitrage opportunities for both before and after τ. For other examples of honest times, and associated classical arbitrages we refer the reader to [64].

Last passage time at a given level

Proposition 8.9.3 *Consider the following random times*

$$\tau := \sup\{t \,:\, S_t = a\} \qquad \textit{and} \qquad \nu := \inf\{t > \tau \;\big|\; S_t \leq \frac{a}{2}\},$$

where $0 < a < 1$. Then, the following assertions hold.
(a) The model $(S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process

$$\frac{1}{a} 1\!\!1_{\{S<a\}} I_{]\!]0,\tau]\!]}.$$

(b) The model $(S - S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by

$$-\frac{1}{a} 1\!\!1_{\{S<a\}} I_{]\!]\tau,\nu]\!]}.$$

Proof: It is clear that τ is a finite honest time, is not a stopping time. Thus τ fulfills the assumptions of assertions of Theorem 8.9.2. We now compute the predictable process φ such that $m = 1 + \varphi \centerdot S$. Using [81, exercise 1.2.3.10], we obtain

$$1 - Z_t := \mathbb{P}\left(\tau \leq t | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_{t<u} S_u \leq a | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_{u} \widetilde{S}_u \leq \frac{a}{S_t} | \mathcal{F}_t\right) = \Phi\left(\frac{a}{S_t}\right)$$

where $\widetilde{S}_u = \exp(\sigma \widetilde{W}_u - \frac{1}{2}\sigma^2 u)$, $\widetilde{W}$ independent of $\mathcal{F}_t$ and $\Phi(x) = \mathbb{P}\left(\sup_u \widetilde{S}_u \leq x\right) = \mathbb{P}(\frac{1}{U} \leq x) = \mathbb{P}(\frac{1}{x} \leq U) = (1 - \frac{1}{x})^+$ (where U is a random variable with uniform law (See Proposition 1.1.13)). Thus we get $Z_t = 1 - (1 - \frac{S_t}{a})^+$ (in particular $Z_\tau = \widetilde{Z}_\tau = 1$), and

$$dZ_t = 1\!\!1_{\{S_t<a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell^a_t$$

where ℓ^a is the local time of the S at the level a. Therefore, we deduce that

$$m = 1 + \varphi \centerdot S.$$

This ends the proof of the proposition. $\square$

Last passage time at a level before maturity

Our second example of random time, in this subsection, takes into into account when one is working in finite horizon. In this example, we introduce the following notation

$$H(z,y,s) := e^{-zy}\mathcal{N}\left(\frac{zs-y}{\sqrt{s}}\right) + e^{zy}\mathcal{N}\left(\frac{-zs-y}{\sqrt{s}}\right) \quad \text{and} \quad V_t := \alpha + \frac{\sigma}{2}t - W_t = (a - X_t)/\sigma, \tag{8.9.2}$$

where $\mathcal{N}(x)$ is the cumulative distribution function of the standard normal distribution.

Proposition 8.9.4 *Consider the following random time (an honest time)*

$$\tau_1 := \sup\{t \leq 1 \,:\, S_t = b\}$$

where b is a positive real number, $0 < b < 1$. Then the $\mathbb{G}$-predictable process

$$\varphi_t := \frac{1}{\sigma S_t}\beta_t I_{]\!]0,\tau_1]\!]},$$

is an arbitrage opportunity for the model $(S^{\tau_1}, \mathbb{G})$, and $-\varphi I_{]\!]\tau_1,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^{\tau_1}, \mathbb{G})$. Here β is given by

$$\beta_t := e^{\gamma V_t}\left(\gamma H(\gamma, |V_t|, 1-t) - sgn(V_t)\, H'_x(\gamma, |V_t|, 1-t)\right), \qquad \gamma = -\frac{\sigma}{2},$$

V and H are defined in (8.9.2), and ν is defined in (8.9.1).

Proof: The proof of this proposition follows from Theorem 8.9.2 as long as we can write the martingale m as an integral stochastic with respect to S. This is the main focus of this remaining part of this proof. The time τ_1 is honest and finite. Let $X_t = \ln S_0 - \frac{1}{2}\sigma^2 t + \sigma W_t$ and $a = \ln b$. We assume $\sigma > 0$. Then,

$$\begin{aligned} \tau_1 &= \sup\{t \leq 1 \,:\, X_t = a\} \\ &= \sup\{t \leq 1 \,:\, \gamma t + W_t = \alpha\} \end{aligned}$$

where $\gamma = -\frac{1}{2}\sigma$ and $\alpha = (a - x)/\sigma$ with $x = \ln S_0$.

Setting $T_0(V) = \inf\{t \,:\, V_t = 0\}$ where V is given by (8.9.2), we obtain, using standard computations (see [3M])

$$1 - Z_t = \mathbb{P}(\tau_1 \leq t|\mathcal{F}_t) = (1 - e^{\nu V_t} H(\gamma, |V_t|, 1-t))1\!\!1_{\{T_0(V) \leq t\}},$$

where H is given in (8.9.2). In particular $Z_\tau = \widetilde{Z}_\tau = 1$. Using Itô's lemma, we obtain the decomposition of $1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t)$ as a semi-martingale.

The martingale part of Z is given by $dm_t = \beta_t dW_t = \frac{1}{\sigma S_t}\beta_t dS_t$ where

$$\beta_t = e^{\gamma V_t}\left(\nu H(\gamma, |V_t|, 1-t) - \operatorname{sgn}(V_t)\, H'_x(\gamma, |V_t|, 1-t)\right).$$

□

8.9.3 Arbitrage opportunities in a Poissonnian filtration

Throughout this subsection, we suppose given a Poisson process N, with intensity rate $\lambda > 0$, and natural filtration $\mathbb{F}$. The stock price process is given by

$$dS_t = S_{t-}\psi dM_t,\; S_0 = 1, \qquad M_t := N_t - \lambda t, \tag{8.9.3}$$

or equivalently $S_t = \exp(-\lambda\psi t + \ln(1+\psi)N_t)$, where $\psi > -1$. In what follows, we introduce the following notations

$$\alpha := \ln(1+\psi) > 0, \quad a := -\frac{1}{\alpha}\ln b, \qquad \mu := \frac{\lambda\psi}{\ln(1+\psi)} \ \text{ and } \ Y_t := \mu t - N_t, \tag{8.9.4}$$

so that $S_t = \exp(-\ln(1+\psi)Y_t)$. To the process Y, we associate its ruin probability, denoted by $\Psi(x)$ given by

$$\Psi(x) = \mathbb{P}(T^x < \infty), \quad \text{ with } \quad T^x = \inf\{t : x + Y_t < 0\} \quad \text{ and } \quad x \geq 0. \tag{8.9.5}$$

We set $\theta = \frac{\mu}{\lambda} - 1$, and deduce that $\Psi(0) = (1+\theta)^{-1}$ (see [14]).

Below, we describe our first example of honest time and the associated arbitrage opportunity.

Last passage time

Proposition 8.9.5 *For $0 < b < 1$, consider the following random time*

$$\tau := \sup\{t : \ S_t \geq b\} = \sup\{t : Y_t \leq a\}. \tag{8.9.6}$$

Suppose that $\psi > 0$, then the following assertions hold.
a) τ is a honest time.
b) The process

$$\varphi := \frac{1}{\psi S_-}\left(\Psi(Y_- - a - 1)1\!\!1_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)1\!\!1_{\{Y_- \geq a\}} + 1\!\!1_{\{Y_- < a+1\}} - 1\!\!1_{\{Y_- < a\}}\right),$$

is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $-\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here Ψ is defined in (8.9.5) and ν is defined in the same manner as in (8.9.1).

PROOF: Since $\psi > 0$, one has $\mu > \lambda$ so that Y goes to $+\infty$ as t goes to infinity, and τ is finite. The Azéma supermartingale associated with the time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \Psi(Y_t - a)1\!\!1_{\{Y_t \geq a\}} + 1\!\!1_{\{Y_t < a\}} = 1 + 1\!\!1_{\{Y_t \geq a\}}\left(\Psi(Y_t - a) - 1\right),$$

where Ψ is defined in (8.9.5). We obtain $Z_\tau = \frac{1}{1+\theta} < 1$.
Define $\vartheta_1 = \inf\{t > 0 : Y_t = a\}$ and then, for each $n > 1$, $\vartheta_n = \inf\{t > \vartheta_{n-1} : Y_t = a\}$. It can be proved that the times ϑ_n are predictable $\mathbb{F}$-stopping times, and $[\![\tau]\!] \subset \cup_n [\![\theta_n]\!]$. For any optional increasing process K, one has

$$\mathbb{E}(K_\tau) = \mathbb{E}(\sum 1\!\!1_{\tau=\vartheta_n} K_{\vartheta_n}) = \mathbb{E}(\sum \mathbb{E}(1\!\!1_{\tau=\vartheta_n}|\mathcal{F}_{\vartheta_n})K_{\vartheta_n})$$

and $\mathbb{E}(1\!\!1_{\tau=\vartheta_n}|\mathcal{F}_{\vartheta_n}) = \mathbb{P}(T^0 = \infty) = 1 - \Psi(0)$. It follows that the dual optional projection A^o of the process $1\!\!1_{[\tau,\infty)}$ equals

$$A^o = \frac{\theta}{1+\theta}\sum_n 1\!\!1_{[\vartheta_n,\infty)}.$$

Note that $\tilde{Z}_\tau = Z_\tau + \Delta A^o_\tau = 1 + (\Psi(0) - 1) + \frac{\theta}{1+\theta} = 1$, hence τ is honest.

As a result the process A^o is predictable, and hence we have $Z = m - A^o$ is the Doob-Meyer decomposition of Z. Thus we can get

$$\Delta m = Z - {}^pZ$$

where ${}^pZ$ is the predictable projection of Z. To calculate ${}^pZ$, we write the process Z in a more adequate form. To this end, we first remark that

$$1\!\!1_{\{Y \geq a\}} = 1\!\!1_{\{Y_- \geq a+1\}}\Delta N + (1 - \Delta N)1\!\!1_{\{Y_- \geq a\}} \quad \text{ and } \quad 1\!\!1_{\{Y < a\}} = 1\!\!1_{\{Y_- < a+1\}}\Delta N + (1 - \Delta N)1\!\!1_{\{Y_- < a\}}.$$

Then, we obtain easily

$$\Delta m = \left(\Psi(Y_- - a - 1)1\!\!1_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)1\!\!1_{\{Y_- \geq a\}} + 1\!\!1_{\{Y_- < a+1\}} - 1\!\!1_{\{Y_- < a\}}\right) \Delta N$$
$$= \psi S_- \varphi \Delta M = \varphi \Delta S.$$

Since the two martingales m and S are purely discontinuous, we deduce that $m - m_0 = \varphi \centerdot S$. Therefore, the proposition follows from Theorem 8.9.2. $\square$

Time of supremum on fixed time horizon

The following example requires the following notations

$$S_t^* := \sup_{s \leq t} S_s, \quad \Psi(x,t) := \mathbb{P}(S_t^* > x), \quad \widehat{\Phi}(t) := \mathbb{P}(\sup_{s<t} S_s \leq 1), \quad \widetilde{\Phi}(x,t) := \mathbb{P}(\sup_{s<t} S_s < x) \quad (8.9.7)$$

Proposition 8.9.6 *Consider the random time τ defined by*

$$\tau = \sup\{t \leq 1 : S_t = S_t^*\}, \tag{8.9.8}$$

where $S_t^* = \sup_{s\leq t} S_s$. Then, the following assertions hold.
a) τ is a honest time.
b) For $\psi > 0$, the $\mathbb{G}$-predictable process

$$\varphi_t \;:=\; 1\!\!1_{\{t<1\}} \left[\Psi\left(\max(\frac{S_{t-}^*}{S_{t-}(1+\psi)}, 1), 1-t\right) - \Psi\left(\frac{S_{t-}^*}{S_{t-}}, 1-t\right)\right] + 1\!\!1_{\{S_{t-}^* < S_{t-}(1+\psi)\}} \widehat{\Phi}(1-t)$$
$$+ \left[1\!\!1_{\{\max(S_{1-}^*, S_{1-}(1+\psi)) = S_0\}} - 1\!\!1_{\{\max(S_{1-}^*, S_{1-}) = S_0\}}\right] 1\!\!1_{\{t=1\}}$$

is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $-\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here Ψ and $\widehat{\Phi}$ are defined in (8.9.7), and ν is defined similarly as in (8.9.1).
c) For $-1 < \psi < 0$, the $\mathbb{G}$-predictable process

$$\varphi_t := \frac{1}{\psi S_{t-}} \left(\psi 1\!\!1_{\{S_t^* = S_{t-}\}} \widehat{\Phi}(\frac{1}{1+\psi}, 1-t) + \Psi(\frac{S_t^*}{S_{t-}(1+\psi)}, 1-t) - \Psi(\frac{S_t^*}{S_{t-}}, 1-t)\right),$$

is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $-\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$.

Proof: Note that, if $-1 < \psi < 0$ $S_\tau < S_\tau^* = \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$ and $S_{\tau-} = S_{\tau-}^* = \sup_{t\in[0,1]} S_t$, and the process S^* is continuous.
If $\psi > 0$, $S_{\tau-} < S_{\tau-}^* < \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$.

Define the sets $(E_n)_{n=0}^\infty$ with

$$E_0 = \{\tau = 1\} \qquad \text{and} \qquad E_n = \{\tau = T_n\} \quad \text{with} \quad n \geq 1.$$

This defines a partition of Ω. Then, $\tau = 1\!\!1_{E_0} + \sum_{n=1}^\infty T_n 1\!\!1_{E_n}$.
Note that τ is not an $\mathbb{F}$ stopping time since $E_n \notin \mathcal{F}_{T_n}$ for any $n \geq 1$.

The Azéma supermartingale associated with the honest time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,1]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,1-t]} \widehat{S}_s > \frac{S_t^*}{S_t}|\mathcal{F}_t) = 1\!\!1_{(t<1)}\Psi(\frac{S_t^*}{S_t}, 1-t),$$

with $\widehat{S}$ an independent copy of S and $\Psi(x,t)$ is given by (8.9.7).

As $\{\tau = T_n\} \subset \{\tau \leq T_n\} \subset \{Z_{T_n} < 1\}$, we have

$$Z_\tau = 1\!\!1_{\{\tau=1\}} Z_1 + \sum_{n=1}^{\infty} 1\!\!1_{\{\tau=T_n\}} Z_{T_n} < 1, \quad \text{and} \ \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

In the following we will prove assertion b). Thus, we suppose that $\psi > 0$, and we calculate

$$\begin{aligned}
A^o_t &= \mathbb{P}(\tau = 1|\mathcal{F}_1) 1\!\!1_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}} \\
&= 1\!\!1_{\{S^*_1=S_0\}} 1\!\!1_{\{t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1\}} 1\!\!1_{\{S^*_{T_n-}<S_{T_n}\}} \mathbb{P}(\sup_{s\in[T_n,1[} S_s \leq S_{T_n}|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}} \\
&= 1\!\!1_{\{S^*_1=S_0\}} 1\!\!1_{\{t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1\}} 1\!\!1_{\{S^*_{T_n-}<S_{T_n-}(1+\psi)\}} \widehat{\Phi}(1-T_n) 1\!\!1_{\{t\geq T_n\}},
\end{aligned}$$

with $\widehat{\Phi}$ is given by (8.9.7). As before, we write

$$\begin{aligned}
A^o_t &= 1\!\!1_{\{S^*_1=S_0\}} 1\!\!1_{\{t\geq 1\}} + \sum_{s\leq t} 1\!\!1_{\{s<1\}} 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}} \widehat{\Phi}(1-s)\Delta N_s \\
&= 1\!\!1_{\{S^*_1=S_0\}} 1\!\!1_{\{t\geq 1\}} + \int_0^{t\wedge 1} 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}} \widehat{\Phi}(1-s)\, dM_s + \lambda \int_0^{t\wedge 1} 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}} \hat{\Phi}(1-s) ds.
\end{aligned}$$

Remark that we have

$$1\!\!1_{\{S^*_1=S_0\}} = \left[1\!\!1_{\{\max(S^*_{1-},S_{1-}(1+\psi))=S_0\}} - 1\!\!1_{\{\max(S^*_{1-},S_{1-})=S_0\}}\right] \Delta M_1 + 1\!\!1_{\{\max(S^*_{1-},S_{1-})=S_0\}}.$$

and

$$\Delta m = \Delta Z + \Delta A^o = Z - \ {}^p(Z) + \Delta A^o - \ {}^p(\Delta A^o).$$

Then we re-write the process Z as follows

$$Z = 1\!\!1_{[\![0,1[\![} \Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)}, 1), 1-t\right) \Delta M + (1-\Delta M) I_{[\![0,1[\![} \Psi\left(\frac{S^*_-}{S_-}, 1-t\right).$$

This implies that

$$Z - \ {}^p(Z) = 1\!\!1_{[\![0,1[\![} \left[\Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)}, 1), 1-t\right) - \Psi\left(\frac{S^*_-}{S_-}, 1-t\right)\right] \Delta M.$$

Thus by combining all these remarks, we deduce that

$$\Delta m = Z - \ {}^p(Z) + \Delta A^o - \ {}^p(\Delta A^o) = \varphi \Delta S.$$

Then, the assertion b) follows immediately from Theorem 8.9.2.
Next, we will prove assertion c). Suppose that $-1 < \psi < 0$, and we calculate

$$\begin{aligned}
A^o_t &= \mathbb{P}(\tau = 1|\mathcal{F}_1) 1\!\!1_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}} \\
&= 1\!\!1_{\{S^*_1=S_1\}} 1\!\!1_{\{t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1\}} 1\!\!1_{\{S^*_{T_n}=S_{T_n-}\}} \mathbb{P}(\sup_{s\in[T_n,1[} S_s < S_{T_n-}|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}} \\
&= 1\!\!1_{\{S^*_1=S_1\}} 1\!\!1_{t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1\}} 1\!\!1_{\{S^*_{T_n}=S_{T_n-}\}} \widetilde{\Phi}(\frac{S_{T_n-}}{S_{T_n}}, 1-T_n) 1\!\!1_{\{t\geq T_n\}},
\end{aligned}$$

with $\widetilde{\Phi}(x,t)$ is given by (8.9.7). In order to find the compensator of A^o, we write

$$\begin{aligned}
A^o_t &= 1\!\!1_{\{S^*_1=S_1\}} 1\!\!1_{\{t\geq 1\}} + \sum_{s\leq t} 1\!\!1_{\{s<1\}} 1\!\!1_{\{S^*_s=S_{s-}\}} \widetilde{\Phi}(\frac{1}{1+\psi}, 1-s)\, \Delta N_s \\
&= 1\!\!1_{\{S^*_1=S_1\}} 1\!\!1_{\{t\geq 1\}} + \int_0^{t\wedge 1} 1\!\!1_{\{S^*_s=S_{s-}\}} \widetilde{\Phi}(\frac{1}{1+\psi}, 1-s)\, dM_s + \lambda \int_0^{t\wedge 1} 1\!\!1_{\{S^*_s=S_{s-}\}} \widetilde{\Phi}(\frac{1}{1+\psi}, 1-s)\, ds.
\end{aligned}$$

As a result, due to the continuity of the process S^*, we get

$$\begin{aligned} A^o_t - {}^p(A^o)_t &= I_{\{S^*_t = S_{t-}\}} \widetilde{\Phi}(\frac{1}{1+\psi}, 1-t)\Delta M_t, \\ Z_t - {}^pZ_t &= \left[\Psi(\frac{S^*_t}{S_{t-}(1+\psi)}, 1-t) - \Psi(\frac{S^*_t}{S_{t-}}, 1-t)\right]\Delta N_t. \end{aligned}$$

This implies that

$$\begin{aligned} \Delta m_t &= Z_t - {}^pZ_t + A^o_t - {}^p(A^o)_t \\ &= \left\{\psi I_{\{S^*_t = S_{t-}\}} \widetilde{\Phi}(\frac{1}{1+\psi}, 1-t) + \Psi\left(\frac{S^*_t}{S_{t-}(1+\psi)}, 1-t\right) - \Psi\left(\frac{S^*_t}{S_{t-}}\right), 1-t\right\}\Delta N_t. \end{aligned}$$

Since m and S are pure discontinuous local martingales, we conclude that m can be written in the form of

$$m = m_0 + \varphi \cdot S,$$

and the proof of the assertion c) follows immediately from Theorem 8.9.2. This ends the proof of the proposition. □

Remark 8.9.7 The fact that τ is an honest time can be also obtained by the equivalent characterization that is is the end of a predictable set, namely the end of $\Gamma = [0,1] \cap (S_- = S^*_-)$.

Time of supremum

Below, we will present our last example of this subsection. The analysis of this example is based on the following three functions.

$$\Psi(x) = \mathbb{P}(S^* > x) = \mathbb{P}(\sup_s S_s > x), \quad \widehat{\Phi} = \mathbb{P}(\sup_s S_s \le 1), \text{ and } \widetilde{\Phi}(x) = \mathbb{P}(\sup_s S_s < x). \tag{8.9.9}$$

Proposition 8.9.8 *Consider the random time τ given by*

$$\tau = \sup\{t : S_t = S^*_t\}. \tag{8.9.10}$$

Then, the following assertions hold.
a) τ is a honest time.
b) For $\psi > 0$, the $\mathbb{G}$-predictable process

$$\varphi_t := \frac{\mathbb{1}_{\{S^*_{t-} < S_{t-}(1+\psi)\}}\widehat{\Phi} + \Psi\left(\max(\frac{S^*_{t-}}{S_{t-}(1+\psi)}, 1\right) - \Psi(\frac{S^*_{t-}}{S_{t-}})}{S_{t-}\psi}$$

is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$ and $-\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here Ψ and $\widehat{\Phi}$ are defined in (8.9.9), and ν is defined in similar way as in (8.9.1).
c) For $-1 < \psi < 0$, the $\mathbb{G}$-predictable process

$$\varphi := \frac{\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-}) + \mathbb{1}_{\{S^* = S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\psi}{\psi S_-},$$

is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$ and $-\varphi I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$, where again ν is defined similarly as in (8.9.1).

PROOF: It is clear that τ satisfies the definition of an $\mathbb{F}$-honest time.
Let us note that τ is finite and, as before, if $-1 < \psi < 0$ $S_\tau < S^*_\tau = \sup_t S_t$ and S^* is continuous and if $\psi > 0$, $S_\tau = S^*_\tau = \sup_t S_t$.
The Azéma supermartingale associated with the honest time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,\infty]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,\infty]} \widehat{S}_s > \frac{S^*_t}{S_t}|\mathcal{F}_t) = \Psi(\frac{S^*_t}{S_t}),$$

with $\widehat{S}$ an independent copy of S and Ψ is given by (8.9.9). As a result, we deduce that $Z_\tau < 1$.In the following, we will prove assertion b). We suppose that $\psi > 0$, denoting by T_n the sequence of jumps of the Poisson process N, we derive

$$\begin{aligned} A^o_t &= \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}} = \sum_n 1\!\!1_{\{S^*_{T_n-}<S_{T_n}\}}\mathbb{P}(\sup_{s\geq T_n} S_s \leq S_{T_n}|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}} \\ &= \sum_n 1\!\!1_{\{S^*_{T_n-}<S_{T_n-}(1+\psi)\}}\widehat{\Phi}1\!\!1_{\{t\geq T_n\}}, \end{aligned}$$

with $\widehat{\Phi} = \mathbb{P}(\sup_s S_s \leq 1)$ given by (8.9.9).
We continue to find compensator of A^o

$$\begin{aligned} A^o_t &= \sum_{s\leq t} 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}\Delta N_s \\ &= \int_0^t 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}dM_s + \lambda\int_0^t 1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi}ds. \end{aligned}$$

Now as we did for the previous propositions, we calculate the jumps of m. To this end, we re-write Z as follows

$$Z = \left[\Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M + \Psi(\frac{S^*_-}{S_-}).$$

This implies that

$$Z - {}^pZ = \left[\Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M.$$

Hence, we derive

$$\Delta m = \left[1\!\!1_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Phi} + \Psi\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi(\frac{S^*_-}{S_-})\right]\Delta M.$$

Since both martingales m and M are purely discontinuous, we deduce that $m = m_0 + \varphi \centerdot S$. Then, the proposition follows immediately from Theorem 8.9.2.
In the following, we will prove assertion c). To this end, we suppose that $\psi < 0$, and we calculate

$$\begin{aligned} A^o_t &= \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}} = \sum_n 1\!\!1_{\{S^*_{T_n}=S_{T_n-}\}}\mathbb{P}(\sup_{s\geq T_n} S_s < S_{T_n-}|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}} \\ &= \sum_n 1\!\!1_{\{S^*_{T_n}=S_{T_n-}\}}\widetilde{\Phi}(\frac{S_{T_n-}}{S_{T_n}})1\!\!1_{\{t\geq T_n\}}, \end{aligned}$$

with $\widetilde{\Phi}(x) = \mathbb{P}(\sup_s S_s < x)$. Therefore,

$$\begin{aligned} A^o_t &= \sum_{s\leq t} 1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})\Delta N_s \\ &= \int_0^t 1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})dM_s + \lambda\int_0^t 1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Phi}(\frac{1}{1+\psi})ds. \end{aligned}$$

Since in the case of $\psi < 0$, the process S^* is continuous, we obtain

$$Z - {}^pZ = \left[\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-})\right]\Delta N, \qquad A^o - {}^p(A^o) = 1\!\!1_{\{S^*=S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\Delta M.$$

Therefore, we conclude that

$$\Delta m = Z - {}^pZ + A^o - {}^p(A^o) = \left\{\Psi(\frac{S^*}{S_-(1+\psi)}) - \Psi(\frac{S^*}{S_-}) + 1\!\!1_{\{S^*=S_-\}}\widetilde{\Phi}(\frac{1}{1+\psi})\psi\right\}\Delta N.$$

This implies that the martingale m has the form of $m = 1+\varphi\cdot S$, and assertion c) follows immediately from Theorem 8.9.2, and the proof of the proposition is completed. □

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8.10 Last Passage Times

We now present the study of the law (and the conditional law) of some last passage times for diffusion processes. In this section, W is a standard Brownian motion and its natural filtration is $\mathbb{F}$. These random times have been studied in Jeanblanc and Rutkowski [78] as theoretical examples of default times, in Imkeller [72] as examples of insider private information and, in a pure mathematical point of view, in Pitman and Yor [117] and Salminen [120].

TY

We show that, in a diffusion setup, the Doob-Meyer decomposition of the Azéma supermartingale may be computed explicitly for some random times τ.

8.10.1 Last Passage Time of a Transient Diffusion

Proposition 8.10.1 *Let X be a transient homogeneous diffusion such that $X_t \to +\infty$ when $t \to \infty$, and s a scale function such that $s(+\infty) = 0$ (hence, $s(x) < 0$ for $x \in \mathbb{R}$) and $\Lambda_y = \sup\{t : X_t = y\}$ the last time that X hits y. Then,*

$$\mathbb{P}_x(\Lambda_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1\,.$$

PROOF: We follow Pitman and Yor [117] and Yor [135, p.48], and use that under the hypotheses of the proposition, one can choose a scale function such that $s(x) < 0$ and $s(+\infty) = 0$ (see Sharpe [121]).

Observe that

$$\begin{aligned}\mathbb{P}_x\big(\Lambda_y > t|\mathcal{F}_t\big) &= \mathbb{P}_x\Big(\inf_{u\ge t} X_u < y \,\Big|\, \mathcal{F}_t\Big) = \mathbb{P}_x\Big(\sup_{u\ge t}(-s(X_u)) > -s(y) \,\Big|\, \mathcal{F}_t\Big)\\ &= \mathbb{P}_{X_t}\Big(\sup_{u\ge 0}(-s(X_u)) > -s(y)\Big) = \frac{s(X_t)}{s(y)} \wedge 1,\end{aligned}$$

where we have used the Markov property of X, and the fact that if M is a continuous local martingale with $M_0 = 1$, $M_t \ge 0$, and $\lim_{t\to\infty} M_t = 0$, then

$$\sup_{t\ge 0} M_t \overset{\text{law}}{=} \frac{1}{U}\,,$$

where U has a uniform law on $[0, 1]$ (see Lemma 1.1.13). □

The time Λ_y is honest: defining $\Lambda^t_y = \sup\{s \le t : X_s = y\}$, one has $\Lambda_y = \Lambda^t_y$ on the set $\{\Lambda_y \le t\}$.

Lemma 8.10.2 *The $\mathbb{F}^X$-predictable compensator A associated with the random time Λ_y is the process A defined as $A_t = -\frac{1}{2s(y)} L_t^{s(y)}(Y)$, where $L(Y)$ is the local time process of the continuous martingale $Y = s(X)$.*

PROOF: From $x \wedge y = x - (x-y)^+$, Proposition 8.10.1 and Tanaka's formula, it follows that

$$\frac{s(X_t)}{s(y)} \wedge 1 = M_t + \frac{1}{2s(y)} L_t^{s(y)}(Y) = M_t + \frac{1}{s(y)} \ell_t^y(X)$$

where M is a martingale. The required result is then easily obtained. □

We deduce the law of the last passage time:

$$\begin{aligned} \mathbb{P}_x(\Lambda_y > t) &= \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \mathbb{E}_x(\ell_t^y(X)) \\ &= \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \int_0^t du \, p_u^{(m)}(x,y) \,. \end{aligned}$$

Hence, for $x < y$

$$\begin{aligned} \mathbb{P}_x(\Lambda_y \in dt) &= -\frac{dt}{s(y)} p_t^{(m)}(x,y) = -\frac{dt}{s(y)m(y)} p_t(x,y) \\ &= -\frac{\sigma^2(y)s'(y)}{2s(y)} p_t(x,y) dt \,. \end{aligned} \tag{8.10.1}$$

For $x > y$, we have to add a mass at point 0 equal to

$$1 - \left(\frac{s(x)}{s(y)} \wedge 1\right) = 1 - \frac{s(x)}{s(y)} = \mathbb{P}_x(T_y < \infty) \,.$$

Example 8.10.3 Last Passage Time for a Transient Bessel Process: For a Bessel process of dimension $\delta > 2$ and index ν (see [3M] Chapter 6), starting from 0,

$$\begin{aligned} \mathbb{P}_0^\delta(\Lambda_a < t) &= \mathbb{P}_0^\delta(\inf_{u \geq t} R_u > a) = \mathbb{P}_0^\delta(\sup_{u \geq t} R_u^{-2\nu} < a^{-2\nu}) \\ &= \mathbb{P}_0^\delta\left(\frac{R_t^{-2\nu}}{U} < a^{-2\nu}\right) = \mathbb{P}_0^\delta(a^{2\nu} < U R_t^{2\nu}) = \mathbb{P}_0^\delta\left(\frac{a^2}{R_1^2 U^{1/\nu}} < t\right) \,. \end{aligned}$$

Thus, the r.v. $\Lambda_a = \frac{a^2}{R_1^2 U^{1/\nu}}$ is distributed as $\frac{a^2}{2\gamma(\nu+1)\beta_{\nu,1}} \overset{\text{law}}{=} \frac{a^2}{2\gamma(\nu)}$ where $\gamma(\nu)$ is a gamma variable with parameter ν:

$$\mathbb{P}(\gamma(\nu) \in dt) = \mathbb{1}_{\{t \geq 0\}} \frac{t^{\nu-1} e^{-t}}{\Gamma(\nu)} dt \,.$$

Hence,

$$\mathbb{P}_0^\delta(\Lambda_a \in dt) = \mathbb{1}_{\{t \geq 0\}} \frac{1}{t\Gamma(\nu)} \left(\frac{a^2}{2t}\right)^\nu e^{-a^2/(2t)} dt \,. \tag{8.10.2}$$

We might also find this result directly from the general formula (8.10.1).

Proposition 8.10.4 *For H a positive predictable process*

$$\mathbb{E}_x(H_{\Lambda_y} | \Lambda_y = t) = \mathbb{E}_x(H_t | X_t = y)$$

and, for $y > x$,

$$\mathbb{E}_x(H_{\Lambda_y}) = \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt) \, \mathbb{E}_x(H_t | X_t = y) \,.$$

In the case $x > y$,

$$\mathbb{E}_x(H_{\Lambda_y}) = H_0 \left(1 - \frac{s(x)}{s(y)}\right) + \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt) \, \mathbb{E}_x(H_t | X_t = y) \,.$$

PROOF: We have shown in the previous Proposition 8.10.1 that

$$\mathbb{P}_x(\Lambda_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1\,.$$

From Itô-Tanaka's formula

$$\frac{s(X_t)}{s(y)} \wedge 1 = \frac{s(x)}{s(y)} \wedge 1 + \int_0^t \mathbb{1}_{\{X_u > y\}}\, d\frac{s(X_u)}{s(y)} - \frac{1}{2} L_t^{s(y)}(s(X))\,.$$

It follows, using Lemma 8.1.1 that

$$\begin{aligned}\mathbb{E}_x(H_{\Lambda_x}) &= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty H_u\, d_u L_u^{s(y)}(s(X))\right)\\ &= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty \mathbb{E}_x(H_u|X_u = y)\, d_u L_u^{s(y)}(s(X))\right)\,.\end{aligned}$$

Therefore, replacing H_u by $H_u g(u)$, we get

$$\mathbb{E}_x\left(H_{\Lambda_x} g(\Lambda_x)\right) = \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty g(u)\, \mathbb{E}_x\left(H_u|X_u = y\right)\, d_u L_u^{s(y)}(s(X))\right)\,. \tag{8.10.3}$$

Consequently, from (8.10.3), we obtain

$$\begin{aligned}\mathbb{P}_x\left(\Lambda_y \in du\right) &= \frac{1}{2} d_u \mathbb{E}_x\left(L_u^{s(y)}(s(X))\right)\\ \mathbb{E}_x\left(H_{\Lambda_y}|\Lambda_y = t\right) &= \mathbb{E}_x(H_t|X_t = y)\,.\end{aligned}$$

□

Exercise 8.10.5 Let X be a drifted Brownian motion with positive drift ν and Λ_y^ν its last passage time at level y. Prove that

$$\mathbb{P}_x(\Lambda_y^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x - y + \nu t)^2\right) dt\,,$$

and

$$\mathbb{P}_x(\Lambda_y^{(\nu)} = 0) = \begin{cases} 1 - e^{-2\nu(x-y)}, & \text{for} \quad x > y \\ 0 & \text{for} \quad x < y\,. \end{cases}$$

Prove, using time inversion that, for $x = 0$,

$$\Lambda_y^{(\nu)} \overset{\text{law}}{=} \frac{1}{T_\nu^{(y)}}$$

where

$$T_a^{(b)} = \inf\{t\,:\, B_t + bt = a\}$$

See Madan et al. [105]. ◁

8.10.2 Last Passage Time Before Hitting a Level

Let $X_t = x + \sigma W_t$ where the initial value x is positive and σ is a positive constant. We consider, for $0 < a < x$ the last passage time at the level a before hitting the level 0, given as $g_{T_0}^a(X) = \sup\{t \le T_0\,:\, X_t = a\}$, where

$$T_0 = T_0(X) = \inf\{t \ge 0\,:\, X_t = 0\}\,.$$

(In a financial setting, T_0 can be interpreted as the time of bankruptcy.) Then, setting $\alpha = (a-x)/\sigma$, $T_{-x/\sigma}(W) = \inf\{t : W_t = -x/\sigma\}$ and $d_t^\alpha(W) = \inf\{s \geq t : W_s = \alpha\}$

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \mathbb{P}\big(d_t^\alpha(W) > T_{-x/\sigma}(W)|\mathcal{F}_t\big)$$

on the set $\{t < T_{-x/\sigma}(W)\}$. It is easy to prove that

$$\mathbb{P}\big(d_t^\alpha(W) < T_{-x/\sigma}(W)|\mathcal{F}_t\big) = \Psi(W_{t\wedge T_{-x/\sigma}(W)}, \alpha, -x/\sigma),$$

where the function $\Psi(\cdot, a, b) : \mathbb{R} \to \mathbb{R}$ equals, for $a > b$,

$$\Psi(y, a, b) = \mathbb{P}_y(T_a(W) > T_b(W)) = \begin{cases} (a-y)/(a-b) & \text{for} \quad b < y < a, \\ 1 & \text{for} \quad a < y, \\ 0 & \text{for} \quad y < b. \end{cases}$$

(See Proposition **??** for the computation of Ψ.) Consequently, on the set $\{T_0(X) > t\}$ we have

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \frac{(\alpha - W_{t\wedge T_0})^+}{a/\sigma} = \frac{(\alpha - W_t)^+}{a/\sigma} = \frac{(a - X_t)^+}{a}. \tag{8.10.4}$$

As a consequence, applying Tanaka's formula, we obtain the following result.

Lemma 8.10.6 *Let $X_t = x + \sigma W_t$, where $\sigma > 0$. The $\mathbb{F}$-predictable compensator associated with the random time $g^a_{T_0(X)}$ is the process A defined as $A_t = \dfrac{1}{2\alpha} L^\alpha_{t\wedge T_{-x/\sigma}(W)}(W)$, where $L^\alpha(W)$ is the local time of the Brownian Motion W at level $\alpha = (a-x)/\sigma$.*

8.10.3 Last Passage Time Before Maturity

In this subsection, we study the last passage time at level a of a diffusion process X before the fixed horizon (maturity) T. We start with the case where $X = W$ is a Brownian motion starting from 0 and where the level a is null:

$$g_T = \sup\{t \leq T : W_t = 0\}.$$

Lemma 8.10.7 *The $\mathbb{F}$-predictable compensator associated with the random time g_T equals*

$$A_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_s}{\sqrt{T-s}},$$

where L is the local time at level 0 of the Brownian motion W.

Proof: It suffices to give the proof for $T = 1$, and we work with $t < 1$. Let G be a standard Gaussian variable. Then

$$\mathbb{P}\Big(\frac{a^2}{G^2} > 1 - t\Big) = \Phi\Big(\frac{|a|}{\sqrt{1-t}}\Big),$$

where $\Phi(x) = \sqrt{\dfrac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2})du$. For $t < 1$, the set $\{g_1 \leq t\}$ is equal to $\{d_t > 1\}$. It follows (see [3M]) that

$$\mathbb{P}(g_1 \leq t|\mathcal{F}_t) = \Phi\left(\frac{|W_t|}{\sqrt{1-t}}\right).$$

Then, the Itô-Tanaka formula combined with the identity

$$x\Phi'(x) + \Phi''(x) = 0$$

leads to

$$\begin{aligned}
\mathbb{P}(g_1 \le t|\mathcal{F}_t) &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) d\left(\frac{|W_s|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \frac{ds}{1-s}\,\Phi''\left(\frac{|W_s|}{\sqrt{1-s}}\right) \\
&= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\,dW_s + \int_0^t \frac{dL_s}{\sqrt{1-s}}\,\Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \\
&= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\,dW_s + \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s}{\sqrt{1-s}}\,.
\end{aligned}$$

It follows that the $\mathbb{F}$-predictable compensator associated with g_1 is

$$A_t = \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s}{\sqrt{1-s}},\ (t<1)\,.$$

□

These results can be extended to the last time before T when the Brownian motion reaches the level α, i.e., $g_T^\alpha = \sup\{t \le T\,:\, W_t = \alpha\}$, where we set $\sup(\emptyset) = T$. The predictable compensator associated with g_T^α is

$$A_t = \sqrt{\frac{2}{\pi}}\int_0^{t\wedge T} \frac{dL_s^\alpha}{\sqrt{T-s}},$$

where L^α is the local time of W at level α.

We now study the case where $X_t = x + \mu\, t + \sigma\, W_t$, with constant coefficients μ and $\sigma > 0$. Let

$$\begin{aligned}
g_1^a(X) &= \sup\{t \le 1\,:\, X_t = a\} \\
&= \sup\{t \le 1\,:\, \nu t + W_t = \alpha\}
\end{aligned}$$

where $\nu = \mu/\sigma$ and $\alpha = (a-x)/\sigma$. Setting

$$V_t = \alpha - \nu t - W_t = (a - X_t)/\sigma\,,$$

we obtain, using standard computations (see [3M])

$$\mathbb{P}(g_1^a(X) \le t|\mathcal{F}_t) = (1 - e^{\nu V_t} H(\nu, |V_t|, 1-t))\mathbb{1}_{\{T_0(V)\le t\}},$$

where

$$H(\nu, y, s) = e^{-\nu y}\mathcal{N}\left(\frac{\nu s - y}{\sqrt{s}}\right) + e^{\nu y}\mathcal{N}\left(\frac{-\nu s - y}{\sqrt{s}}\right)\,.$$

Using Itô's lemma, we obtain the decomposition of $1 - e^{\nu V_t}H(\nu, |V_t|, 1-t)$ as a semi-martingale $M_t + C_t$.

We note that C increases only on the set $\{t\,:\, X_t = a\}$. Indeed, setting $g_1^a(X) = g$, for any predictable process H, one has

$$\mathbb{E}(H_g) = \mathbb{E}\left(\int_0^\infty dC_s H_s\right)$$

hence, since $X_g = a$,

$$0 = \mathbb{E}(\mathbb{1}_{X_g \ne a}) = \mathbb{E}\left(\int_0^\infty dC_s \mathbb{1}_{X_s \ne a}\right)\,.$$

Therefore, $dC_t = \kappa_t dL_t^a(X)$ and, since L increases only at points such that $X_t = a$ (i.e., $V_t = 0$), one has

$$\kappa_t = H_x'(\nu, 0, 1-t)\,.$$

The martingale part is given by $dM_t = m_t dW_t$ where

$$m_t = e^{\nu V_t} \left(\nu H(\nu, |V_t|, 1-t) - \mathrm{sgn}(V_t)\, H'_x(\nu, |V_t|, 1-t) \right) .$$

Therefore, the predictable compensator associated with $g_1^a(X)$ is

$$\int_0^t \frac{H'_x(\nu, 0, 1-s)}{e^{\nu V_s} H(\nu, 0, 1-s)} dL_s^a \,.$$

Exercise 8.10.8 The aim of this exercise is to compute, for $t < T < 1$, the quantity $\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{G}_t)$, which is the price of the claim $h(S_T)$ with barrier condition $\mathbb{1}_{\{T<g_1\}}$.

Prove that

$$\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{F}_t) = \mathbb{E}(h(W_T)|\mathcal{F}_t) - \mathbb{E}\Big(h(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big) \,\Big|\, \mathcal{F}_t\Big) \,,$$

where

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp\left(-\frac{u^2}{2}\right) du \,.$$

Define $k(w) = h(w)\Phi(|w|/\sqrt{1-T})$. Prove that $\mathbb{E}\Big(k(W_T) \,\Big|\, \mathcal{F}_t\Big) = \widetilde{k}(t, W_t)$, where

$$\begin{aligned} \widetilde{k}(t,a) &= \mathbb{E}\Big(k(W_{T-t}+a)\Big) \\ &= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} h(u)\Phi\Big(\frac{|u|}{\sqrt{1-T}}\Big) \exp\Big(-\frac{(u-a)^2}{2(T-t)}\Big)\, du. \end{aligned}$$

$\triangleleft$

8.10.4 Time When the Supremum is Reached

Let W be a Brownian motion, $M_t = \sup_{s \le t} W_s$ and let τ be the time when the supremum on the interval $[0,1]$ is reached, i.e.,

$$\tau = \inf\{t \le 1 \,:\, W_t = M_1\} = \sup\{t \le 1 \,:\, M_t - W_t = 0\} \,.$$

Let us denote by ζ the positive continuous semimartingale

$$\zeta_t = \frac{M_t - W_t}{\sqrt{1-t}},\ t < 1.$$

Let $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$. Since $F_t = \Phi(\zeta_t)$, (where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2}) du$, (see Exercise in Chapter 4 in [3M]) using Itô's formula, we obtain the canonical decomposition of F as follows:

$$\begin{aligned} F_t &= \int_0^t \Phi'(\zeta_u)\, d\zeta_u + \frac{1}{2}\int_0^t \Phi''(\zeta_u)\, \frac{du}{1-u} \\ &\overset{(i)}{=} -\int_0^t \Phi'(\zeta_u) \frac{dW_u}{\sqrt{1-u}} + \sqrt{\frac{2}{\pi}} \int_0^t \frac{dM_u}{\sqrt{1-u}} \overset{(ii)}{=} U_t + \tilde{F}_t, \end{aligned}$$

where $U_t = -\int_0^t \Phi'(\zeta_u) \dfrac{dW_u}{\sqrt{1-u}}$ is a martingale and $\widetilde{F}$ a predictable increasing process. To obtain (i), we have used that $x\Phi' + \Phi'' = 0$; to obtain (ii), we have used that $\Phi'(0) = \sqrt{2/\pi}$ and also that the process M increases only on the set

$$\{u \in [0,t] : M_u = W_u\} = \{u \in [0,t] : \zeta_u = 0\}.$$

8.10.5 Last Passage Times for Particular Martingales

We now study the Azéma supermartingale associated with the random time L, a last passage time or the end of a predictable set Γ, i.e.,

$$L(\omega) = \sup\{t \,:\, (t, \omega) \in \Gamma\}\,.$$

Proposition 8.10.9 *Let L be the end of a predictable set. Assume that all the $\mathbb{F}$-martingales are continuous and that L avoids the $\mathbb{F}$-stopping times. Then, there exists a continuous and nonnegative local martingale N, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$Z_t = \mathbb{P}\left(L > t \mid \mathcal{F}_t\right) = \frac{N_t}{\Sigma_t}$$

where $\Sigma_t = \sup_{s\le t} N_s$. The Doob-Meyer decomposition of Z is

$$Z_t = m_t - A_t$$

and the following relations hold

$$\begin{aligned} N_t &= \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) \\ \Sigma_t &= \exp(A_t) \\ m_t &= 1 + \int_0^t \frac{dN_s}{\Sigma_s} = \mathbb{E}(\ln S_\infty | \mathcal{F}_t) \end{aligned}$$

PROOF: As recalled previously, the Doob-Meyer decomposition of Z reads $Z_t = m_t - A_t$ with m and A continuous, and dA_t is carried by $\{t \,:\, Z_t = 1\}$. Then, for $t < T_0 := \inf\{t \,:\, Z_t = 0\}$

$$-\ln Z_t = -\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) + A_t$$

From Skorokhod's reflection lemma we deduce that

$$A_t = \sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)$$

Introducing the local martingale N defined by

$$N_t = \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right)\,,$$

it follows that

$$Z_t = \frac{N_t}{\Sigma_t}$$

and

$$\Sigma_t = \sup_{u\le t} N_u = \exp\left(\sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)\right) = e^{A_t}$$

□

The three following exercises are from the work of Bentata and Yor [22].

Exercise 8.10.10 Let M be a positive martingale, such that $M_0 = 1$ and $\lim_{t\to\infty} M_t = 0$. Let $a \in [0, 1[$ and define $G_a = \sup\{t \,:\, M_t = a\}$. Prove that

$$\mathbb{P}(G_a \le t | \mathcal{F}_t) = \left(1 - \frac{M_t}{a}\right)^+$$

Assume that, for every $t>0$, the law of the r.v. M_t admits a density $(m_t(x), x\geq 0)$, and $(t,x)\to m_t(x)$ may be chosen continuous on $(0,\infty)^2$ and that $d\langle M\rangle_t = \sigma_t^2 dt$, and there exists a jointly continuous function $(t,x)\to\theta_t(x) = \mathbb{E}(\sigma_t^2|M_t = x)$ on $(0,\infty)^2$. Prove that

$$\mathbb{P}(G_a\in dt) = (1-\frac{M_0}{a})\delta_0(dt) + 1\!\!1_{\{t>0\}}\frac{1}{2a}\theta_t(a)m_t(a)dt$$

`Hint:` Use Tanaka's formula to prove that the result is equivalent to $d_t\mathbb{E}(L_t^a(M)) = dt\theta_t(a)m_t(a)$ where L is the Tanaka-Meyer local time. $\triangleleft$

Exercise 8.10.11 Let B be a Brownian motion and

$$\begin{aligned} T_a^{(\nu)} &= \inf\{t\,:\,B_t+\nu t = a\}\\ G_a^{(\nu)} &= \sup\{t\,:\,B_t+\nu t = a\}\end{aligned}$$

Prove that

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{\text{law}}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}}\right)$$

Give the law of the pair $(T_a^{(\nu)}, G_a^{(\nu)})$. $\triangleleft$

Exercise 8.10.12 Let X be a transient diffusion, such that

$$\begin{aligned} \mathbb{P}_x(T_0<\infty) &= 0, x>0\\ \mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) &= 1, x>0\end{aligned}$$

and note s the scale function satisfying $s(0^+) = -\infty, s(\infty) = 0$. Prove that for all $x,t>0$,

$$\mathbb{P}_x(G_y\in dt) = \frac{-1}{2s(y)}p_t^{(m)}(x,y)dt$$

where $p^{(m)}$ is the density transition w.r.t. the speed measure m. $\triangleleft$

8.11 NUPBR

In this section, we study another kind of arbitrages in progressive enlargement. (See Section 1.5.2). We do not present the full theory (for which we refer the reader to [7, 6, 4] and [1]).

8.11.1 Before τ

Let $\widehat{m}$ be the $\mathbb{G}$-martingale stopped at time τ associated with m as in Theorem 8.3.1, on $\{t\leq\tau\}$

$$\widehat{m}_t := m_t^\tau - \int_0^t \frac{d\langle m,m\rangle_s^{\mathbb{F}}}{Z_{s-}}\,.$$

Case of Continuous Filtration

We start with the particular case of continuous martingales and prove that, for any random time τ, NUPBR holds before τ. As a consequence, in the case of honest times, all the deflators are strict local martingales

We recall that the continuity assumption implies that the martingale part of Z is continuous and that the optional and Doob-Meyer decompositions of Z are the same.

Proposition 8.11.1 *Assume that all $\mathbb{F}$ martingales are continuous. Then, for any random time τ, NUPBR holds before τ. A deflator is given by $L = \mathcal{E}(-\frac{1}{Z} \centerdot \widehat{m})$.*

PROOF: Define the positive $\mathbb{G}$ local martingale L as $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$. Then, if SL is a $\mathbb{G}$-local martingale, NUPBR holds. Recall that, using 8.3.1 again,

$$\widehat{S}_t := S_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

is a $\mathbb{G}$ local martingale. From integration by parts, we obtain (using that the bracket of continuous martingales is continuous and does not depend on the filtration)

$$\begin{aligned} d(LS^\tau)_t &= L_{t-}dS_t^\tau + S_{t-}dL_t + d\langle L, S^\tau\rangle_t^{\mathbb{G}} \\ &\overset{\text{mart}}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle_t^{\mathbb{F}} + \frac{1}{Z_{t-}} L_{t-} d\langle S, \widehat{m}\rangle_t^{\mathbb{G}} \\ &\overset{\text{mart}}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t \right) = 0 \end{aligned}$$

where $X \overset{\text{mart}}{=} Y$ is a notation for $X - Y$ is a $\mathbb{G}$ local martingale. □

Exercise 8.11.2 Prove that, if $Z = N/N^*$ is the multiplicative decomposition of Z, then $L = \frac{1}{N}$.◁

Case of a Poisson Filtration

We assume that S is an $\mathbb{F}$ martingale of the form $dS_t = S_{t-}\psi_t dM_t$, where ψ is a predictable process, satisfying $\psi > -1$ and $\psi \neq 0$, where M is the compensated martingale of a standard Poisson process.

In a Poisson setting, from PRP, $dm_t = \nu_t dM_t$ for some predictable process ν, so that, on $t \leq \tau$,

$$d\widehat{m}_t = dm_t - \frac{1}{Z_{t-}} d\langle m\rangle_t = dm_t - \frac{1}{Z_{t-}} \lambda \nu_t^2 dt$$

Proposition 8.11.3 *In a Poisson setting, for any random time τ, NUPBR holds before τ. Furthermore,*

$$L = \mathcal{E}\left(-\frac{1}{Z_- + \nu} \centerdot \widehat{m}\right) = \mathcal{E}\left(-\frac{\nu}{Z_- + \nu} \centerdot \widehat{M}\right),$$

is a $\mathbb{G}$-local martingale deflator for S^τ.

PROOF: We are looking for a deflator of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\nu_t\kappa_t > -1$) so that L is positive and $S^\tau L$ is a $\mathbb{G}$ local martingale. Integration by parts formula leads to (on $t \leq \tau$)

$$\begin{aligned} d(LS)_t &= L_{t-}dS_t + S_{t-}dL_t + d[L, S]_t \\ &\overset{\text{mart}}{=} L_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}S_{t-}\kappa_t\psi_t\nu_t dN_t \\ &\overset{\text{mart}}{=} L_{t-}S_{t-}\psi_t \frac{1}{Z_{t-}} \nu_t \lambda dt + L_{t-}S_{t-}\kappa_t\psi_t\nu_t\lambda(1 + \frac{1}{Z_{t-}}\nu_t)dt \\ &= L_{t-}S_{t-}\psi_t\nu_t\lambda \left(\frac{1}{Z_{t-}} + \kappa_t(1 + \frac{1}{Z_{t-}}\nu_t) \right) dt. \end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+\nu_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = -L_{t-}\frac{1}{Z_{t-} + \nu_t} \nu_t d\widehat{M}_t$$

is indeed a positive martingale, since $\frac{1}{Z_{t-}+\nu_t}\nu_t < 1$. This last equality follows from the fact that

$$dN_t = dM_t + \lambda dt = d\widehat{M}_t + \lambda(1 + \frac{\nu_t}{Z_t})dt$$

is the Doob-Meyer decomposition of the submartingale N, hence the predictable bounded variation part (in $\mathbb{G}$) part is increasing. It is also possible to note that

$$\Delta(\frac{\nu}{Z_- + \nu} \centerdot \widehat{M}) = -\frac{\nu \Delta M}{Z_- + \nu \Delta M} = \frac{\Delta m}{\widetilde{Z}_-} = -1 + \frac{Z_-}{\widetilde{Z}} > -1$$

□

□

Lévy Processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its compensator. Here, $\psi \star (\mu - \nu)$ is the process $\int_0^{\cdot} \int \psi(x,s)(\mu(dx,ds) - \nu(dx,ds)$. The martingale m admits a representation as $m = \psi^m \star (\mu - \nu)$. Then, the $\mathbb{G}$ compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt,dx) = \frac{1}{Z_{t-}}\left(Z_{t-} + \psi^m(t,x)\right)\nu(dt,dx)$$

i.e., S admits a $\mathbb{G}$-semi-martingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Proposition 8.11.4 *Consider the positive $\mathbb{G}$-local martingale*

$$L := \mathcal{E}\left(-\frac{\psi^m}{Z_- + \psi^m} I_{]\!]0,\tau]\!]} \star (\nu - \nu^{\mathbb{G}})\right).$$

Then L is a $\mathbb{G}$-local martingale deflator for S^τ, and hence S^τ satisfies NUPBR.

PROOF: Our goal is to find a positive martingale L of the form

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t$$

so that LS is a local martingale.

From integration by parts formula

$$\begin{aligned} d(SL) &\overset{\text{mart}}{=} -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + d[S,L] = -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + L_-\psi\psi^m\kappa \star \mu \\ &\overset{\text{mart}}{=} -L_-\psi \star (\nu - \nu^{\mathbb{G}}) + L_-\psi\psi^m\kappa \star \nu^{\mathbb{G}} \\ &= -L_-\psi\left(1 - (1+\psi^m\kappa)\frac{1}{Z_-}\left(Z_- + \psi^m\right)\right) \star \nu \end{aligned}$$

Hence the possible choice $\kappa = -\frac{1}{Z_- + \psi^m}$. It can be checked that indeed, L is a positive martingale. See [6] □

8.11.2 After τ

We now assume that τ is an honest time, which satisfies $Z_\tau < 1$. Note that, in the case of continuous filtration, and $Z_\tau = 1$, NUPBR fails to hold after τ (see [64]).

For any $\mathbb{F}$ local martingale X (in particular for m and S)

$$\widehat{X}_t := X^\tau_t - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle^{\mathbb{F}}_s}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle^{\mathbb{F}}_s}{1-Z_s}$$

is a $\mathbb{G}$ local martingale.

We prove that, under the above conditions, NUPBR holds after τ. As a consequence, all the deflators are strict local martingales

Case of Continuous Filtration

We start with the particular case of continuous martingales and prove that, for any honest time τ, NUPBR holds after τ.

Proposition 8.11.5 *Assume that τ is an honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$ martingales are continuous. Then, for any honest time τ, NUPBR holds after τ. A deflator is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.*

PROOF: The proof is based on Itô's calculus. Looking for a deflator of the form $dL_t = L_t\kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1-Z)^{-1}$, the process $L(S - S^\tau)$ is a $\mathbb{G}$-local martingale. □

Case of a Poisson Filtration

We assume that S is an $\mathbb{F}$ martingale of the form $dS_t = S_{t-}\psi_t dM_t$, with ψ is a predictable process, satisfying $\psi > -1$.

The decomposition formula reads, after τ as

$$\widehat{S}_t = (1\!\!1_{]\tau,\infty[} \cdot S)_t + \int_{t\vee\tau}^t \frac{1}{1-Z_{s-}} d\langle S, m\rangle_s = (1\!\!1_{]\tau,\infty[} \cdot S)_t + \lambda \int_{t\vee\tau}^t \frac{1}{1-Z_{s-}} \nu_s\psi_s S_{s-} ds$$

Proposition 8.11.6 *Let $\mathbb{F}$ be a Poisson filtration and τ be an honest time satisfying $Z_\tau < 1$. Then, NUPBR holds after τ. Furthermore,*

$$L = \mathcal{E}\left(\frac{1}{1-Z_- - \nu} \centerdot \widehat{m}\right) = \mathcal{E}\left(\frac{\nu}{1-Z_- - \nu} 1\!\!1_{]\tau,\infty[} \centerdot \widehat{M}\right),$$

is a $\mathbb{G}$-martingale deflator for $S - S^\tau$.

PROOF: We are looking for a RN density of the form $dL_t = L_{t-}\kappa_t d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that L is positive $\mathbb{G}$ local martingale and $(S - S^\tau)L$ is a $\mathbb{G}$ local martingale. Integration by parts formula leads to

$$\begin{aligned}
d(L(S-S^\tau))_t &= L_{t-}d(S-S^\tau)_t + (S_{t-} - S^\tau_{t-})dL_t + d[L, S-S^\tau]_t \\
&\overset{\text{mart}}{=} -\lambda L_{t-}S_{t-}\nu_t\psi_t \frac{1}{1-Z_{t-}} 1\!\!1_{\{t>\tau\}}dt + L_{t-}S_{t-}\kappa_t\psi_t\nu_t 1\!\!1_{\{t>\tau\}}dN_t \\
&\overset{\text{mart}}{=} -\lambda L_{t-}S_{t-}\nu_t\psi_t \frac{1}{1-Z_{t-}} 1\!\!1_{\{t>\tau\}}dt + \lambda L_{t-}S_{t-}\kappa_t\psi_t\nu_t 1\!\!1_{\{t>\tau\}}(1 - \frac{1}{1-Z_{t-}}\nu_t)dt \\
&= \lambda L_{t-}S_{t-}\psi_t\nu_t 1\!\!1_{\{t>\tau\}} \left(-\frac{1}{1-Z_{t-}} + \kappa_t(1 - \frac{1}{1-Z_{t-}}\nu_t)\right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = \frac{1}{1-Z_{t-}-\nu_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = L_{t-}\frac{1}{1-Z_{t-}-\nu_t}\nu_t 1\!\!1_{\{t>\tau\}} d\widehat{M}_t$$

is indeed a positive martingale, since $\frac{1}{1-Z_{t-}-\nu_t}\nu_t \Delta N_t > -1$. This last fact can be proved using the same argument as before.

Lévy Processes

Assume that $S = \psi \star (\mu - \nu)$ where μ is the jump measure of a Lévy process and ν its compensator.

Then, the $\mathbb{G}$ compensator of μ is $\nu^{\mathbb{G}}$ where

$$\nu^{\mathbb{G}}(dt,dx) = \left(1 + 1\!\!1_{\{t\le\tau\}}\frac{1}{Z_{t-}}\psi^m(t,x) - 1\!\!1_{\{t>\tau\}}\frac{1}{1-Z_{t-}}\psi^m(t,x)\right)\nu(dt,dx)$$

i.e., S admits a $\mathbb{G}$-semi-martingale decomposition of the form

$$S = \psi \star (\mu - \nu^{\mathbb{G}}) - \psi \star (\nu - \nu^{\mathbb{G}})$$

Proposition 8.11.7 *Assume that τ be an honest time satisfying $Z_\tau < 1$ in a Lévy framework. Then, the positive $\mathbb{G}$-local martingale*

$$L := \mathcal{E}\left(\frac{\psi^m}{1-Z_- - \psi^m} I_{]\!]\tau,\infty[\![} \star (\nu - \nu^{\mathbb{G}})\right),$$

is a $\mathbb{G}$-martingale density for $S - S^\tau$, and hence $S - S^\tau$ satisfies NUPBR.

Proof: Our goal is to find a positive martingale L of the form

$$dL_t = L_{t-}\kappa_t 1\!\!1_{\{t>\tau\}} d\widehat{m}_t$$

so that $L(S - S^\tau)$ is a local martingale.

From integration by parts formula

$$\begin{aligned}
d(L(S-S^\tau)) &\overset{\text{mart}}{=} -L_- d(S - S^\tau) + d[S,L] \\
&= -L_-\psi\frac{\psi^m}{1-Z_-}1\!\!1_{]\tau,\infty[} \star \nu + L_-\kappa\psi\psi^m 1\!\!1_{]\tau,\infty[} \star \mu \\
&\overset{\text{mart}}{=} -L_-\psi\frac{\psi^m}{1-Z_-}1\!\!1_{]\tau,\infty[} \star \nu + L_-\kappa\psi\psi^m 1\!\!1_{]\tau,\infty[} \star \nu^{\mathbb{G}} \\
&= -L_-\psi\psi^m 1\!\!1_{]\tau,\infty[}\left(-\frac{1}{1-Z_-} + \kappa(1 - \frac{\psi^m}{1-Z_-})\right) \star \nu
\end{aligned}$$

Hence the possible choice $\kappa = \frac{1}{1-Z_- - \psi^m}$. □

8.11.3 General Results

We give here some general results. We refer the reader [6, 7, 1] for the proof of the first result (before τ) and to [4] for the second result. We recall that a set $A \subset \Omega \times \mathbb{R}^+$ is evanescent if the process $1\!\!1_A$ is indistinguishable from 0. A set A is thin if it is contained in the union of graphs of stopping times.

The following are equivalent:
(a) The thin set $\left\{\widetilde{Z} = 0 \;\&\; Z_- > 0\right\}$ is evanescent.
(b) For any (bounded) X satisfying NUPBR($\mathbb{F}$), X^τ satisfies NUPBR($\mathbb{G}$).

Suppose that τ is an honest time such that $Z_\tau < 1$. Then, the following assertions are equivalent.
(a) For any S satisfying NUPBR($\mathbb{F}$), the process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).
(b) The thin set $\{\widetilde{Z} = 1 \;\&\; Z_- < 1\}$ is evanescent.

Chapter 9

Initial and Progressive Enlargements with $(\mathcal{E})$-times

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses of right-continuity and completeness, and where $\mathcal{F}_0$ is the trivial σ-field. Let τ be a finite random time (i.e., a finite non-negative random variable) with law ν, $\nu(du) = \mathbb{P}(\tau \in du)$.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})$) the predictable (resp. optional) σ-algebra corresponding to $\mathbb{F}$ on $\mathbb{R}^+ \times \Omega$. We consider the three nested filtrations

$$\mathbb{F} \subset \mathbb{G} \subset \mathbb{F}^{(\tau)}$$

where $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$ stand, respectively, for the *progressive* and the *initial* enlargement of $\mathbb{F}$ with the random time τ.

In this chapter, our standing assumption is the following:

Hypothesis 9.0.8 $(\mathcal{E})$-***Hypothesis***
The $\mathbb{F}$-(regular) conditional law of τ is equivalent to the law of τ. Namely,

$$\mathbb{P}(\tau \in du|\mathcal{F}_t) \sim \nu(du) \quad \textit{for every } t \geq 0, \quad \mathbb{P} - a.s.$$

We assume that ν has no atoms and has $\mathbb{R}_+$ as support.

We shall call $(\mathcal{E})$-times random times which satisfy $(\mathcal{E})$-Hypothesis. This assumption, in the case when $t \in [0, T]$, corresponds to the *equivalence assumption* in Föllmer and Imkeller [60] and in Amendinger's thesis [9, Assumption 0.2] and to hypothesis (HJ) in the papers by Grorud and Pontier (see, e.g., [66]). Under the $(\mathcal{E})$-Hypothesis, we address the following problems:

- Characterization of $\mathbb{G}$-martingales and $\mathbb{F}^{(\tau)}$-martingales in terms of $\mathbb{F}$-martingales;
- Canonical decomposition of an $\mathbb{F}$-martingale, as a semimartingale, in $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$;
- Predictable Representation Theorem in $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$.

This chapter is based on [9] and [31].

9.1 Preliminaries

The exploited idea is the following: assuming that the $\mathbb{F}$-conditional law of τ is equivalent to the law of τ, after an *ad hoc* change of probability measure, the problem reduces to the case where τ

and $\mathbb{F}$ are independent. Under this newly introduced probability measure, working in the initially enlarged filtration is "easy". Then, under the original probability measure, for the initially enlarged filtration, the results are achieved by means of Girsanov's theorem. Finally, by projection, one obtains the results of interest in the progressively enlarged filtration (notice that, alternatively, they can be obtained with another application of Girsanov's theorem, starting from the newly introduced probability measure, with respect to the progressively enlarged filtration).

The "change of probability measure" viewpoint for treating problems on enlargement of filtrations was remarked in the early 80's and developed by Song in [122] (see also Jacod [76, Section 5]). This is also the point of view adopted by Gasbarra et al. in [65] while applying the Bayesian approach to study the impact of the initial enlargement of filtration on the characteristic triplet of a semimartingale. For what concerns the idea of recovering the results in the progressively enlarged filtration starting from the ones in the initially enlarged one, we have to cite Yor [133].

Amongst the consequences of the ($\mathcal{E}$)-Hypothesis, one has the existence and regularity of the conditional density, for which we refer to Amendinger's reformulation (see [9, Remarks, p. 17]) of Jacod's result [76, Lemma 1.8]: there exists a strictly positive $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \to p_t(\omega, u)$, such that for every $u \in \mathbb{R}^+$, $p(u)$ is a càdlàg $(\mathbb{P}, \mathbb{F})$-martingale and

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty p_t(u)\nu(du) \quad \text{for every } t \geq 0, \ \ \mathbb{P} - a.s.$$

In particular, $p_0(u) = 1$ for every $u \in \mathbb{R}^+$ and $\int_0^\infty p_t(u)\nu(du) = 1$, $\forall t$. This family of processes p is called the $(\mathbb{P}, \mathbb{F})$-*conditional density* of τ with respect to ν, or the *density* of τ if there is no ambiguity.

Furthermore, under the ($\mathcal{E}$)-Hypothesis, the assumption that ν has no atoms implies that the default time τ avoids the $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \xi) = 0$ for every $\mathbb{F}$-stopping time ξ (see, e.g., El Karoui et al. [49, Corollary 2.2]).

It was shown in [9, Proposition 1.10] that the strict positiveness of p implies the right-continuity of the filtration $\mathbb{F}^{(\tau)}$.

In the sequel, we will consider the right-continuous version of all the martingales.

Now, we consider the change of probability measure introduced, independently, by Grorud and Pontier in [66] and by Amendinger in [9] (for an initial enlargement with a random variable L instead of with a random time τ).

Lemma 9.1.1 *Let L be the $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale defined as $L_t = \frac{1}{p_t(\tau)}$, $t \geq 0$, and $\mathbb{P}^*$ the probability measure defined on $\mathbb{F}^{(\tau)}$ as*

$$d\mathbb{P}^*{}_{|\mathcal{F}_t^{(\tau)}} = L_t \, d\mathbb{P}_{|\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)} d\mathbb{P}_{|\mathcal{F}_t^{(\tau)}} \, .$$

Under $\mathbb{P}^*$, the random time τ is independent of $\mathcal{F}_t$ for any $t \geq 0$ and, moreover

$$\mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t} \ \textit{for any } t \geq 0, \qquad \mathbb{P}^*{}_{|\sigma(\tau)} = \mathbb{P}_{|\sigma(\tau)}.$$

Proof: In a first step, we prove that L is an $\mathbb{F}^{(\tau)}$-martingale. We shall denote by $L_t(x) = \frac{1}{p_t(x)}$. Indeed, $\mathbb{E}(L_t | \mathcal{F}_s^{(\tau)}) = L_s$ if (and only if) $\mathbb{E}(L_t h(\tau) A_s) = \mathbb{E}(L_s h(\tau) A_s)$ for any (bounded) Borel function h and any $\mathcal{F}_s$-measurable (bounded) random variable A_s. From definition of p, one has

$$\begin{aligned} \mathbb{E}(L_t h(\tau) A_s) &= \mathbb{E}(\int_\mathbb{R} L_t(x) h(x) p_t(x) \nu(dx) A_s) = \mathbb{E}(\int_\mathbb{R} h(x)\nu(dx) A_s) \\ &= \int_\mathbb{R} h(x)\nu(dx) \mathbb{E}(A_s) = \mathbb{E}(A_s)\,\mathbb{E}(h(\tau)) \end{aligned}$$

The particular case $t = s$ leads to $\mathbb{E}(L_s h(\tau) A_s) = \mathbb{E}(h(\tau))\,\mathbb{E}(A_s)$, hence $\mathbb{E}(L_s h(\tau) A_s) = \mathbb{E}(L_t h(\tau) A_s)$. Note that, since $p_0(x) = 1$, one has $\mathbb{E}(1/p_t(\tau) | \mathcal{F}_0^{(\tau)}) = 1/p_0(\tau) = 1$.

Now, we prove the required independence. From the above,

$$\mathbb{E}_{\mathbb{Q}}(h(\tau)A_s) = \mathbb{E}_{\mathbb{P}}(L_s h(\tau)A_s) = \mathbb{E}_{\mathbb{P}}(h(\tau))\,\mathbb{E}_{\mathbb{P}}(A_s)$$

For $h = 1$ (resp. $A_t = 1$), one obtains $\mathbb{E}_{\mathbb{Q}}(A_s) = \mathbb{E}_{\mathbb{P}}(A_s)$ (resp. $\mathbb{E}_{\mathbb{Q}}(h(\tau)) = \mathbb{E}_{\mathbb{P}}(h(\tau))$) and we are done. □

Lemma 9.1.2 *Let X be a $(\mathbb{P},\mathbb{F})$ martingale. The process $\widetilde{X}(\tau)$ defined by $\widetilde{X}_t(\tau) = X_t/p_t(\tau)$ is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale and satisfies $\mathbb{E}(\widetilde{X}_t(\tau)|\mathcal{F}_t) = X_t$.*

PROOF: To establish the martingale property, it suffices to check that for $s < t$ and $A \in \mathcal{F}_s^{(\tau)}$, one has $\mathbb{E}_{\mathbb{P}}(\widetilde{X}_t(\tau)\mathbb{1}_A) = \mathbb{E}_{\mathbb{P}}(\widetilde{X}_s(\tau)\mathbb{1}_A)$, which is equivalent to $\mathbb{E}_{\mathbb{Q}}(X_t\mathbb{1}_A) = \mathbb{E}_{\mathbb{Q}}(X_s\mathbb{1}_A)$. The last equality follows from the fact that the $(\mathbb{F},\mathbb{P})$ martingale X is also a $(\mathbb{F},\mathbb{Q})$ martingale (indeed $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathbb{F}$), hence a $(\mathbb{F}^{(\tau)},\mathbb{Q})$ martingale (by independence of τ and $\mathbb{F}$ under $\mathbb{Q}$. Bayes criteria shows that m^τ) is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale. Noting that $\mathbb{E}(1/p_t(\tau)|\mathcal{F}_t) = 1$ (take $A_s = 1$ and $h = 1$ in the preceeding proof), the equality

$$\mathbb{E}\widetilde{X}_t(\tau)|\mathcal{F}_t) = X_t\mathbb{E}(1/p_t(\tau)|\mathcal{F}_t) = m_t$$

ends the proof. □

Of course, the reverse holds true: if there exists a probability equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}$, the r.v. τ is independent to $\mathcal{F}_\infty$, then $(\mathbb{P},\mathbb{F})$-martingales are $(\mathbb{P},\mathbb{F}^{(\tau)})$-semi martingales.

The above properties imply that $\mathbb{P}^*(\tau \in du|\mathcal{F}_t) = \mathbb{P}^*(\tau \in du)$, so that the $(\mathbb{P}^*,\mathbb{F})$-density of τ, denoted by $p^*(u), u \geq 0$, is a constant equal to one, $\mathbb{P}^* \otimes \nu$-a.s.

Remark 9.1.3 If one assumes only abssolute continuity Jacod's hypothesis, the process $1/p_t(\tau)$ is well defined, but is no more a martingale. See [8].

Remark 9.1.4 The probability measure $\mathbb{P}^*$, being defined on $\mathcal{F}_t$ for $t \geq 0$, is (uniquely) defined on $\mathcal{F}_\infty = \bigvee_{t\geq 0}\mathcal{F}_t$. Then, as τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, it immediately follows that τ is also independent of $\mathcal{F}_\infty$, under $\mathbb{P}^*$. However, one can not claim that: "$\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty^{(\tau)}$", since we do not know *a priori* whether $\frac{1}{p(\tau)}$ is a *closed* $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale or not. A similar problem is studied by Föllmer and Imkeller in [60] (it is therein called "paradox") in the case where the reference (canonical) filtration is enlarged by means of the information about the endpoint at time $t = 1$. In our setting, it corresponds to the case where $\tau \in \mathcal{F}_\infty$ and $\tau \notin \mathcal{F}_t, \forall\ t$. In the Brownian bridge case, the conditional law of B_1 w.r.t. $\mathcal{F}_t$ is the Dirac measure for $t = 1$.

Notation 9.1.5 *In this chapter, as we mentioned, we deal with three different levels of information and two equivalent probability measures. In order to distinguish objects defined under $\mathbb{P}$ and under $\mathbb{P}^*$, we will use, in this chapter, a superscript $*$ when working under $\mathbb{P}^*$. For example, $\mathbb{E}$ and $\mathbb{E}^*$ stand for the expectations under $\mathbb{P}$ and $\mathbb{P}^*$, respectively. For what concerns the filtrations, when necessary, we will use the following illustrating notation: $x, X, X^{(\tau)}$ to denote processes adapted to $\mathbb{F}, \mathbb{G}$ and $\mathbb{F}^{(\tau)}$, respectively.*

Let $x = (x_t, t \geq 0)$ be a $(\mathbb{P},\mathbb{F})$-martingale. Since $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathbb{F}$, x is a $(\mathbb{P}^*,\mathbb{F})$-martingale, hence, using the fact that τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, a $(\mathbb{P}^*,\mathbb{G})$-martingale (and also a $(\mathbb{P}^*,\mathbb{F}^{(\tau)})$-martingale). Because of these facts, the measure $\mathbb{P}^*$ is called by Amendinger *"martingale preserving probability measure under initial enlargement of filtrations"*.

Exercise 9.1.6 Prove that $(Y_t(\tau), t \geq 0)$ is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale if and only if $Y_t(x)p_t(x)$ is a family of $\mathbb{F}$-martingales. ◁

Exercise 9.1.7 Let $\mathbb{F}$ be a Brownian filtration. Prove that, if X is a square integrable $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale, then, there exists a function h and a process ψ such that

$$X_t = h(\tau) + \int_0^t \psi_s(\tau) dB_s$$

$\triangleleft$

9.2 Expectation and projection tools

9.2.1 Optional Projection, initial enlargement

Lemma 9.2.1 *Let $Y_t^{(\tau)} = y_t(\tau)$ be an $\mathcal{F}_t^{(\tau)}$-measurable random variable.*
(i) If $y_t(\tau)$ is $\mathbb{P}$-integrable and $y_t(\tau) = 0$ $\mathbb{P}$-a.s. then, for ν-a.e. $u \geq 0$, $y_t(u) = 0$ $\mathbb{P}$-a.s.
(ii) For $s \leq t$ one has, $\mathbb{P}$-a.s. (or, equivalently, $\mathbb{P}^*$-a.s.):
if $y_t(\tau)$ is $\mathbb{P}^*$-integrable and if $y_t(u)$ is $\mathbb{P}$ (or $\mathbb{P}^*$)-integrable for any $u \geq 0$,

$$\mathbb{E}^*\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \mathbb{E}^*\big(y_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau} = \mathbb{E}\big(y_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau}; \tag{9.2.1}$$

if $y_t(\tau)$ is $\mathbb{P}$-integrable

$$\mathbb{E}\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \frac{1}{p_s(\tau)} \mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau}. \tag{9.2.2}$$

Proof: (i) We have, by applying Fubini-Tonelli's Theorem,

$$0 \;=\; \mathbb{E}\big(|y_t(\tau)|\big) = \mathbb{E}\Big(\mathbb{E}\big(|y_t(\tau)|\big|\mathcal{F}_t\big)\Big) = \mathbb{E}\Big(\int_0^\infty |y_t(u)|p_t(u)\nu(du)\Big).$$

Then $\int_0^\infty |y_t(u)|p_t(u)\nu(du) = 0$ $\mathbb{P}$-a.s. and, given that $p_t(u)$ is strictly positive for ν almost every u, we have that, for ν-almost every u, $y_t(\cdot, u) = 0$ $\mathbb{P}$-a.s.
(ii) The first equality in (9.2.1) is straightforward for elementary random variables of the form $f(\tau)x_t$, given the independence between τ and $\mathcal{F}_t$, for any $t \geq 0$. It is extended to $\mathcal{F}_t^{(\tau)}$-measurable r.vs via the monotone class theorem. The second equality follows from the fact that $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathcal{F}_t$, for any $t \geq 0$.

The result (9.2.2) is an immediate consequence of (9.2.1), since it suffices, by means of (conditional) Bayes' formula, to pass under the measure $\mathbb{P}^*$. More precisely, for $s < t$, we have

$$\mathbb{E}\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \frac{\mathbb{E}^*\big(y_t(\tau)p_t(\tau)|\mathcal{F}_s^{(\tau)}\big)}{\mathbb{E}^*\big(p_t(\tau)|\mathcal{F}_s^{(\tau)}\big)} = \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)_{|u=\tau},$$

where in the last equality we have used the previous result (9.2.1) and the fact that $p(\tau)$ is a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale. Note that if $y_t(\tau)$ is $\mathbb{P}$-integrable, then $\mathbb{E}(\int_0^\infty |y_t(u)|p_t(u)\nu(du)) = \mathbb{E}(|y_t(\tau)|) < \infty$, which implies that $\mathbb{E}(|y_t(u)|p_t(u)) < \infty$. $\square$

9.2.2 Optional Projection, progressive enlargement

The Azéma supermartingale associated with τ under the probability measure $\mathbb{P}$ (resp. $\mathbb{P}^*$) is

$$Z_t \;:=\; \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty p_t(u)\nu(du), \tag{9.2.3}$$

$$Z^*(t) \;:=\; \mathbb{P}^*(\tau > t|\mathcal{F}_t) = \mathbb{P}^*(\tau > t) = \mathbb{P}(\tau > t) = \int_t^\infty \nu(du) = G(t). \tag{9.2.4}$$

Note, in particular, that Z is a $(\mathbb{P},\mathbb{F})$ super-martingale, whereas $Z^*(\cdot)$ is a (deterministic) continuous and decreasing function. Furthermore, it is clear that, under the $(\mathcal{E})$-Hypothesis and the hypothesis that the support of ν is $\mathbb{R}_+$, Z and Z^* do not vanish.

Lemma 9.2.2 *Let $Y_t^{(\tau)} = y_t(\tau)$ be an $\mathcal{F}_t^{(\tau)}$-measurable, $\mathbb{P}$-integrable random variable. Then, for $s \le t$,*

$$\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_s) = \mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \tilde{y}_s \mathbb{1}_{s<\tau} + \hat{y}_s(\tau)\mathbb{1}_{\tau\le s},$$

with

$$\begin{aligned}\tilde{y}_s &= \frac{1}{Z_s}\mathbb{E}\Big(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big)\,,\\ \hat{y}_s(u) &= \frac{1}{p_s(u)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\,.\end{aligned}$$

Proof: From the above Proposition 8.2.1, it is clear that $\mathbb{E}(y_t(\tau)|\mathcal{G}_s)$ can be written in the form $\tilde{y}_s\mathbb{1}_{s<\tau} + \hat{y}_s(\tau)\mathbb{1}_{\tau\le s}$. On the set $\{s<\tau\}$, we have, applying the key Lemma 8.2.3 and using the $(\mathcal{E})$-Hypothesis,

$$\begin{aligned}\mathbb{1}_{s<\tau}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) &= \mathbb{1}_{s<\tau}\frac{\mathbb{E}\left[\mathbb{E}(y_t(\tau)\mathbb{1}_{s<\tau}|\mathcal{F}_t)|\mathcal{F}_s\right]}{Z_s}\\ &= \mathbb{1}_{s<\tau}\frac{1}{Z_s}\mathbb{E}\left(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\right) =: \mathbb{1}_{s<\tau}\tilde{y}_s.\end{aligned}$$

On the complementary set, we have, by applying Lemma 9.2.1,

$$\mathbb{1}_{\tau\le s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \mathbb{1}_{\tau\le s}\mathbb{E}\left[\mathbb{E}(y_t(\tau)|\mathcal{G}_s^{\tau})|\mathcal{G}_s\right] = \mathbb{1}_{\tau\le s}\frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau} =: \mathbb{1}_{\tau\le s}\hat{y}_s(\tau).$$

□

For $s>t$, we obtain $\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_s) = \frac{1}{Z_s}\int_s^\infty y_t(u)p_s(u)\nu(du)\mathbb{1}_{s<\tau} + y_t(\tau)\mathbb{1}_{\tau\le s}$.

As an application, projecting the martingale L (defined earlier as $L_t = \frac{1}{p_t(\tau)}, t\ge 0$) on $\mathbb{G}$ yields to the corresponding Radon-Nikodým density on $\mathbb{G}$:

$$d\mathbb{P}^*{}_{|\mathcal{G}_t} = \ell_t\, d\mathbb{P}_{|\mathcal{G}_t}\,,$$

with

$$\begin{aligned}\ell_t &:= \mathbb{E}(L_t|\mathcal{G}_t) = \mathbb{1}_{t<\tau}\frac{1}{Z_t}\int_t^\infty \nu(du) + \mathbb{1}_{\tau\le t}\frac{1}{p_t(\tau)}\\ &= \mathbb{1}_{t<\tau}\frac{G(t)}{Z_t} + \mathbb{1}_{\tau\le t}\frac{1}{p_t(\tau)}\,.\end{aligned}$$

Proposition 9.2.3 *The Azéma super-martingale Z, introduced in Equation (9.2.3), admits the Doob-Meyer decomposition $Z_t = \mu_t - \int_0^t p_u(u)\nu(du), t\ge 0$, where μ is the $\mathbb{F}$-martingale defined as*

$$\mu_t := 1 - \int_0^t \left(p_t(u) - p_u(u)\right)\nu(du)$$

The intensity of τ is $\lambda_t = \frac{p_t(t)}{Z_t}$.

Proof: From the definition of Z and using the fact that $p(u)$ is martingale,

$$Z_t + \int_0^t p_u(u)\nu(du) = \int_t^\infty p_t(u)\nu(du) + \int_0^t p_u(u)\nu(du) = \mathbb{E}(\int_0^\infty p_u(u)\nu(du)|\mathcal{F}_t)$$

□

We now recall some useful facts concerning the compensated martingale of H. We know, from the general theory (see Proposition 2.2.7), that denoting by H the default indicator process $H_t = 1\!\!1_{\tau\leq t}, t \geq 0$, the process M defined as

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_s \,\nu(ds), \quad t \geq 0, \tag{9.2.5}$$

with $\lambda_t = \frac{p_t(t)}{Z_t}$, is a $(\mathbb{P}, \mathbb{G})$-martingale and that

$$M_t^* := H_t - \int_0^{t\wedge\tau} \lambda^*(s) \,\nu(ds), \quad t \geq 0, \tag{9.2.6}$$

with $\lambda^*(t) = \frac{1}{G(t)}$, is a $(\mathbb{P}^*, \mathbb{G})$-martingale. Furthermore, since λ^* is deterministic, M^* (being $\mathbb{H}$-adapted) is a $(\mathbb{P}^*, \mathbb{H})$-martingale, too.

9.2.3 Predictable projections

We conclude this subsection with the following two propositions, concerning the predictable projection, respectively on $\mathbb{F}$ and on $\mathbb{G}$, of a $\mathbb{F}^{(\tau)}$-predictable process. The first result is due to Jacod [76, Lemme 1.10].

Proposition 9.2.4 *Let $Y^{(\tau)} = y(\tau)$ be an $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process. Then, the $\mathbb{P}$-predictable projection of $Y^{(\tau)}$ on $\mathbb{F}$ is given by*

$$^{(p,\mathbb{F})}\big(Y^{(\tau)}\big)_t = \int_0^\infty y_t(u) p_{t-}(u)\nu(du)\ .$$

Proof: It is obtained by a monotone class argument and by using the definition of density of τ, writing, for "elementary" processes, $Y_t^{(\tau)} := y_t f(\tau)$, with y a bounded $\mathbb{F}$-predictable process and f a bounded Borel function. For this, we refer to the proof in Jacod [76, Lemme 1.10]. □

Proposition 9.2.5 *Let $Y^{(\tau)} = y(\tau)$ be an $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process. Then, the $\mathbb{P}$-predictable projection of $Y^{(\tau)}$ on $\mathbb{G}$ is given by*

$$^{(p,\mathbb{G})}\big(Y^{(\tau)}\big)_t = 1\!\!1_{t\leq\tau}\frac{1}{Z_{t-}}\int_t^\infty y_t(u) p_{t-}(u)\nu(du) + 1\!\!1_{\tau<t} y_t(\tau)\ .$$

Proof: By the definition of predictable projection, we know (from Proposition 8.2.1 (ii)) that we are looking for a (unique) process of the form

$$^{(p,\mathbb{G})}\big(Y^{(\tau)}\big)_t = \tilde{y}_t 1\!\!1_{t\leq\tau} + \hat{y}_t(\tau) 1\!\!1_{\tau<t}, \quad t \geq 0,$$

where $\tilde{y}$ is $\mathbb{F}$-predictable, positive or bounded, and $(t,\omega,u) \mapsto \hat{y}_t(\omega,u)$ is a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable positive or bounded function, to be identified.

- On the predictable set $\{\tau < t\}$, being $Y^{(\tau)}$ an $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process (recall Proposition 6.1.1 (ii)), we immediately find $\hat{y}(\tau) = y(\tau)$;
- On the complementary set $\{t \leq \tau\}$, introducing the $\mathbb{G}$-predictable process

  $$Y := {}^{(p,\mathbb{G})}\big(Y^{(\tau)}\big)$$

  it is possible to use Jeulin [82, Remark 4.5, page 64] (see also Dellacherie et al. [41, Ch. XX, page 186]), to write

  $$Y 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{Z_-}\ {}^{(p,\mathbb{F})}\big(Y 1\!\!1_{]\!]0,\tau]\!]}\big) 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{Z_-}\ {}^{(p,\mathbb{F})}\Big({}^{(p,\mathbb{G})}\big(Y^{(\tau)}\big) 1\!\!1_{]\!]0,\tau]\!]}\Big) 1\!\!1_{]\!]0,\tau]\!]}.$$

We then have, being $1\!\!1_{]\!]0,\tau]\!]}$, by definition, $\mathbb{G}$-predictable (recall that τ is a $\mathbb{G}$-stopping time),

$$Y 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{Z_-} \; {}^{(p,\mathbb{F})}\Big(Y^{(\tau)} 1\!\!1_{]\!]0,\tau]\!]}\Big) 1\!\!1_{]\!]0,\tau]\!]},$$

where the last equality follows by the definition of predictable projection, being $\mathbb{F} \subset \mathbb{G}$. Finally, given the result in Proposition 9.2.4, we have

$${}^{(p,\mathbb{F})}\Big(Y^{(\tau)} 1\!\!1_{]\!]0,\tau]\!]}\Big)_t = \int_t^{+\infty} y_t(u) p_{t-}(u) \nu(du)$$

and the proposition is proved.

□

9.3 Martingales' characterization

The aim of this section is to characterize $(\mathbb{P}, \mathbb{F}^{(\tau)})$ and $(\mathbb{P}, \mathbb{G})$-martingales in terms of $(\mathbb{P}, \mathbb{F})$-martingales.

Proposition 9.3.1 ***Characterization of*** $(\mathbb{P}, \mathbb{F}^{(\tau)})$***-martingales in terms of*** $(\mathbb{P}, \mathbb{F})$***-martingales***
A process $Y^{(\tau)} = y(\tau)$ *is a* $(\mathbb{P}, \mathbb{F}^{(\tau)})$*-martingale if and only if* $(y_t(u)p_t(u), t \geq 0)$ *is a* $(\mathbb{P}, \mathbb{F})$*-martingale, for* ν*-almost every* $u \geq 0$.

Proof: The sufficiency is a direct consequence of Proposition 6.1.1 and Lemma 9.2.1 (ii).
Conversely, assume that $y(\tau)$ is an $\mathbb{F}^{(\tau)}$-martingale. Then, for $s \leq t$, from Lemma 9.2.1 (ii),

$$y_s(\tau) = \mathbb{E}\left(y_t(\tau) | \mathcal{F}_s^{(\tau)}\right) = \frac{1}{p_s(\tau)} \mathbb{E}\big(y_t(u)p_t(u) | \mathcal{F}_s\big)_{|u=\tau}$$

and the result follows from Lemma 9.2.1 (i). □

Passing to the progressive enlargement setting, we state and prove a martingale characterization result, established by El Karoui et al. in [49, Theorem 5.7].

Proposition 9.3.2 ***Characterization of*** $(\mathbb{P}, \mathbb{G})$ ***martingales in terms of*** $(\mathbb{P}, \mathbb{F})$***-martingales***
A $\mathbb{G}$*-adapted process* $Y_t := \tilde{y}_t 1\!\!1_{t<\tau} + \hat{y}_t(\tau) 1\!\!1_{\tau \leq t}, t \geq 0$, *is a* $(\mathbb{P}, \mathbb{G})$*-martingale if and only if the following two conditions are satisfied*

(i) for ν*-a.e* u, $\big(\hat{y}_t(u)p_t(u), t \geq u\big)$ *is a* $(\mathbb{P}, \mathbb{F})$*-martingale;*

(ii) the process $m = (m_t, t \geq 0)$, *given by*

$$m_t := \mathbb{E}(Y_t | \mathcal{F}_t) = \tilde{y}_t Z_t + \int_0^t \hat{y}_t(u) p_t(u) \nu(du) \;, \tag{9.3.1}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale.*

Proof: For the necessity, in a first step, we show that we can reduce our attention to the case where Y is u.i.: indeed, let Y be a $(\mathbb{P}, \mathbb{G})$-martingale. For any T, let $Y^T = (Y_{t\wedge T}, t \geq 0)$ be the associated stopped martingale, which is u.i. Assuming that the result is established for u.i. martingales will prove that the processes in *(i)* and *(ii)* are martingales up to time T. Since T can be chosen as large as possible, we shall have the result.

Assume, then, that Y is a u.i. $(\mathbb{P},\mathbb{G})$-martingale. From Proposition 1.2.3, $Y_t = \mathbb{E}(Y^{(\tau)}_t|\mathcal{G}_t)$ for some $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale $Y^{(\tau)}$. Proposition 9.3.1, then, implies that $Y^{(\tau)}_t = y_t(\tau)$, where for ν-almost every $u\geq 0$ the process $\big(y_t(u)p_t(u), t\geq 0\big)$ is a $(\mathbb{P},\mathbb{F})$-martingale. One then has

$$\mathbb{1}_{\tau\leq t}\hat{y}_t(\tau) = \mathbb{1}_{\tau\leq t}Y_t = \mathbb{1}_{\tau\leq t}\mathbb{E}(Y^{(\tau)}_t|\mathcal{G}_t) = \mathbb{E}(\mathbb{1}_{\tau\leq t}Y^{(\tau)}_t|\mathcal{G}_t) = \mathbb{1}_{\tau\leq t}y_t(\tau)\ ,$$

which implies, in view of Lemma 9.2.1(i), that for ν-almost every $u\leq t$, the identity $y_t(u)=\hat{y}_t(u)$ holds $\mathbb{P}$-almost surely. So, *(i)* is proved. Moreover, Y being a $(\mathbb{P},\mathbb{G})$-martingale, its projection on the smaller filtration $\mathbb{F}$, namely the process m in (9.3.1), is a $(\mathbb{P},\mathbb{F})$-martingale.

Conversely, assuming *(i)* and *(ii)*, we verify that $\mathbb{E}(Y_t|\mathcal{G}_s)=Y_s$ for $s\leq t$. We start by noting that

$$\mathbb{E}(Y_t|\mathcal{G}_s) = \mathbb{1}_{s<\tau}\frac{1}{Z_s}\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) + \mathbb{1}_{\tau\leq s}\mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{G}_s)\ . \tag{9.3.2}$$

We then compute the two conditional expectations in (9.3.2):

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) &= \mathbb{E}(Y_t|\mathcal{F}_s) - \mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{F}_s)\\
&= \mathbb{E}(m_t|\mathcal{F}_s) - \mathbb{E}\big(\mathbb{E}(\hat{y}_t(\tau)\mathbb{1}_{\tau\leq s}|\mathcal{F}_t)|\mathcal{F}_s\big)\\
&= m_s - \mathbb{E}\Big(\int_0^s \hat{y}_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big)\\
&= \tilde{y}_sZ_s + \int_0^s \hat{y}_s(u)p_s(u)\nu(du) - \int_0^s \hat{y}_s(u)p_s(u)\nu(du) = \tilde{y}_sZ_s\ ,
\end{aligned}$$

where we used Fubini's theorem and the condition *(i)* to obtain the next-to-last identity. Also, an application of Lemma 9.2.2 yields to

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{G}_s) &= \mathbb{E}(\hat{y}_t(\tau)\mathbb{1}_{\tau\leq s}|\mathcal{G}_s) = \mathbb{1}_{\tau\leq s}\frac{1}{p_s(\tau)}\mathbb{E}\big(\hat{y}_t(u)p_t(u)|\mathcal{F}_s\big)_{|u=\tau}\\
&= \mathbb{1}_{\tau\leq s}\frac{1}{p_s(\tau)}\hat{y}_s(\tau)p_s(\tau) = \mathbb{1}_{\tau\leq s}\hat{y}_s(\tau)
\end{aligned}$$

where the next-to-last identity holds in view of the condition *(ii)*. □

9.4 Canonical decomposition in the enlarged filtrations

In this section, we work under $\mathbb{P}$ and we show that any $\mathbb{F}$-local martingale x is a semi-martingale in both the initially enlarged filtration $\mathbb{F}^{(\tau)}$ and in the progressively enlarged filtration $\mathbb{G}$, and that any $\mathbb{G}$-martingale is a $\mathbb{F}^{(\tau)}$-semi-martingale. We also provide the canonical decomposition of any $\mathbb{F}$-local martingale as a semi-martingale in $\mathbb{F}^{(\tau)}$ and in $\mathbb{G}$. Under the assumption that the $\mathbb{F}$-conditional law of τ is absolutely continuous w.r.t. the law of τ, these questions were answered in Chapter 6, in the initial enlargement setting, and in [49] and [77], in the progressive enlargement case. Our aim here is to retrieve their results in an alternative manner.

We will need the following technical result, concerning the existence of the predictable bracket $\langle x, p_\cdot(u)\rangle$. From [76, Theorem 2.5 a)], it follows immediately that, under the $(\mathcal{E})$-Hypothesis, for every $(\mathbb{P},\mathbb{F})$-(local)martingale x, there exists a ν-negligible set B (depending on x), such that $\langle x, p_\cdot(u)\rangle$ is well-defined for $u\notin B$. Hereafter, by $\langle x, p_\cdot(\tau)\rangle_s$ we mean $\langle x, p_\cdot(u)\rangle_s|_{u=\tau}$.

Furthermore, according to [76, Theorem 2.5 b)], under the $(\mathcal{E})$-Hypothesis, there exists an $\mathbb{F}$-predictable increasing process A and a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable function $(t,\omega,u)\to k_t(\omega,u)$ such that, for any $u\notin B$ and for all $t\geq 0$,

$$\langle x, p_\cdot(u)\rangle_t = \int_0^t k_s(u)p_{s-}(u)dA_s \quad \text{a.s.} \tag{9.4.1}$$

(the two processes A and k depend on x, however, to keep simple notation, we do not write $A^{(x)}$, nor $k^{(x)}$).

Moreover,

$$\int_0^t |k_s(\tau)| dA_s \ < \ \infty \quad \text{a.s., for any } t > 0. \tag{9.4.2}$$

The following two propositions provide, under the $(\mathcal{E})$-Hypothesis, the canonical decomposition of any $(\mathbb{P}, \mathbb{F})$- local martingale x in the enlarged filtrations $\mathbb{F}^{(\tau)}$ and $\mathbb{G}$, respectively. The first result is due to Jacod [76, Theorem 2.5 c)]). Our proof is easier (mainly because we do not prove the difficult regularity results obtained by Jacod), but less general. The interest is that we show the power of the change of probability methodology.

Proposition 9.4.1 ***Canonical Decomposition in*** $\mathbb{F}^{(\tau)}$
Any $(\mathbb{P}, \mathbb{F})$-local martingale x is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-semimartingale with canonical decomposition

$$x_t = X_t^{(\tau)} + \int_0^t \frac{d\langle x, p_{.}(\tau)\rangle_s}{p_{s-}(\tau)},$$

where $X^{(\tau)}$ is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-local martingale.

Proof: If x is a $(\mathbb{P}, \mathbb{F})$-martingale, it is a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale, too (Indeed, since $\mathbb{P}$ and $\mathbb{P}^*$ are equal on $\mathbb{F}$, x is a $(\mathbb{P}^*, \mathbb{F})$ martingale, hence, using the fact that τ is $\mathbb{P}^*$ independant of $\mathbb{F}$, it is a $(\mathbb{P}^*, \mathbb{G})$ martingale). Noting that $d\mathbb{P} = p_t(\tau) d\mathbb{P}^*$ on $\mathcal{G}_t^{\tau}$, Girsanov's theorem tells us that the process $X^{(\tau)}$, defined by

$$X_t^{(\tau)} = x_t - \int_0^t \frac{d\langle x, p_{.}(\tau)\rangle_s}{p_{s-}(\tau)}$$

is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale. $\square$

Now, any $(\mathbb{P}, \mathbb{F})$-local martingale is a $\mathbb{G}$-adapted process and a $(\mathbb{P}, \mathbb{F}^{(\tau)})$ semi-martingale (from the above Proposition 9.4.1), so in view of Stricker's theorem in [127], it is also a $\mathbb{G}$ semi-martingale. The following proposition aims to obtain the $\mathbb{G}$-canonical decomposition of an $\mathbb{F}$-local martingale. We refer to [77] for an alternative proof.

The following lemma provides a formula for the predictable quadratic covariation process $\langle x, G\rangle = \langle x, \mu\rangle$ in terms of the density p.

Lemma 9.4.2 *Let x be a $(\mathbb{P}, \mathbb{F})$-local martingale and μ the $\mathbb{F}$-martingale part in the Doob-Meyer decomposition of G. If kp_- is $dA \otimes d\nu$-integrable, where A is defined in (9.4.1), then*

$$\langle x, \mu\rangle_t = \int_0^t dA_s \int_s^\infty \nu(du) k_s(u) p_{s-}(u), \tag{9.4.3}$$

where k was introduced in Equation (9.4.1).

Proof: First consider the right-hand-side of (9.4.3), that is, by definition, predictable, and apply

Fubini's Theorem

$$\begin{aligned}
\xi_t &:= \int_0^t dA_s \int_s^\infty k_s(u) p_{s-}(u)\nu(du) \\
&= \int_0^t dA_s \int_s^t k_s(u) p_{s-}(u)\nu(du) + \int_0^t dA_s \int_t^\infty k_s(u) p_{s-}(u)\nu(du) \\
&= \int_0^t \nu(du) \int_0^u k_s(u) p_{s-}(u) dA_s + \int_t^\infty \nu(du) \int_0^t k_s(u) p_{s-}(u) dA_s \\
&= \int_0^t \langle x, p.(u)\rangle_u \ \nu(du) + \int_t^\infty \langle x, p.(u)\rangle_t \ \nu(du) \\
&= \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_u - \langle x, p.(u)\rangle_t\right) \nu(du) \, .
\end{aligned}$$

To verify (9.4.3), it suffices to show that the process $x\mu - \xi$ is an $\mathbb{F}$-local martingale (since ξ is a predictable, finite variation process). By definition, for ν-almost every $u \in \mathbb{R}^+$, the process $(m_t(u) := x_t p_t(u) - \langle x, p.(u)\rangle_t, t \geq 0)$ is an $\mathbb{F}$-local martingale. Then, given that $1 = \int_0^\infty p_t(u)\nu(du)$ for every $t \geq 0$, a.s., we have

$$\begin{aligned}
x_t\mu_t - \xi_t &= x_t \int_0^\infty p_t(u)\nu(du) - x_t \int_0^t \left(p_t(u) - p_u(u)\right)\nu(du) \\
&- \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_t - \langle x, p.(u)\rangle_u\right)\nu(du) \\
&= \int_0^\infty m_t(u)\nu(du) - \int_0^t \left(m_t(u) - m_u(u)\right)\nu(du) + x_t \int_0^t p_u(u)\nu(du) - \int_0^t p_u(u) x_u \nu(du) \, .
\end{aligned}$$

The first two terms are martingales (this follows easily from the martingale property of $m(u)$). As for the last term, using the fact that ν has no atoms, we find

$$\begin{aligned}
& d\left(x_t \int_0^t p_u(u)\nu(du) - \int_0^t p_u(u) x_u \nu(du)\right) \\
&= \left(\int_0^t p_u(u)\nu(du)\right) dx_t + x_t p_t(t)\nu(dt) - p_t(t) x_t \nu(dt) = \left(\int_0^t p_u(u)\nu(du)\right) dx_t
\end{aligned}$$

and we have, indeed, proved that $x\mu - \xi$ is an $\mathbb{F}$-local martingale. □

Proposition 9.4.3 ***Canonical Decomposition in*** $\mathbb{G}$
Any (càdlàg) $(\mathbb{P}, \mathbb{F})$*-local martingale* x *is a* $(\mathbb{P}, \mathbb{G})$ *semi-martingale with canonical decomposition*

$$x_t = X_t + \int_0^{t\wedge\tau} \frac{d\langle x, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^t \frac{d\langle x, p.(\tau)\rangle_s}{p_{s-}(\tau)}, \tag{9.4.4}$$

where X *is a* $(\mathbb{P}, \mathbb{G})$*-local martingale.*

PROOF: The proof follows from Theorem 8.3.1 and Proposition 9.4.1. See [8] or [31] for details. □

Exercise 9.4.4 Give a direct check of Proposition 9.4.3 in a Brownian filtration ◁

We end this section proving that any $(\mathbb{P}^*, \mathbb{G})$-martingale remains a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-semimartingale, but it is not necessarily a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale. Indeed, we have the following result.

Lemma 9.4.5 *Any* $(\mathbb{P}^*, \mathbb{G})$*-martingale* Y^* *is a* $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ *semi-martingale which can have a non-null bounded variation part.*

PROOF: The result follows immediately from Proposition 9.3.2 (under $\mathbb{P}^*$), noticing that the $(\mathbb{P}^*, \mathbb{G})$ martingale Y^* can be written as $Y_t^* = \tilde{y}_t^* 1\!\!1_{t<\tau} + \hat{y}_t^*(\tau) 1\!\!1_{\tau \leq t}$. Therefore, in the filtration $\mathbb{F}^{(\tau)}$, it is the sum of two $\mathbb{F}^{(\tau)}$ semi-martingales: the processes $1\!\!1_{t<\tau}$ and $1\!\!1_{\tau\leq t}$ are $\mathbb{F}^{(\tau)}$ semi-martingales, as well as the processes $\tilde{y}, \hat{y}^*(\tau)$. Indeed, from Proposition 9.3.2, recalling that the $(\mathbb{P}^*, \mathbb{F})$-density of τ is a constant equal to one, we know that, for every $u > 0$, $\big(\hat{y}_t^*(u), t \geq u\big)$ is an $\mathbb{F}$-martingale and that the process $\big(\tilde{y}_t^* G(t) + \int_0^t \hat{y}_u^*(u)\nu(du), t \geq 0\big)$ is an $\mathbb{F}$-martingale, hence $\tilde{y}^*$ is a $\mathbb{G}$-semi-martingale.

□

As in Lemma 9.4.5, we deduce that any $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-semi-martingale. Note that this result can also be proved using Lemma 9.4.5 and a change of probability argument: a $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}^*, \mathbb{G})$-semi-martingale (from Girsanov's theorem), thus also a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-semi-martingale in view of Lemma 9.4.5. By another use of Girsanov's theorem, it is thus a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-semi-martingale.

9.5 Predictable Representation Theorems

The aim of this section is to obtain Predictable Representation Property (PRP hereafter) in the enlarged filtrations $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$, both under $\mathbb{P}$ and $\mathbb{P}^*$. We start by assuming that there exists a $(\mathbb{P}, \mathbb{F})$-local martingale y (possibly multidimensional), such that the PRP holds in $(\mathbb{P}, \mathbb{F})$. Notice that y is not necessarily continuous.

Beforehand we introduce some notation: $\mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ denotes the set of $(\mathbb{P}, \mathbb{F})$-local martingales, and $\mathcal{M}^2(\mathbb{P}, \mathbb{F})$ denotes the set of $(\mathbb{P}, \mathbb{F})$-martingales x, such that

$$\mathbb{E}\left(x_t^2\right) < \infty, \quad \forall\, t \geq 0. \tag{9.5.1}$$

Also, for a $(\mathbb{P}, \mathbb{F})$-local martingale m, we denote by $\mathcal{L}(m, \mathbb{P}, \mathbb{F})$ the set of $\mathbb{F}$-predictable processes which are integrable with respect to m (in the sense of local martingale), namely (see, e.g., Definition 9.1 and Theorem 9.2. in [67])

$$\mathcal{L}(m, \mathbb{P}, \mathbb{F}) = \left\{ \varphi \in \mathcal{P}(\mathbb{F}) : \ \left(\int_0^\cdot \varphi_s^2 d[m]_s \right)^{1/2} \text{ is } \mathbb{P} - \text{locally integrable} \right\}.$$

Hypothesis 9.5.1 ***PRP for*** $(\mathbb{P}, \mathbb{F})$
There exists a process $y \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ *such that every* $x \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ *can be represented as*

$$x_t = x_0 + \int_0^t \varphi_s dy_s$$

for some $\varphi \in \mathcal{L}(y, \mathbb{P}, \mathbb{F})$.

We start investigating what happens under the measure $\mathbb{P}^*$, in the initially enlarged filtration $\mathbb{F}^{(\tau)}$.

Recall that, assuming the immersion property, Kusuoka [100] has established a PRP for the progressively enlarged filtration, in the case where $\mathbb{F}$ is a Brownian filtration.

Also, under the *equivalence assumption* in $[0, T]$ and assuming a PRP in the reference filtration $\mathbb{F}$, Amendinger (see [9, Th. 2.4]) proved a PRP in $(\mathbb{P}^*, \mathbb{F}^{(\tau)}$ and extended the result to $(\mathbb{P}, \mathbb{F}^{(\tau)})$, in the case where the underlying (local) martingale in the reference filtration is continuous.

Proposition 9.5.2 ***PRP for*** $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$
Under Assumption 9.5.1, every $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ *admits a representation*

$$X_t^{(\tau)} = X_0^{(\tau)} + \int_0^t \Phi_s^\tau dy_s \tag{9.5.2}$$

where $\Phi^\tau \in \mathcal{L}(y, \mathbb{P}^*, \mathbb{F}^{(\tau)})$*. In the case where* $X^{(\tau)} \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{F}^{(\tau)})$*, one has* $\mathbb{E}^*\big(\int_0^t (\Phi_s^\tau)^2 d[y]_s\big) < \infty$*, for all* $t \geq 0$ *and the representation is unique.*

Proof: From Theorem 13.4 in [67], it suffices to prove that any bounded martingale admits a predictable representation in terms of y. Let $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ be bounded by K. From Proposition 9.3.1, $X_t^{(\tau)} = x_t(\tau)$ where, for ν-almost every $u \in \mathbb{R}^+$, the process $\big(x_t(u), t \geq 0\big)$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale, hence a $(\mathbb{P}, \mathbb{F})$-martingale. Thus Assumption 9.5.1 implies that (for ν-almost every $u \in \mathbb{R}^+$),

$$x_t(u) = x_0(u) + \int_0^t \varphi_s(u) dy_s \ ,$$

where $(\varphi_t(u), t \geq 0)$ is an $\mathbb{F}$-predictable process.

The process $X^{(\tau)}$ being bounded by K, it follows by an application of Lemma 9.2.1(i) that for ν-almost every $u \geq 0$, the process $(x_t(u), t \geq 0)$ is bounded by K. Then, using the Itô isometry,

$$\begin{aligned}
\mathbb{E}^*\big(\int_0^t \varphi_s^2(u) d[y]_s\big) &= \mathbb{E}^*\big(\int_0^t \varphi_s(u) dy_s\big)^2 \\
&= \mathbb{E}^*\big((x_t(u) - x_0(u))^2\big) \leq \mathbb{E}^*(x_t^2(u)) \leq K^2 \ .
\end{aligned}$$

Also, from [126, Lemma 2], one can consider a version of the process $\int_0^\cdot \varphi_s^2(u) d[y]_s$ which is measurable with respect to u. Using this fact,

$$\mathbb{E}^*\Big[\big(\int_0^t \varphi_s^2(\tau) d[y]_s\big)^{1/2}\Big] = \int_0^\infty \nu(du)\Big(\mathbb{E}^*\big(\int_0^t \varphi_s^2(u) d[y]_s\big)\Big)^{1/2} \leq \int_0^\infty \nu(du) K = K \ .$$

The process $\Phi^{(\tau)}$ defined by $\Phi_t^{(\tau)} = \varphi_t(\tau)$ is $\mathbb{F}^{(\tau)}$-predictable, according to Proposition 6.1.1, it satisfies (9.5.2), with $X_0^{(\tau)} = x_0(\tau)$, and it belongs to $\mathcal{L}(y, \mathbb{P}^*, \mathbb{F}^{(\tau)})$.

If $X^{(\tau)} \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{F}^{(\tau)})$, from Itô's isometry,

$$\mathbb{E}^*\left(\int_0^t (\Phi_s^{(\tau)})^2 d[y]_s\right) = \mathbb{E}^*\left(\int_0^t \Phi_s^{(\tau)} dy_s\right)^2 = \mathbb{E}^*(X_t^{(\tau)} - X_0^{(\tau)})^2 < \infty \ .$$

Also, from this last equation, if $X^{(\tau)} \equiv 0$ then $\Phi^{(\tau)} \equiv 0$, from which the uniqueness of the representation follows. □

Passing to the progressively enlarged filtration $\mathbb{G}$, which consists of two filtrations, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, intuitively one needs two martingales to establish a PRP. Apart from y, intuition tells us that a candidate for the second martingale might be the compensated martingale of H, that was introduced, respectively under $\mathbb{P}$ (it was denoted by M) and under $\mathbb{P}^*$ (denoted by M^*), in Equation (9.2.5) and in Equation (9.2.6).

Proposition 9.5.3 ***PRP for*** $(\mathbb{P}^*, \mathbb{G})$
Under Assumption 9.5.1, every $X \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{G})$ *admits a representation*

$$X_t = X_0 + \int_0^t \Phi_s dy_s + \int_0^t \Psi_s dM_s^*$$

for some processes $\Phi \in \mathcal{L}(y, \mathbb{P}^*, \mathbb{G})$ *and* $\Psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{G})$*. Moreover, if* $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$*, one has, for any t*e0*,*

$$\mathbb{E}^*\left(\int_0^t \Phi_s^2 d[y]_s\right) < \infty \quad , \quad \mathbb{E}^*\left(\int_0^t \Psi_s^2 \lambda^*(s) \nu(ds)\right) < \infty \ ,$$

and the representation is unique.

PROOF: It is known that any $(\mathbb{P}^*, \mathbb{H})$ local martingale ξ can be represented as $\xi_t = \xi_0 + \int_0^t \psi_s dM_s^*$ for some process $\psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{H})$ (see, e.g., the proof in [33]). Notice that ψ has a role only before τ and, for this reason (recall that $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is the natural filtration of the indicator process H), ψ can be chosen deterministic.

Under $\mathbb{P}^*$, we then have

- the PRP holds in $\mathbb{F}$ with respect to y,
- the PRP holds in $\mathbb{H}$ with respect to M^*,
- the filtration $\mathbb{F}$ and $\mathbb{H}$ are independent.

From classical literature (see Lemma 9.5.4.1(ii) in [3M], for instance) the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ enjoys the PRP under $\mathbb{P}^*$ with respect to the pair (y, M^*).

Now suppose that $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$. We find

$$\begin{aligned}\infty &> \mathbb{E}^*(X_t - X_0)^2 = \mathbb{E}^*\left(\int_0^t \Phi_s dy_s + \int_0^t \Psi_s dM_s^*\right)^2 \\ &= \mathbb{E}^*\left(\int_0^t \Phi_s^2 d[y]_s\right) + 2\mathbb{E}^*\left(\int_0^t \Phi_s dy_s \int_0^t \Psi_s dM_s^*\right) + \mathbb{E}^*\left(\int_0^t \Psi_s^2 \lambda^*(s)\nu(ds)\right),\end{aligned}$$

where in the last equality we used the Itô isometry. The cross-product term in the last equality is zero due to the orthogonality of y and M^* (under $\mathbb{P}^*$). From this inequality, the desired integrability conditions hold and the uniqueness of the representation follows (as in the previous proposition). □

Remark 9.5.4 In order to establish a PRP for the initially enlarged filtration $\mathbb{F}^{(\tau)}$ and under $\mathbb{P}^*$, one could have proceeded as in the proof of Proposition 9.5.3, noting that any martingale ξ in the "constant" filtration $\sigma(\tau)$ satisfies $\xi_t = \xi_0 + 0$ and that under $\mathbb{P}^*$ the two filtrations $\mathbb{F}$ and $\sigma(\tau)$ are independent.

Proposition 9.5.5 ***PRP under*** $\mathbb{P}$
Under Assumption 9.5.1, one has:

(i) Every $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F}^{(\tau)})$ can be represented as

$$X_t^{(\tau)} = X_0^{(\tau)} + \int_0^t \Phi_s^{(\tau)} dy_s^{(\tau)}$$

where $y^{(\tau)}$ is the martingale part in the $\mathbb{F}^{(\tau)}$-canonical decomposition of y and $\Phi \in \mathcal{L}(y^{(\tau)}, \mathbb{P}, \mathbb{F}^{(\tau)})$.

(ii) Every $X \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{G})$ can be represented as

$$X_t = X_0 + \int_0^t \Phi_s dY_s + \int_0^t \Psi_s dM_s,$$

where Y is the martingale part in the $\mathbb{G}$-canonical decomposition of y, M is the $(\mathbb{P}, \mathbb{G})$-compensated martingale associated with H and $\Phi \in \mathcal{L}(Y, \mathbb{P}, \mathbb{G})$, $\Psi \in \mathcal{L}(M, \mathbb{P}, \mathbb{G})$.

PROOF: The assertion (i) (resp. (ii)) follows from Proposition 9.5.2 (resp. Proposition 9.5.3) and the stability of PRP under an equivalent change of measure (see for example Theorem 13.12 in [67]).

For part (ii), it is important to note that, if y is a $(\mathbb{P}, \mathbb{F})$-martingale, it is a $(\mathbb{P}^*, \mathbb{G})$-martingale, too. Hence, by a Girsanov type transformation, Y defined as $dZY_t := dy_t - \frac{1}{\ell^*_{t-}} d\langle y, \ell^* \rangle_t$, $Y_0 = y_0$, is a $(\mathbb{P}, \mathbb{G})$-martingale, where $\ell^* := 1/\ell$ is a $(\mathbb{P}^*, \mathbb{G})$-martingale (in fact $d\mathbb{P}_{|\mathcal{G}_t} = \ell_t^* \, d\mathbb{P}^*{}_{|\mathcal{G}_t}$). From the uniqueness of the canonical decomposition of the $(\mathbb{P},\mathbb{G})$-semimartingale y (which is indeed special) and from Proposition 9.4.3, it follows that the $(\mathbb{P},\mathbb{G})$-martingale Y is in particular given by

$$Y_t = y_t - \int_0^{t\wedge\tau} \frac{d\langle y, G\rangle_s}{G_{s_-}} - \int_{t\wedge\tau}^t \frac{d\langle y, p.(\tau)\rangle_s}{p_{s_-}(\tau)} .$$

□

This result is extended under absolute continuity Jacod's hypothesis in [63].

9.6 Change of probability

In this section, we show how the various quantities associated with a random time τ are transformed under a change of probability. We recall that the intensity is the $\mathbb{F}$ adapted process λ such that $H_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a martingale and that the Azéma supermartingale factorizes as $G_t = N_t e^{-\Lambda_t}$.

Theorem 9.6.1 *Let $Y^{\mathbb{G}}_t = y_t 1\!\!1_{\{\tau>t\}} + y_t(\tau)1\!\!1_{\{\tau\le t\}}$ be a positive $\mathbb{G}$-martingale with $Y^{\mathbb{G}}_0 = 1$ and let $Y^{\mathbb{F}}_t = y_t G_t + \int_0^t y_t(u)p_t(u)\nu(du)$ be its $\mathbb{F}$ projection.*
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Y^{\mathbb{G}}_t d\mathbb{P}$. Then,
(i) for $t \ge \theta$ $p^{\mathbb{Q}}_t(\theta) = p_t(\theta)\frac{y_t(\theta)}{Y^{\mathbb{F}}_t}$,
(ii) the $\mathbb{Q}$-Azéma's supermartingale is defined by $G^{\mathbb{Q}}_t = G_t \dfrac{y_t}{Y^{\mathbb{F}}_t}$
(iii) the $(\mathbb{F},\mathbb{Q})$-intensity process is $\lambda^{\mathbb{F},\mathbb{Q}}_t = \lambda^{\mathbb{F}}_t \dfrac{y_t(t)}{y_{t-}}$, dt- a.s.;
(iv) $N^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F},\mathbb{Q})$-local martingale

$$N^{\mathbb{F},\mathbb{Q}}_t = N^{\mathbb{F}}_t \frac{y_t}{Y^{\mathbb{F}}_t} \exp\int_0^t (\lambda^{\mathbb{F},\mathbb{Q}}_s - \lambda^{\mathbb{F}}_s)ds$$

PROOF: From change of probability

$$\mathbb{Q}(\tau>\theta|\mathcal{F}_t) = \frac{1}{\mathbb{E}(Y^{\mathbb{G}}_t|\mathcal{F}_t)}\mathbb{E}_{\mathbb{P}}(Y^{\mathbb{G}}_t 1\!\!1_{\tau>\theta}|\mathcal{F}_t) = \frac{1}{Y^{\mathbb{F}}_t}\mathbb{E}_{\mathbb{P}}(y_t(\tau)1\!\!1_{\tau>\theta}|\mathcal{F}_t) = \frac{1}{Y^{\mathbb{F}}_t}\int_\theta^\infty y_t(u)p_t(u)\nu(du)$$

The form of the survival process follows immediately by differentiation. The form of the intensity is obvious. The form of N is obtained follows from the definition

$$G^{\mathbb{Q}}_t = N^{\mathbb{Q}}_t e^{-\Lambda^{\mathbb{Q}}_t} = G_t \frac{y_t}{Y^{\mathbb{F}}_t} e^{-\Lambda^{\mathbb{P}}_t}$$

□

Girsanov's tranform with Doléans Dade exponential

We restrict our attention to the case where τ is constructed on a probability space $(\Omega,\mathbb{F},\mathbb{P})$ with a given intensity λ as in the Cox process model, where $\mathbb{F}$ is a Brownian filtration generated by W. Any strictly positive martingale can be written as

$$dL_t = L_{t-}(\Psi_t dW_t + \Phi_t dM_t)$$

where Ψ and Ψ are $\mathbb{G}$ predictable processes, of the form

$$\begin{aligned}\Psi_t &= \psi_t 1\!\!1_{t<\tau} + \psi_t(\tau)1\!\!1_{\tau\le t}\\ \Phi_t &= \phi_t 1\!\!1_{t<\tau} + \phi_t(\tau)1\!\!1_{\tau\le t}\end{aligned}$$

where ψ and ϕ are $\mathbb{F}$-predictable. It follows that

$$\begin{aligned} L_t &= \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \exp\left(-\int_0^t \lambda_s \gamma_s ds\right) := \widetilde{L}_t,\, t < \tau \\ &= L_{\tau-}(1+\gamma_\tau)\exp\left(\int_\tau^t \psi_s(\tau) dW_s - \frac{1}{2}\int_\tau^t (\psi_s(\tau))^2 ds\right) = L_{\tau-}(1+\gamma_\tau)\Upsilon_t(\tau),\, \tau < t \end{aligned}$$

where $\Upsilon_t(u) = \exp\left(\int_u^t \psi_s(u) dW_s - \frac{1}{2}\int_u^t (\psi_s(u))^2 ds\right)$ Let $d\mathbb{Q} = L_t d\mathbb{P}$. Then, setting

$$\ell_t = \mathbb{E}(L_t|\mathcal{F}_t) = \widetilde{L}_t e^{-\Lambda_t} + \int_0^t \widetilde{L}_u (1+\gamma_u)\Upsilon_t(u)\lambda_u e^{-\Lambda_u} du$$

$$\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \frac{1}{\ell_t}\left(\widetilde{L}_t e^{-\Lambda_t} + \int_\theta^t \widetilde{L}_u (1+\gamma_u)\Upsilon_t(u)\lambda_u e^{-\Lambda_u} du\right)$$

It remains to differentiate w.r.t. θ

$$\alpha_t(\theta) = \frac{1}{\ell_t}\widetilde{L}_\theta (1+\gamma_\theta)\Upsilon_t(\theta)\lambda_\theta e^{-\Lambda_\theta}$$

Exercise 9.6.2 Prove that the change of probability measure generated by the two processes

$$z_t = (L_t^{\mathbb{F}})^{-1}, \qquad z_t(\theta) = \frac{p_\theta(\theta)}{p_t(\theta)}$$

provides a model where the immersion property holds true, and where the intensity processes does not change $\triangleleft$

Exercise 9.6.3 Check that

$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_s}{G_{s-}} - \int_{t\wedge\tau}^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s-}(\theta)}\bigg|_{\theta=\tau} |\mathcal{F}_t)$$

is an $\mathbb{F}$-martingale.
Check that that

$$\mathbb{E}(\int_0^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s-}(\theta)}\bigg|_{\theta=\tau} |\mathcal{G}_t)$$

is a $\mathbb{G}$ martingale. $\triangleleft$

Exercise 9.6.4 Let λ be a positive $\mathbb{F}$-adapted process and $\Lambda_t = \int_0^t \lambda_s ds$ and Θ be a strictly positive random variable such that there exists a family $\gamma_t(u)$ which satisfies $\mathbb{P}(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty \gamma_t(u) du$. Let $\tau = \inf\{t > 0 : \Lambda_t \geq \Theta\}$.Prove that the density of τ is given by

$$p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta) \text{ if } t \geq \theta \qquad \text{and} \qquad p_t(\theta) = \mathbb{E}[\lambda_\theta \gamma_\theta(\Lambda_\theta)|\mathcal{F}_t] \text{ if } t < \theta.$$

Conversely, if we are given a density p, prove that it is possible to construct a threshold Θ such that τ has p as density. $\triangleleft$

9.7 Applications to Finance

9.7.1 Defaultable Zero-Coupon Bonds

A defaultable zero-coupon with maturity T associated with the default time τ is an asset which pays one monetary unit at time T if (and only if) the default has not occurred before T. We assume that

$\mathbb{P}$ is the pricing measure and that the interest rate is null. By definition, the risk-neutral price under $\mathbb{P}$ of the T-maturity defaultable zero-coupon bond with zero recovery equals, for every $t \in [0,T]$,

$$D(t,T) := \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{P}(\tau > T \,|\, \mathcal{F}_t)}{G_t} = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)}{G_t} \tag{9.7.1}$$

where N is the martingale part in the multiplicative decomposition of G (see Proposition). Using (9.7.1), we obtain

$$D(t,T) \quad := \quad \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}} \, \mathbb{E}_{\mathbb{P}}(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t)$$

However, using a change of probability, one can get rid of the martingale part N, assuming that there exists p such that

$$\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty p_t(u)du$$

Let $\mathbb{P}^*$ be defined as

$$d\mathbb{P}^*|_{\mathcal{G}_t} = Z_t^* d\mathbb{P}|_{\mathcal{G}_t}$$

where Z^* is the $(\mathbb{P}, \mathbb{G})$-martingale defined as

$$Z_t^* = \mathbb{1}_{\{t<\tau\}} + \mathbb{1}_{\{t\geq\tau\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}$$

Note that

$$d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$$

and that $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$.
Indeed,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_t^*|\mathcal{F}_t) &= G_t + \int_0^t \lambda_u e^{-\Lambda_u} \frac{N_t}{p_t(u)} p_t(u)\eta(du) \\
&= N_t e^{-\Lambda_t} + N_t \int_0^t \lambda_u e^{-\Lambda_u} \eta(du) = N_t e^{-\Lambda_t} + N_t(1 - e^{-\Lambda_t})
\end{aligned}$$

Then, for $t > \theta$,

$$\begin{aligned}
\mathbb{P}^*(\theta < \tau|\mathcal{F}_t) &= \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(Z_t^* \mathbb{1}_{\theta<\tau}|\mathcal{F}_t) = \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(\mathbb{1}_{t<\tau} + \mathbb{1}_{\{t\geq\tau>\theta\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}|\mathcal{F}_t) \\
&= \frac{1}{N_t}\left(N_t e^{-\Lambda_t} + \int_\theta^t \lambda_u e^{-\Lambda_u} \frac{N_t}{p_t(u)} p_t(u)du\right) \\
&= \frac{1}{N_t}\left(N_t e^{-\Lambda_t} + N_t(e^{-\Lambda_\theta} - e^{-\Lambda_t})\right) = e^{-\Lambda_\theta}
\end{aligned}$$

which proves that immersion holds true under $\mathbb{P}^*$, and the intensity of τ is the same under $\mathbb{P}$ and $\mathbb{P}^*$. It follows that

$$\mathbb{E}_{\mathbb{P}}(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{E}^*(X\mathbb{1}_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*(e^{-\Lambda_T} X|\mathcal{F}_t)$$

Note that, if the intensity is the same under $\mathbb{P}$ and $\mathbb{P}^*$, its dynamics under $\mathbb{P}^*$ will involve a change of driving process, since $\mathbb{P}$ and $\mathbb{P}^*$ do not coincide on $\mathcal{F}_\infty$.
Let us now study the pricing of a recovery. Let Z be an $\mathbb{F}$-predictable bounded process.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t) &= \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}}(-\int_t^T Z_u dG_u|\mathcal{F}_t) \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_{\mathbb{P}}(\int_t^T Z_u N_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t) \\
&= \mathbb{E}^*(Z_\tau \mathbb{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t) \\
&= \mathbb{1}_{\{t<\tau\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*(\int_t^T Z_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t)
\end{aligned}$$

The problem is more difficult for pricing a recovery paid at maturity, i.e. for $X \in \mathcal{F}_T$

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\tau<T}|\mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) - \mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\tau>T}|\mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) - 1\!\!1_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}} \mathbb{E}_{\mathbb{P}}\big(X N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big) \\ &= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) - 1\!\!1_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*\big(X e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)\end{aligned}$$

Since immersion holds true under $\mathbb{P}^*$

$$\begin{aligned}\mathbb{E}^*(X 1\!\!1_{\tau<T}|\mathcal{G}_t) &= \mathbb{E}^*(X|\mathcal{G}_t) - 1\!\!1_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*\big(X N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big) \\ &= \mathbb{E}^*(X|\mathcal{F}_t) - 1\!\!1_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*\big(X N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)\end{aligned}$$

If both quantities $\mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\tau<T}|\mathcal{G}_t)$ and $\mathbb{E}^*(X 1\!\!1_{\tau<T}|\mathcal{G}_t)$ are the same, this would imply that $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) = \mathbb{E}^*(X|\mathcal{F}_t)$ which is impossible: this would lead to $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_t)$, i.e. immersion holds under P. Hence, non-immersion property is important while evaluating recovery paid at maturity ($\mathbb{P}^*$ and $\mathbb{P}$ do not coincide on $\mathcal{F}_\infty$).

9.7.2 Forward intensity

By using the density approach, we adopt an additive point of view to represent the conditional probability of τ: the conditional survival function $G_t(\theta) = \mathbb{P}(\tau > \theta \,|\, \mathcal{F}_t)$ is written in the form $G_t(\theta) = \int_\theta^\infty p_t(u)\nu(du)$. In the default framework, the "intensity" point of view is often preferred, and one uses the multiplicative representation $G_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)\nu(du))$. In the particular case where ν denotes the Lebesgue measure (in that case, the law of τ is $p_0(u)$, and we shall), the family of $\mathcal{F}_t$-measurable random variables $\lambda_t(\theta) = -\partial_\theta \ln G_t(\theta)$ is called the "forward intensity". We shall discuss and compare these two points of view further on.

We now consider $(G_t(\theta), t \geq 0)$ as in the classical HJM models where its dynamics is given in multiplicative form. By using the forward intensity $\lambda_t(\theta)$ of τ, the density can then be calculated as $p_t(\theta) = \lambda_t(\theta) G_t(\theta)$. It follows that the forward intensity is non-negative. As noted before, $\lambda(\theta)$ plays the same role as the spot forward rate in the interest rate models.

Proposition 9.7.1 *Let $dG_t(\theta) = Z_t(\theta) dW_t$ be the martingale representation of $(G_t(\theta), t \geq 0)$ and assume that the processes $(Z_t(\theta); t \geq 0)$ are differentiable in the following sense: there exists a family of processes $(z_t(\theta), t \geq 0)$ such that $Z_t(\theta) = \int_0^\theta z_t(u)\nu(du)$, $Z_t(0) = 0$. Then, under regularity conditions,*

1) the density processes have the following dynamics $dp_t(\theta) = -z_t(\theta) dW_t$ where $z(\theta)$ is subjected to the constraint $\int_0^\infty z_t(\theta)\nu(d\theta) = 0$ for any $t \geq 0$.

2) The survival process G evolves as $dG_t = -\alpha_t(t)\nu(dt) + Z_t(t) dW_t$.

3) With more regularity assumptions, if $(\partial_\theta p_t(\theta))_{\theta=t}$ is simply denoted by $\partial_\theta p_t(t)$, then the process $p_t(t)$ follows :

$$dp_t(t) = \partial_\theta p_t(t)\nu(dt) - z_t(t) dW_t.$$

Proof: 1) Observe that $Z(0) = 0$ since $G(0) = 1$, hence the existence of z is related with some smoothness conditions. Then using the stochastic Fubini theorem , one has

$$G_t(\theta) = G_0(\theta) + \int_0^t Z_u(\theta) dW_u = G_0(\theta) + \int_0^\theta \nu(dv) \int_0^t z_u(v) dW_u.$$

So 1) follows. Using the fact that for any $t \geq 0$,

$$1 = \int_0^\infty p_t(u)\nu(du) = \int_0^\infty \nu(du)\big(P_0(u) - \int_0^t z_s(u) dW_s\big) = 1 - \int_0^t dW_s \int_0^\infty z_s(u)\nu(du),$$

one gets $\int_0^\infty z_t(u)\nu(du) = 0$.
2) By using Proposition **??** and integration by parts,

$$M_t^{\mathbb{F}} = -\int_0^t (p_t(u) - p_u(u))\nu(du) = \int_0^t \nu(du)\int_u^t z_s(u)dW_s = \int_0^t dW_s\Big(\int_0^s z_s(u)\nu(du)\Big),$$

which implies 2).
3) We follow the same way as for the decomposition of G, by studying the process

$$p_t(t) - \int_0^t (\partial_\theta p_s)(s)\nu(ds) = p_t(0) + \int_0^t (\partial_\theta p_t)(s)\nu(ds) - \int_0^t (\partial_\theta p_s)(s)\nu(ds)$$

where the notation $\partial_\theta p_t(t)$ is defined in 3). Using the martingale representation of $p_t(\theta)$ and integration by parts (assuming that smoothness hypothesis allows these operations), the integral in the RHS is a stochastic integral,

$$\begin{aligned}
&\int_0^t \Big((\partial_\theta p_t)(s) - (\partial_\theta p_s)(s)\Big)\nu(ds) = -\int_0^t \nu(ds)\partial_\theta(\int_s^t z_u(\theta)dW_u) \\
&= -\int_0^t \nu(ds)\int_s^t \partial_\theta z_u(s)dW_u = -\int_0^t dW_u\int_0^u \nu(ds)\partial_\theta z_u(s) = -\int_0^t dW_u(z_u(u) - z_u(0))
\end{aligned}$$

The stochastic integral $\int_0^t z_u(0)dW_u$ is the stochastic part of the martingale $p_t(0)$, and so the property 3) holds true. □
Classically, HJM framework is studied for time smaller than maturity, i.e. $t \le T$. Here we consider all positive pairs (t,θ).

Proposition 9.7.2 *We keep the notation and the assumptions in Proposition 9.7.1. For any $t,\theta \ge 0$, let $\Psi_t(\theta) = \frac{Z_t(\theta)}{G_t(\theta)}$. We assume that there exists a family of processes ψ such that $\Psi_t(\theta) = \int_0^\theta \psi_t(u)\nu(du)$. Then*

1) $G_t(\theta) = G_0(\theta)\exp\Big(\int_0^t \Psi_s(\theta)dW_s - \frac{1}{2}\int_0^t |\Psi_s(\theta)|^2 ds\Big)$;

2) the forward intensity $\lambda(\theta)$ has the following dynamics:

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)dW_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)ds; \tag{9.7.2}$$

3) $S_t = \exp\Big(-\int_0^t \lambda_s^{\mathbb{F}}\nu(ds) + \int_0^t \Psi_s(s)dW_s - \frac{1}{2}\int_0^t |\Psi_s(s)|^2 ds\Big)$;

Proof: By choice of notation, 1) holds since the process $G_t(\theta)$ is the solution of the equation

$$\frac{dG_t(\theta)}{G_t(\theta)} = \Psi_t(\theta)dW_t, \qquad \forall\, t,\theta \ge 0. \tag{9.7.3}$$

2) is the consequence of 1) and the definition of $\lambda(\theta)$.
3) This representation is the multiplicative version of the additive decomposition of G in Proposition 9.7.1. We recall that $\lambda_t^{\mathbb{F}} = p_t(t)G_t^{-1}$. There are no technical difficulties because G is continuous. □

9.7.3 Multidefault

Lemma 9.7.3 *Assume that $\mathbb{P}(\tau_i > t_i, i = 1,\dots,n|\mathcal{F}_t) = \int_{t_1}^\infty \dots \int_{t_n}^\infty g_t(u_1,\dots,u_n)du_1\dots du_n$. Prove that $\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $g_t(t_1,\dots,t_n) = g_u(t_1,\dots,t_n)$ for $u > t > \max(t_i)$*

In the multi-dimensional case, that is when $\tau = (\tau_1, \ldots, \tau_d)$ is a vector of finite random times, the same machinery can be applied. More precisely, under the assumption

$$\mathbb{P}(\tau_1 \in d\theta_1, \ldots, \tau_d \in \; d\theta_d | \mathcal{F}_t) \sim \mathbb{P}(\tau_1 \in d\theta_1, \ldots, \tau_d \in d\theta_d)$$

one defines the probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ on $\mathcal{F}_t^{(\tau)} = \mathcal{F}_t \vee \sigma(\tau_1) \vee \cdots \vee \sigma(\tau_d)$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}_{|\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau_1, \ldots, \tau_d)},$$

where $p_t(\tau_1, \ldots, \tau_d)$ is the (multidimensional) analog to $p_t(\tau)$, and the results for the initially enlarged filtration are obtained in the same way as for the one-dimensional case.

As for the progressively enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_1 \wedge t) \vee \cdots \vee \sigma(\tau_d \wedge t)$, one has to note that, in this case, a measurable process is decomposed into 2^d terms, corresponding to the measurability of the process on the various sets $\{\tau_i \leq t < \tau_j, i \in I, j \in I^c\}$ for all the subsets I of $\{1, ..., d\}$.

See many applications in El Karoui et al [50, 48], Jiao and Li [87, 88].

9.7.4 Concluding Remarks

- In this study, *honest* times are automatically excluded, as we explain now. Under the probability $\mathbb{P}^*$, the Azéma supermartingale associated with τ being a continuous decreasing function, it has a trivial Doob-Meyer decomposition $G^* = 1 - A^*$ with $A_t^* = \int_0^t \nu(du)$. So, $A_\infty^* = 1$ and, in particular, τ can not be an honest time: recall that in our setting, τ avoids the $\mathbb{F}$-stopping times and therefore, from a result due to Azéma [16], if τ is an honest time, the random variable A_∞^* should have exponential law with parameter 1, which is not the case (note that the notion of honest time does not depend on the probability measure).

- Under immersion property and under the $(\mathcal{E})$-Hypothesis, $p_t(u) = p_u(u), t \geq u$. In particular, as expected, the canonical decomposition's formulae presented in Section 9.4 are trivial, i.e., the "drift" terms vanish.

- Predictable representation theorems can be obtained in the more general case, where any $(\mathbb{P}, \mathbb{F})$-martingale x admits a representation as

$$x_t = x_0 + \int_0^t \int_E \varphi(s, \theta) \tilde{\mu}(ds, d\theta),$$

  for a compensated martingale associated with a point process.

9.8 Conditional Laws of Random Times

In this section, we are interested in models for the conditional law of a random time τ: more precisely, our goal is to give examples of processes $g(u)$ so that one can construct a random time τ satisfying $G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty g_t(u) du$. The process $g(u)$ is called the (un-normalized) density. (the density being $p(u)$ such that $\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty p_t(u) \nu(du)$, where ν is the law of τ, i.e., $p_t(u) = \frac{g_t(u)}{g_0}$ where $g_0 = \int_0^\infty G_0(u) du$). We recall the classical construction of default times as first hitting time of a barrier, independent of the reference filtration, and we extend this construction to the case where the barrier is no more independent of the reference filtration. It is then natural to characterize the dependence of this barrier and the filtration by means of its conditional law.

9.8.1 General thresholds

In the classical model, the barrier Θ as an $\mathcal{A}$-measurable random variable independent of $\mathbb{F}$, and to consider

$$\tau := \inf\{t \,:\, \Gamma_t \geq \Theta\}\,. \tag{9.8.1}$$

The $\mathbb{F}$-conditional law of τ is

$$P(\tau > \theta|\mathcal{F}_t) = G^{\Theta}(\Gamma_\theta),\ \theta \leq t$$

where G^{Θ} is the survival probability of Θ given by $G^{\Theta}(t) = P(\Theta > t)$. We recall that in this particular case, $P(\tau > \theta|\mathcal{F}_t) = P(\tau > \theta|\mathcal{F}_\infty)$ for any $\theta \leq t$, which means that the H-hypothesis is satisfied and that the martingale survival processes remain constant after θ (i.e., $G_t(\theta) = G_\theta(\theta)$ for $t \geq \theta$). This result is stable by increasing transformation of the barrier, so that we can assume without loss of generality that the barrier is the standard exponential random variable $-\log(G^{\Theta}(\Theta))$.

If the increasing process Γ is assumed to be absolutely continuous w.r.t. the Lebesgue measure with Radon-Nikodým density γ and if G^{Θ} is differentiable, then the random time τ admits a density process given by

$$\begin{aligned} g_t(\theta) &= -(G^{\Theta})'(\Gamma_\theta)\gamma_\theta = g_\theta(\theta),\ \theta \leq t \\ &= E(g_\theta(\theta)|\mathcal{F}_t),\ \theta > t. \end{aligned} \tag{9.8.2}$$

In the widely used Cox process model, the independent barrier Θ follows the exponential law and $\Gamma_t = \int_0^t \gamma_s ds$ represents the default compensator process. As a direct consequence of (9.8.2),

$$g_t(\theta) = \gamma_\theta e^{-\Gamma_\theta},\ \theta \leq t.$$

We now relax the assumption that the threshold Θ is independent of $\mathcal{F}_\infty$. We assume that the barrier Θ is a strictly positive random variable whose conditional distribution w.r.t. $\mathbb{F}$ admits a density process, i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $p_t(u)$ such that

$$G_t^{\Theta}(\theta) := P(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty p_t(u)du\,. \tag{9.8.3}$$

We assume in addition that the process Γ is absolutely continuous w.r.t. the Lebesgue measure, i.e., $\Gamma_t = \int_0^t \gamma_s ds$. We still consider τ defined as in (9.8.1) and we say that a random time constructed in such a setting is given by *a generalized threshold*.

Proposition 9.8.1 *Let τ be given by a generalized threshold. Then τ admits the density process $g(\theta)$ where*

$$g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta),\ \theta \leq t. \tag{9.8.4}$$

PROOF: By definition and by the fact that Γ is strictly increasing and absolutely continuous, we have for $t \geq \theta$,

$$G_t(\theta) := P(\tau > \theta|\mathcal{F}_t) = P(\Theta > \Gamma_\theta|\mathcal{F}_t) = G_t^{\Theta}(\Gamma_\theta) = \int_{\Gamma_\theta}^\infty p_t(u)du = \int_\theta^\infty p_t(\Gamma_u)\gamma_u du,$$

which implies $g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta)$ for $t \geq \theta$.

Obviously, in the particular case where the threshold Θ is independent of $\mathcal{F}_\infty$, we recover the classical results (9.8.2) recalled above.

Conversely, if we are given a density process g, then it is possible to construct a random time τ by a generalized threshold, that is, to find Θ such that the associated τ has g as density, as we show now. It suffices to define $\tau = \inf\{t \,:\, t \geq \Theta\}$ where Θ is a random variable with conditional density $p_t = g_t$. Of course, for any increasing process Γ, $\tau = \inf\{t \,:\, \Gamma_t \geq \Delta\}$ where $\Delta := \Gamma_\Theta$ is a different way to obtain a solution!

9.8.2 Dynamic Gaussian Copula

This example, despite its simplicity, will allow us to construct a dynamic copula, in a Gaussian framework; more precisely, we construct, for any t, the (conditional) copula of a family of random times $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t)$ and we can chose the parameters so that $P(\tau_i > t_i, i = 1, \dots, n)$ equals a given (static) Gaussian copula. To the best of our knowledge, there are very few explicit constructions of such a model.
In Fermanian and Vigneron [56], the authors apply a copula methodology, using a factor Y. However, the processes they use to fit the conditional probabilities $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t \vee \sigma(Y))$ are not martingales. They show that, using some adequate parametrization, they can produce a model so that $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t)$ are martingales. Our model will satisfy both martingale conditions.
In [32], Carmona is interested in the dynamics of prices of assets corresponding to a payoff which is a Bernoulli random variable (taking values 0 or 1), in other words, he is looking for examples of dynamics of martingales valued in $[0, 1]$, with a given terminal condition. Surprisingly, the example he provides corresponds to the one we gave in Section 6.4.4, up to a particular choice of the parameters to satisfy the terminal constraint.

Let φ be the standard Gaussian probability density, and Φ the Gaussian cumulative function. We recall the results obtained in Section 6.4.4.

Let B be a Brownian motion and consider the random variable $X := \int_0^\infty f(s)dB_s$ where f is a deterministic, square-integrable function. For any real number θ and any positive t, one has

$$P(X > \theta|\mathcal{F}_t^B) = P\Big(m_t > \theta - \int_t^\infty f(s)dB_s|\mathcal{F}_t^B\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathcal{F}_t^B$-measurable. The random variable $\int_t^\infty f(s)dB_s$ follows a centered Gaussian law with variance $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and is independent of $\mathcal{F}_t^B$. Assuming that $\sigma(t)$ does not vanish, one has

$$P(X > \theta|\mathcal{F}_t^B) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big) . \tag{9.8.5}$$

In other words, the conditional law of X given $\mathcal{F}_t^B$ is a Gaussian law with mean m_t and variance $\sigma^2(t)$. We summarize the result in the following proposition, and we give the dynamics of the martingale survival process, obtained with a standard use of Itô's rule.

Proposition 9.8.2 *Let B be a Brownian motion, f an L^2 deterministic function, $m_t = \int_0^t f(s)dB_s$ and $\sigma^2(t) = \int_t^\infty f^2(s)ds$. The family*

$$G_t^X(\theta) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)$$

is a family of $\mathbb{F}^B$-martingales, valued in $[0, 1]$, which is decreasing w.r.t. θ. Moreover

$$dG_t^X(\theta) = \varphi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)}dB_t .$$

We obtain the associated density family by differentiating $G_t^X(\theta)$ w.r.t. θ,

$$g_t^X(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma(t)} \exp\Big(-\frac{(m_t - \theta)^2}{2\sigma^2(t)}\Big)$$

and its dynamics

$$dg_t^X(\theta) = -g_t^X(\theta)\frac{m_t - \theta}{\sigma^2(t)}f(t)dB_t. \tag{9.8.6}$$

In order to provide conditional survival probabilities for positive random variables, we consider $\tau = \psi(X)$ where ψ is a differentiable, positive and strictly increasing function and let $h = \psi^{-1}$. The conditional law of τ is

$$G_t(\theta) = \Phi\Big(\frac{m_t - h(\theta)}{\sigma(t)}\Big).$$

We obtain

$$g_t(\theta) = \frac{1}{\sqrt{2\pi}\sigma(t)} h'(\theta) \exp\Big(-\frac{(m_t - h(\theta))^2}{2\sigma^2(t)}\Big)$$

and

$$\begin{aligned} dG_t(\theta) &= \varphi\Big(\frac{m_t - h(\theta)}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)} dB_t\,, \\ dg_t(\theta) &= -g_t(\theta)\frac{m_t - h(\theta)}{\sigma(t)}\frac{f(t)}{\sigma(t)} dB_t. \end{aligned}$$

Introducing an n-dimensional standard Brownian motion $B = (B^i, i = 1, \dots, n)$ and a factor Y, independent of $\mathbb{F}^B$, gives a dynamic copula approach, as we present now. For h_i an increasing function, mapping $I\!R^+$ into $I\!R$, and setting $\tau_i = (h_i)^{-1}(\sqrt{1-\rho_i^2}\int_0^\infty f_i(s)dB_s^i + \rho_i Y)$, for $\rho_i \in (-1,1)$, an immediate extension of the Gaussian model leads to

$$P(\tau_i > t_i, \forall i = 1, \dots, n | \mathcal{F}_t^B \vee \sigma(Y)) = \prod_{i=1}^n \Phi\left(\frac{1}{\sigma_i(t)}\left(m_t^i - \frac{h_i(t_i) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)\right)$$

where $m_t^i = \int_0^t f_i(s)dB_s^i$ and $\sigma_i^2(t) = \int_t^\infty f_i^2(s)ds$. It follows that

$$P(\tau_i > t_i, \forall i = 1, \dots, n | \mathcal{F}_t^B) = \int_{-\infty}^{\infty} \prod_{i=1}^n \Phi\left(\frac{1}{\sigma_i(t)}\left(m_t^i - \frac{h_i(t_i) - \rho_i y}{\sqrt{1-\rho_i^2}}\right)\right) f_Y(y)dy\,.$$

Note that, in that setting, the random times $(\tau_i, i = 1, \dots, n)$ are conditionally independent given $\mathbb{F}^B \vee \sigma(Y)$, a useful property which is not satisfied in Fermanian and Vigneron model. For $t = 0$, choosing f_i so that $\sigma_i(0) = 1$, and Y with a standard Gaussian law, we obtain

$$P(\tau_i > t_i, \forall i = 1, \dots, n) = \int_{-\infty}^{\infty} \prod_{i=1}^n \Phi\left(-\frac{h_i(t_i) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) \varphi(y)dy$$

which corresponds, by construction, to the standard Gaussian copula ($h_i(\tau_i) = \sqrt{1-\rho_i^2}X_i + \rho_i Y$, where X_i, Y are independent standard Gaussian variables).

Relaxing the independence condition on the components of the process B leads to more sophisticated examples.

9.8.3 Markov processes

Let X be a real-valued Markov process with transition probability $p_T(t, x, y)dy = P(X_T \in dy | X_t = x)$, and Ψ a family of functions $I\!R \times I\!R \to [0, 1]$, decreasing w.r.t. the second variable, such that

$$\Psi(x, -\infty) = 1, \Psi(x, \infty) = 0\,.$$

Then, for any T,

$$G_t(\theta) := E(\Psi(X_T, \theta)|\mathcal{F}_t^X) = \int_{-\infty}^{\infty} p_T(t, X_t, y)\Psi(y, \theta)dy$$

is a family of martingale survival processes on $I\!R$. While modeling $(T; x)$-bond prices, Filipovic et al. [58] have used this approach in an affine process framework. See also Keller-Ressel et al. [93].

Example 9.8.3 Let X be a Brownian motion, and $\Psi(x,\theta) = e^{-\theta x^2}\mathbf{1}_{\theta\geq 0} + \mathbf{1}_{\theta\leq 0}$. We obtain a martingale survival process on $\mathbb{R}_+$, defined for $\theta \geq 0$ and $t < T$ as,

$$G_t(\theta) = E\big[\exp(-\theta X_T^2)|\mathcal{F}_t^X\big] = \frac{1}{\sqrt{1+2(T-t)\theta}}\exp\big(-\frac{\theta X_t^2}{1+2(T-t)\theta}\big)$$

The construction given above provides a martingale survival process $G(\theta)$ on the time interval $[0,T]$. Using a (deterministic) change of time, one can easily deduce a martingale survival process on the whole interval $[0,\infty[$: setting

$$\hat{G}_t(\theta) = G_{h(t)}(\theta)$$

for a differentiable increasing function h from $[0,\infty]$ to $[0,T]$, and assuming that $dG_t(\theta) = G_t(\theta)K_t(\theta)dB_t, t < T$, one obtains

$$d\hat{G}_t(\theta) = \hat{G}_t(\theta)K_{h(t)}(\theta)\sqrt{h'(t)}dW_t$$

where W is a Brownian motion.
One can also randomize the terminal date and consider T as an exponential random variable independent of $\mathbb{F}$. Noting that the previous $G_t(\theta)$'s depend on T, one can write them as $G_pt(\theta,T)$ and consider

$$\widetilde{G}_t(\theta) = \int_0^\infty G_t(\theta,z)e^{-z}dz$$

which is a martingale survival process. The same construction can be done with a random time T with any given density, independent of $\mathbb{F}$.

9.8.4 Diffusion-based model with initial value

Lemma 9.8.4 *Let Ψ be a cumulative distribution function of class C^2, and Y the solution of*

$$dY_t = a(Y_t)dt + \nu(Y_t)dB_t,\ Y_0 = y_0$$

where a and ν are deterministic functions smooth enough to ensure that the solution of the above SDE is unique. Then, the process $(\Psi(Y_t), t\geq 0)$ is a martingale, valued in $[0,1]$, if and only if

$$a(y)\Psi'(y) + \frac{1}{2}\nu^2(y)\Psi''(y) = 0\,. \tag{9.8.7}$$

PROOF: The result follows by applying Itô's formula and noting that $\Psi(Y_t)$ being a (bounded) local martingale is a martingale.

We denote by $Y_t(y)$ the solution of the above SDE with initial condition $Y_0 = y$. Note that, from the uniqueness of the solution, $y \to Y_t(y)$ is increasing (i.e., $y_1 > y_2$ implies $Y_t(y_1) \geq Y_t(y_2)$). It follows that

$$G_t(\theta) := 1 - \Psi(Y_t(\theta))$$

is a family of martingale survival processes.

Example 9.8.5 Let us reduce our attention to the case where Ψ is the cumulative distribution function of a standard Gaussian variable. Using the fact that $\Phi''(y) = -y\Phi'(y)$, Equation (9.8.7) reduces to

$$a(t,y) - \frac{1}{2}y\nu^2(t,y) = 0$$

In the particular the case where $\nu(t,y) = \nu(t)$, straightforward computation leads to

$$Y_t(y) = e^{\frac{1}{2}\int_0^t \nu^2(s)ds}(y + \int_0^t e^{-\frac{1}{2}\int_0^s \nu^2(u)du}\nu(s)dB_s)\,.$$

Setting $f(s) = -\nu(s)\exp(-\frac{1}{2}\int_0^s \nu^2(u)du)$, one deduces that $Y_t(y) = \frac{y-m_t}{\sigma(t)}$, where $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and $m_t =: \int_0^t f(s)dB_s$, and we recover the Gaussian example of Subsection 6.4.4.

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