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EULER EQUATIONS AND REAL HARMONIC ANALYSIS

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Abstract

We reprove various existence theorems of regular solutions for the Euler equations, using classical tools of real harmonic analysis such as singular integrals, atomic decompositions or maximal functions.

Key words. Euler equations – Besov spaces – Triebel–Lizorkin spaces – commutators – singular integrals

1. Introduction

This paper contains no actually new theorem. It aims to give a new proof of well-established results of existence of solutions to the Euler equations in spaces such as Besov spaces or Triebel–Lizorkin spaces. Following the seminal work of J.Y. Chemin [6], a large number of papers were written on that topic, mainly based on the use of the Littlewood–Paley decomposition. This approach is very efficient, especially in the critical case of $B^1_{\infty,1}$ [22], but can lead to tedious computations, as in the case of Triebel–Lizorkin spaces [7].

In this paper, we shall try not to use the Littlewood–Paley decomposition where it can be avoided . More precisely, we shall relax our computations and get rid of the computation of the Littlewood–Paley decomposition of the solution, and replace it by some more or less classical lemmas on transport equations, singular integral operators, atomic decompositions, and interpolation. This will allow us to recover existence results in Besov spaces and in Triebel–Lizorkin spaces.

2. A general scheme for solving Euler equations.

We consider a divergence-free vector field $v_0 = (v_{0,1}, \dots, v_{0,d})$ on $\mathbb{R}^d$:

$$\textbf{div } v_0 = \sum_{i=1}^{d} \partial_i v_{0,i} = 0 \tag{1}$$

and the associated Cauchy problem for the Euler equations

$$\begin{cases} \partial_t v + v.\nabla v = \nabla p \\ \\ \textbf{div } v = 0 \\ \\ v_{|t=0} = v_0 \end{cases} \tag{2}$$

v is assumed to be a bounded Lipschitz vector field (more precisely, we shall consider $v \in (L^\infty((0,T), \textbf{Lip}))^d$, where **Lip** is the space of bounded functions with bounded derivatives).

If we take the divergence of those equations, we find that

$$\Delta p = \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j (v_i v_j) \tag{3}$$

so that

$$\nabla p = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\nabla \partial_i \partial_j}{\Delta} (v_i v_j) + \nabla q \text{ with } \Delta q = 0. \tag{4}$$

For $v \in (\textbf{Lip})^d$ and $\textbf{div } v = 0$, $\sum_{i=1}^d \sum_{j=1}^d \frac{\nabla \partial_i \partial_j}{\Delta} (v_i v_j)$ is a well-defined distribution and may be written as the gradient of a distribution : if K is the kernel of the convolution operator $\frac{1}{\Delta}\nabla$, then we have $|K(x)| \leq C|x|^{1-d}$ and $|\partial_i \partial_j K(x)|^{-d-1}$ [for $|x| \neq 0$], so that we may write, taking $\varphi \in \mathscr{D}$ be equal to 1 on the ball $|x| \leq 1$, that $\sum_{i=1}^d \sum_{j=1}^d \frac{\nabla \partial_i \partial_j}{\Delta} (v_i v_j) = \sum_{i=1}^d \sum_{j=1}^d (\varphi K) * (\partial_j v_i \partial_i v_j) + \sum_{i=1}^d \sum_{j=1}^d \partial_i \partial_j ((1-\varphi)K) * (v_i v_j)$ and hence we get that $\sum_{i=1}^d \sum_{j=1}^d \frac{\nabla \partial_i \partial_j}{\Delta} (v_i v_j)$ belongs to $(L^\infty)^d$. We shall consider only cases where $q = 0$ (excluding the action of harmonic polynomials).

The Euler equations we shall consider will then be

$$\begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d \sum_{j=1}^d \frac{\nabla \partial_i \partial_j}{\Delta} (v_i v_j) \\ \\ \textbf{div } v = 0 \\ \\ v_{|t=0} = v_0 \end{cases} \tag{5}$$

Throughout the paper, we shall look for existence of solutions in $(L^\infty((0,T), E)^d$, where E will be a Banach space embedded into **Lip**; we are not looking for differentiabilty with respect to t, hence the equations will be satisfied in a weak sense (in the distribution sense). The spaces E we shall consider will be actually embedded in a smaller space : $E \subset B^1_{\infty,1} \subset \textbf{Lip}$. It is known that, when v_0 belongs to

$(B^1_{\infty,1})^d$, then (5) has a solution $v \in (\mathscr{C}([0,T), B^0_{\infty,1}) \cap L^\infty((0,T), B^1_{\infty,1}))^d$ and that this solution is unique [22] (see [1] for a larger class of uniqueness obtained by Danchin : $v \in (\mathscr{C}([0,T), B^0_{\infty,\infty}) \cap L^1((0,T), B^1_{\infty,\infty}))^d$). Thus, we shall be interested in the problem of proving existence of solutions keeping the regularity of the initial value $v_0 \in E^d$, and pay no special interest in the uniqueness issue (as it has been settled by Danchin [1]).

While in dimension $d = 2$, the study of the equations is easy through the control of the vorticity $\omega = \text{curl } u$ (classical results are [28] and [30]), the equations are more difficult to deal with when $d \geq 3$. We shall now rewrite equations (5) in a more convenient way for further study. We consider the Leray projection operator $\mathbb{P}$ on the solenoidal vector fields :

$$\mathbb{P}f = f - \nabla \frac{1}{\Delta} \mathbf{div}\, f; \tag{6}$$

this is not defined for all distributions, but at least it is well defined on vector fields of the form $\sum_{i=1}^d \partial_i u_i$ where the u_i are bounded vector fields. For $w = \sum_{i=1}^d \partial_i(v_i v) = v.\nabla v = v.\mathbb{P}\nabla v$, we find that

$$\sum_{i=1}^d \sum_{j=1}^d \frac{\nabla \partial_i \partial_j}{\Delta}(v_i v_j) = w - \mathbb{P}w = \sum_{i=1}^d v_i \mathbb{P}\partial_i v - \mathbb{P}\partial_i(v_i v) \tag{7}$$

so that we get finally

$$\begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ v_{|t=0} = v_0 \\ \mathbf{div}\, v = 0 \end{cases} \tag{8}$$

Equations (8) are the Euler equations we shall study in the rest of the paper.

We shall consider the following linear equations associated to the non-linear problem (8)

$$\begin{cases} \partial_t f + v.\nabla f = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] f \\ f_{|t=0} = v_0 \end{cases} \tag{9}$$

In equations (9), we see two parts. The left-hand part $\partial_t f + v.\nabla f$ is a transport equation through the vector field v; this can be solved through the use of characteristic curves when $v \in L^1_t \mathbf{Lip}$. The right-hand part $\sum_{i=1}^d [v_i, \partial_i \mathbb{P}] f$ is a sum of Calderón's commutators (commutators between pointwise multiplication and singular convolution operators with homogeneous kernels of exponent $-d-1$); those commutators are generalized Calderón–Zygmund operators when the multipliers v_i are Lipschitz functions. Thus, the same kind of minimal regularity on v is required to deal with both parts of the equations (9).

Let us pay now a few words on those two aspects of the equation. The characteristic curves are defined by $s \mapsto X_{t,x}(s)$ where $X_{t,x}$ is the solution of

$$\begin{cases} \frac{d}{ds} X_{t,x}(s) = v(s, X_{t,x}(s)) \\ \\ X_{t,x}(t) = x \end{cases} \tag{10}$$

But, for a divergence-free vector field $v \in L^1_t \mathbf{Lip}$, the homeomorphism $x \mapsto X_{t,x}(s)$ is bi-lipschitzian and preserves the Lebesgue measure, so that it operates on many function spaces. For instance, we have the following lemma :

Lemma 1. *Let $s \mapsto X_{t,x}(s)$ be the characteristic curves associated to a divergence-free vector field $v \in L^1([0,T], (\mathbf{Lip})^d)$. Then there exists two constants C_0 and C_1 such that, for $g \in BMO$ and $0 \le s \le t \le T$, we have*

$$\|g(X_{t,x}(s))\|_{BMO} \le C_0 \|g\|_{BMO} \, e^{C_1 \int_s^t \|\nabla \otimes v\|_\infty \, d\sigma}. \tag{11}$$

Proof. For a measure-preserving bi-Lipschitzian homeomorphism X, we have for any ball $B = B(x_0, r_0)$ and any constant λ

$$\begin{aligned} \frac{1}{|B|} \int_B |g(X(x)) - m_B(g(X))| \, dx &\le 2 \frac{1}{|B|} \int_B |g(X(x)) - \lambda| \, dx \\ &= 2 \frac{1}{|B|} \int_{X(B)} |g(y) - \lambda| \, dy \end{aligned} \tag{12}$$

Let M be the Lipschitz constant of X ($M = \sup_{x \neq y} \frac{\|X(x) - X(y)\|}{\|x-y\|}$) and $B_1 = B(X(x_0), M r_0)$, $\lambda = m_{B_1} g$. We have $X(B) \subset B_1$ so that (12) gives

$$\begin{aligned} \frac{1}{|B|} \int_B |g(X(x)) - m_B(g(X))| \, dx &\le 2 \frac{M^d}{|B_1|} \int_{B_1} |g(y) - m_{B_1} g| \, dx \\ &\le 2 M^d \|g\|_{BMO} \end{aligned} \tag{13}$$

Thus, we have (11). □

A Calderón commutator is a commutator between an operator M_A of pointwise multiplication by a function A and a singular convolution operator T_K with a homogeneous distribution K of exponent $-d-1$ which is smooth outside from $\{0\}$. The distribution kernel of $[M_A, T_K]$ is given by $L(x,y) = (A(x) - A(y)) K(x-y)$. If A is Lipschitz, then $[M_A, T_K]$ is a generalized Calderón–Zygmund operator [4] [21] [16] : T is bounded on L^2 and its kernel satisfies, outside from the diagonal $x = y$,

$$\begin{cases} \sup_{x \neq y} |x-y|^d |L(x,y)| < +\infty \\ \\ \sup_{x \neq y} |x-y|^{d+1} |\nabla_x L(x,y)| < +\infty \\ \\ \sup_{x \neq y} |x-y|^{d+1} |\nabla_y L(x,y)| < +\infty \end{cases} \tag{14}$$

The operator $\mathbb{P}$ is a matrix of scalar operators $(P_{j,k})_{1 \le j,k \le d}$ and thus $\sum_{i=1}^d [v_i, \mathbb{P} \partial_i]$ is a matrix of Calderón–Zygmund operators $T_{j,k} = \sum_{i=1}^d [v_i, P_{j,k} \partial_i]$. But the operators $T_{j,k}$ enjoy further interesting properties. Indeed, we have

$$T_{j,k}(1) = -\sum_{i=1}^d P_{j,k} \partial_i v_i = P_{j,k}(\mathbf{div} \, v) = 0 \tag{15}$$

and similarly $T^*_{j,k}(1) = 0$, so that they operate as well on many function spaces. For instance, a Calderón–Zygmund operator T maps boundedly L^∞ to BMO, but it maps as well boundedly BMO to BMO if and only if $T(1) = 0$ [14]. Thus, we have the following lemma :

Lemma 2. *If $v \in (\mathbf{Lip})^d$ and $\mathbf{div}\ v = 0$, then there exists a constant C_2 such that, for every $g \in BMO$, we have*

$$\| \sum_{i=1}^{d} [v_i, P_{j,k}\partial_i] g \|_{BMO} \le C_2 \|\nabla \otimes v\|_\infty \|g\|_{BMO} \tag{16}$$

Combining Lemmas 1 and 2, we easily get (by an unusual proof) the following (well-known) result about the conservation of the solenoidal character of the vector fields for solutions of equations (9) [1] :

Proposition 1. *Let $f \in (L^\infty((0,T), \mathbf{Lip})^d$ be a solution of the system*

$$\begin{cases} \partial_t f + v.\nabla f = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] f \\ \\ f_{|t=0} = v_0 \end{cases} \tag{17}$$

where $v \in (L^1((0,T), \mathbf{Lip})^d$, $\mathbf{div}\ v = 0$, $v_0 \in (\mathbf{Lip})^d$ and $\mathbf{div}\ v_0 = 0$. Then, we have : $\mathbf{div}\ f = 0$.

Proof. We are going to prove that $f = \mathbb{P}f$ in BMO. Indeed, we have

$$\begin{cases} \partial_t \mathbb{P}f + \mathbb{P}(v.\nabla) f = \mathbb{P}\sum_{i=1}^d [v_i, \mathbb{P}\partial_i] f = \mathbb{P}(v.\nabla)\mathbb{P}f - \mathbb{P}(v.\nabla) f \\ \\ \mathbb{P}f_{|t=0} = v_0 \end{cases} \tag{18}$$

and

$$\begin{cases} \partial_t f + v.\nabla f = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] f = v.\nabla \mathbb{P}f - \mathbb{P}(v.\nabla) f \\ \\ f_{|t=0} = v_0 \end{cases} \tag{19}$$

so that

$$\begin{cases} \partial_t (f - \mathbb{P}f) + v.\nabla (f - \mathbb{P}f) & = \mathbb{P}(v.\nabla)\mathbb{P}f - \mathbb{P}(v.\nabla) f \\ & = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i](f - \mathbb{P}f) \\ \\ f - \mathbb{P}f_{|t=0} = 0 \end{cases} \tag{20}$$

and thus

$$f - \mathbb{P}f = \int_0^t \big(\sum_{i=1}^{d} [v_i, \mathbb{P}\partial_i](f - \mathbb{P}f) \big)(s, X_{t,x}(s))\, ds \tag{21}$$

where X is the solution of

$$\begin{cases} \frac{d}{ds} X_{t,x}(s) = v(s, X_{t,x}(s)) \\ \\ X_{t,x}(t) = x \end{cases} \tag{22}$$

Using Lemmas 1 and 2, we find that

$$(23)\qquad \|f-\mathbb{P}f\|_{BMO} \leq C_0 C_2 \int_0^t e^{C_1 \int_s^t \|\nabla\otimes v\|_\infty \, d\sigma} \|\nabla\otimes v\|_\infty \|f-\mathbb{P}f\|_{BMO}\, ds$$

which is enough (due to the Gronwall lemma) to grant that $\|f-\mathbb{P}f\|_{BMO}=0$. □

Proposition 1 will lead us to choose our way of constructing solutions to equations (8). The classical way [6] [1] is to construct inductively approximations h_n of the solution v as solutions of the problem

$$(24)\qquad \begin{cases} \partial_t h_{n+1} + h_n.\nabla h_{n+1} = \sum_{i=1}^d [h_{n,i}, \mathbb{P}\partial_i] h_n \\ \\ h_{n+1\ |t=0} = v_0 \end{cases}$$

but the intermediate solutions h_n are not divergence-free, so that the operator $T_n = \sum_{i=1}^d [h_{n,i}, \mathbb{P}\partial_i]$ on the left-hand side of (24) doesn't satisfy $T_n(1) = T_n^*(1) = 0$. Thus, we shall prefer the following scheme (as in [7]) :

The scheme we shall follow to sove the Euler equations is then the following one : starting from $f_0 = v_0$, we shall try to find a solution $f_{n+1} \in L_t^\infty \mathrm{Lip}$ of the equation

$$(25)\qquad \begin{cases} \partial_t f_{n+1} + f_n.\nabla f_{n+1} = \sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] f_{n+1} \\ \\ f_{n+1\ |t=0} = v_0 \end{cases}$$

If this can be done, we will have (by induction) $\nabla.f_n = 0$.

In order to compute f_{n+1}, we define inductively $g_{n,k}$ as $g_{n,0} = v_0$ and

$$(26)\qquad \begin{cases} \partial_t g_{n,k+1} + f_n.\nabla g_{n,k+1} = \sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] g_{n,k} \\ \\ g_{n,k+1\ |t=0} = v_0 \end{cases}$$

The problem is now to prove the convergence of $g_{n,k}$ to f_{n+1} (as $k \to +\infty$) and of f_n to v (as $n \to +\infty$).

3. The abstract theory : the Cauchy problem in A^s.

In this section, we are going to solve equations (8) in an abstract space $A^{1+\sigma}$. $A^{1+\sigma}$ will belong to a scale of Banach spaces A^s (where $s > 0$ stands for a regularity index) which satisfies the following hypotheses:

◇ **Hypothesis (H1) : integrability**

$A^s \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (continuous embedding)

◇ **Hypothesis (H2) : monotony**

For $s_1 < s_2$, $A^{s_2} \subset A^{s_1}$

$\diamond$ **Hypothesis (H3) : regularity**

$f \in A^{1+s} \Leftrightarrow f \in A^s$ and $\nabla f \in A^s$ (with equivalence of the norms $\|f\|_{A^{s+1}}$ and $\|f\|_{A^s} + \|\nabla f\|_{A^s}$)

$\diamond$ **Hypothesis (H4) : stability**

If a sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in A^s and converges in $\mathscr{D}'(\mathbb{R}^d)$ then the limit belongs to A^s and we have $\|\lim_{n\to+\infty} f_n\|_{A^s} \leq C_s \liminf_{n\to+\infty} \|f_n\|_{A^s}$. (This is usually checked by using the theorem of Banach–Steinhaus, when A^s is a dual to a Banach space of functions in which $\mathscr{D}$ is densely and continuously embedded)

$\diamond$ **Hypothesis (H5) : invariance**

The map $(f,g) \in \mathscr{D} \times A^s \mapsto f * g$ extends to a bounded bilinear operator from $L^1 \times A^s$ to A^s. (Due to hypothesis (H^4), it is equivalent to the invariance through translations : there exists a constant C_s such that for all $x_0 \in \mathbb{R}^d$ and $f \in A^s$ we have $\|f(x-x_0)\|_{A^s} \leq C_s \|f\|_{A^s}$).

$\diamond$ **Hypothesis (H6) : interpolation**

If T is a linear operator which is bounded from A^{s_1} to A^{s_1} and from A^{s_2} to A^{s_2} then it is bounded from A^s to A^s for every $s \in [s_1, s_2]$ and $\|T\|_{\mathscr{L}(A^s,A^s)} \leq C(s,s_1,s_2) \max(\|T\|_{\mathscr{L}(A^{s_1},A^{s_1})}, \|T\|_{\mathscr{L}(A^{s_2},A^{s_2})})$.

$\diamond$ **Hypothesis (H7) : transport by Lipschitz flows**

Let $u \in L^1((0,T),\mathbf{Lip})$ be a divergence-free vector field and let $f_0 \in A^s$ for some $s \in (0,1)$. Then the solution $f \in \mathscr{C}([0,T], L^1_{loc})$ of the transport equation

$$\begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases} \tag{27}$$

satisfies $\sup_{0\leq t\leq T} \|f(t,.)\|_{A^s} \leq C_s e^{C_s \int_0^T \|u\|_{\mathbf{Lip}}\, dt} \|f_0\|_{A^s}$.

$\diamond$ **Hypothesis (H8) : singular integrals**

Let T be a bounded linear operator from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) which satisfies the following conditions

- T is bounded on L^2 : $\|T(f)\|_2 \leq C_0 \|f\|_2$
- outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \leq C_0 \frac{1}{|x-y|^d(1+|x-y|)}$
- outside from the diagonal, K satisfies $|\nabla_x K(x,y)| \leq C_0 |x-y|^{-d-1}$ and $|\nabla_y K(x,y)| \leq C_0 |x-y|^{-d-1}$
- $T(1) = T^*(1) = 0$ in BMO

Then, T is bounded from A^s to A^s for all $0 < s < 1$ and $\|T\|_{\mathscr{L}(A^s,A^s)} \leq C_s C_0$

We further consider an hypothesis on some $\sigma > 0$:

$\diamond$ **Hypothesis (H9) : pointwise products with A^σ**

$A^\sigma \subset L^\infty$ (continuous embedding) and, for all $s \in (0,\sigma]$, the product $(f,g) \mapsto fg$ is a bounded bilinear operator from $A^\sigma \times A^s$ to A^s.

We then have the following theorem on the Cauchy problem for the Euler equations with initial data in $A^{1+\sigma}$:

Theorem 1. *Let A^s be a scale of spaces satisfying hypotheses (H1) to (H8) and let $\sigma > 0$ satisfy hypothesis (H9). Let $v_0 \in A^{1+\sigma}$ be a divergence free vector field. Then there exists a positive T such that the Cauchy problem*

$$\left\{ \begin{array}{l} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ \\ v_{|t=0} = v_0 \\ \\ \nabla . v = 0 \end{array} \right. \tag{28}$$

has a unique solution $v \in \mathscr{C}([0,T], A^\sigma)$ such that $\sup_{0\le t\le T} \|v\|_{A^{\sigma+1}} < +\infty$.

Proof.

Step 1 : Study of the operator $\sum_{i=1}^d [u_i, \mathbb{P}\partial_i]$

$\mathbb{P}$ is a matrix of singular integral operators $P_{j,k} = \delta_{j,k} Id + R_j R_k$ where R_j is the j-th Riesz transform $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$. We shall prove :

Lemma 3. *Let $u \in A^{1+\sigma}$ with* **div** *$u = 0$. Then the operator $\sum_{i=1}^d [u_i, P_{j,k}\partial_i]$ is bounded on A^s for every $s \in (0, 1+\sigma]$ and we have $\| \sum_{i=1}^d [u_i, P_{j,k}\partial_i] f\|_{A^s} \le C_{s,\sigma} \|f\|_{A^s} \|u\|_{A^{1+\sigma}}$.*

Proof. The operator $T_{i,j,k} = [u_i, P_{j,k}\partial_i]$ is an example of the famous Calderón commutators [4] [16] between a Lipschitz function and an operator of order 1. The operator $P_{j,k}\partial_i$ is a convolution operator with a distribution $K_{i,j,k}$ whose restriction to $\mathbb{R}^d \setminus \{0\}$ is a smooth function which is homogeneous of homogeneity order $-d-1$. The distribution kernel of $T_{i,j,k}$ is given (outside from the diagonal $x = y$) by the function $L_{i,j,k}(x,y) = (u_i(x) - u_i(y)) K_{i,j,k}(x-y)$. Since $u_i \in A^{1+\sigma} \subset$ **Lip**, we have that $|L_{i,j,k}(x,y)| \le C_\sigma \|u_i\|_{A^{1+\sigma}} \frac{1}{|x-y|^d (1+|x-y|)}$ and $|\nabla_x L_{i,j,k}(x,y)| + |\nabla_y L_{i,j,k}(x,y)| \le C_\sigma \|u_i\|_{A^{1+\sigma}} |x-y|^{-d-1}$. Moreover, Calderón's theorem states that $T_{i,j,k}$ is bounded on L^2 with operator norm bounded by $C\|\nabla u_i\|_\infty \le C_\sigma \|u_i\|_{A^{1+\sigma}}$.

The next step is to compute $T_{i,j,k}(1) = T^*_{i,j,k}(1)$. We have $T_{i,j,k}(1) = -P_{j,k}(\partial_i u_i)$. Thus, $\sum_{i=1}^d T_{i,j,k}(1) = P_{j,k}($**div** $u) = 0$. Thus, we can apply (H8) and we get Lemma 3 for $0 < s < 1$.

Now, we consider s such that $1 + s \le 1 + \sigma$ and such that $\sum_{i=1}^d [u_i, P_{j,k}\partial_i]$ is bounded on A^s. We take $f \in A^{1+s}$ and try to estimate $g = \sum_{i=1}^d [u_i, P_{j,k}\partial_i] f$ in A^{s+1}. Due to (H3), we must estimate $\|g\|_{A^s}$ and, for $l = 1, \dots, d$, $\|\partial_l g\|_{A^s}$. We just write

$$\partial_l g = \sum_{i=1}^d [u_i, P_{j,k}\partial_i] \partial_l f + \sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f \tag{29}$$

so that we find

$$\|g\|_{A^{s+1}} \le C_s \big(\| \sum_{i=1}^d [u_i, P_{j,k}\partial_i] \|_{\mathscr{L}(A^s, A^s)} \|f\|_{A^{s+1}} + \sum_{l=1}^d \| \sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f \|_{A^s} \big). \tag{30}$$

We thus need to estimate $\|\sum_{i=1}^{d}[\partial_l u_i, P_{j,k}\partial_i]f\|_{A^s}$. This will be done by distinguishing the low frequencies and the high frequencies. If $S_0 f$ is the low-frequency block in the Littlewood–Paley decomposition $f = S_0 f + \sum_{j=1}^{+\infty}\Delta_j f$, then we write (using the fact that u is divergence-free)

$$\text{(31)}\qquad \begin{cases} \sum_{i=1}^{d}[\partial_l u_i, P_{j,k}\partial_i]f &= A+B+C+D \\ A &= \sum_{i=1}^{d}\partial_l u_i S_0 P_{j,k}\partial_i f \\ B &= -\sum_{i=1}^{d}\partial_i S_0 P_{j,k}(\partial_l u_i f) \\ C &= \sum_{i=1}^{d}\partial_l u_i (Id-S_0)P_{j,k}\partial_i f \\ D &= -\sum_{i=1}^{d}(Id-S_0)P_{j,k}(\partial_l u_i \partial_i f) \end{cases}$$

$(Id-S_0)P_{j,k}$ satisfies the assumptions of (H8), hence is bounded on every A^τ with $0<\tau<1$; since it is a convolution operator, hence commutes with derivatives, we use (H3) and find that it is bounded on every A^τ with $0<\tau\notin\mathbb{N}$ and finally for every positive τ (by (H6)). Thus, using (H9), we find that $\|C\|_{A^s}+\|D\|_{A^s}$ is controlled by $\|u\|_{A^{1+\sigma}}\|f\|_{A^{1+s}}$. Moreover, $\partial_i S_0 P_{j,k}$ has an integrable kernel; we then use the embedding $A^{s+1}\subset A^s$ (by (H2)) and (H5) to get that $\|A\|_{A^s}+\|B\|_{A^s}$ is controlled by $\|u\|_{A^{1+\sigma}}\|f\|_{A^s}$ and thus by $\|u\|_{A^{1+\sigma}}\|f\|_{A^{1+s}}$.

Thus, by induction, we get Lemma 3 for $0<s\leq 1+\sigma$, $s\notin\mathbb{N}$; the case $s\in\mathbb{N}$ and $0<s<1+\sigma$ then follows by interpolation; if $\sigma\in\mathbb{N}$, we obtain the final case $s=1+\sigma$ by induction from $s=\sigma$ to $s=1+\sigma$ one more time. □

Step 2 : Transport equations in A^s

In this section, we shall prove :

Lemma 4. *Let $u\in L^1([0,T],A^{1+\sigma})$ with* **div** $u=0$*. Let $f_0\in A^s$ for some exponent $s\in(0,1+\sigma]$. Then the solution $f\in\mathscr{C}([0,T],L^1_{loc})$ of the transport equation*

$$\text{(32)}\qquad \begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases}$$

satisfies $\sup_{0\leq t\leq T}\|f(t,.)\|_{A^s}\leq C_{s,\sigma}e^{C_{s,\sigma}\int_0^T\|u(t,.)\|_{A^{1+\sigma}}\,dt}\|f_0\|_{A^s}$

Proof. As for Lemma 3, we shall prove the lemma for $0<s<1$, then we shall prove that it holds for $1+s\leq 1+\sigma$ when it holds for s; this will give that the lemma is valid for $0<s<1+\sigma$, $s\notin\mathbb{N}$; then interpolation will give the case $0<s<1+\sigma$, $s\in\mathbb{N}$ and, if $\sigma\in\mathbb{N}$, a final induction gives the case $s=1+\sigma$.

The case $0<s<1$ is a direct consequence of (H7) since we have (by (H2), (H3) and (H9)) the embedding $A^{1+\sigma}\subset$ **Lip**.

Now, let us assume that Lemma 4 is valid for some $s\in(0,\sigma]$ and let us assume that $f_0\in A^{1+s}$. In particular, f_0 is uniformly locally in $W^{1,1}$ and since u is a Lipschitz vector field, we find that f as well is uniformly locally in $W^{1,1}$ and that its derivatives $(\partial_1 f,\ldots,\partial_d f)$ are solutions of the system

$$\text{(33)}\qquad \text{for } j=1,\ldots,d,\quad \partial_t\partial_j f + u.\nabla\partial_j f = -\sum_{k=1}^{d}\partial_j u_k\partial_k f$$

Thus, writing $M_u = (\partial_j u_k)_{1\le j,k\le d}$ and $\tau \mapsto X_{t,x}(\tau)$ the characteristic curves associated to the vector field u, we find that $H(t,x) = \begin{pmatrix} \partial_1 f \\ \vdots \\ \partial_d f \end{pmatrix}$ is solution of the fixed-point problem

$$H(t,x) = H(0,X_{t,x}(0)) + \int_0^t M_u(\tau,X_{t,x}(\tau))H(\tau,X_{t,x}(\tau))\, d\tau \tag{34}$$

For $\lambda > 0$, let $\mathscr{L}_\lambda$ be the operator $K \mapsto \mathscr{L}_\lambda K = S$ where $S(t,x)$ is given by $S(t,x) = \int_0^t e^{-\lambda(t-\tau)} M_u(\tau,X_{t,x}(\tau))K(\tau,X_{t,x}(\tau))\, d\tau$. $\mathscr{L}_\lambda$ maps $L^\infty((0,T),(L^1_{uloc})^d)$ into itself (where L^1_{uloc} is the space of uniformly locally integrable functions, normed by $\|f\|_{L^1_{uloc}} = \sup_{x_0\in\mathbf{R}^d} \int_{|x-x_0|<1} |f(x)|\, dx$) and we have

$$\begin{aligned} \|\mathscr{L}_\lambda K\|_{L^\infty L^1_{uloc}} \le \quad & C\|K\|_{L^\infty L^1_{uloc}} sup_{0<t<T} \int_0^t e^{-\lambda(t-\tau)} \|u\|_{\mathbf{Lip}}\, e^{C\int_\tau^t \|u\|_{\mathbf{Lip}}\, d\theta} d\tau \\ & = C_{\lambda,u}\|K\|_{L^\infty L^1_{uloc}} \end{aligned} \tag{35}$$

The solution H of (34) may be written as $H = e^{\lambda t}K$ where K is solution of

$$K(t,x) = e^{-\lambda t} H(0,X_{t,x}(0)) + \mathscr{L}_\lambda K \tag{36}$$

For λ large enough, we have $C_{\lambda,u} < 1$ and $\mathscr{L}_\lambda$ is a contraction on $L^\infty((0,T),(L^1_{uloc})^d)$.

Further, we may apply the induction hypothesis and (H9) to see that $\mathscr{L}_\lambda$ maps $L^\infty((0,T),(A^s)^d)$ into itself and that we have

$$\begin{aligned} \|\mathscr{L}_\lambda K\|_{L^\infty A^s} \le \quad & C\|K\|_{L^\infty A^s} \sup_{0<t<T} \int_0^t e^{-\lambda(t-\tau)} \|u\|_{A^{1+\sigma}}\, e^{C\int_\tau^t \|u\|_{A^{1+\sigma}}\, d\theta} d\tau \\ & = D_{\lambda,u}\|K\|_{L^\infty L^1_{uloc}} \end{aligned} \tag{37}$$

For λ large enough, we have $D_{\lambda,u} < 1$ and $\mathscr{L}_\lambda$ is a contraction on $L^\infty((0,T),(A^s)^d)$. Since $H(0,x) = \begin{pmatrix} \partial_1 f_0 \\ \vdots \\ \partial_d f_0 \end{pmatrix}$ belongs to $(L^1_{uloc} \cap A^s)^d$, we get that $H(0,X_{t,x}(0))$ belongs to $L^\infty((0,T),(L^1_{uloc})^d) \cap L^\infty((0,T),(A^s)^d)$ and finally that H itself belongs to $L^\infty((0,T),(A^s)^d)$. This proves that $f \in L^\infty A^{1+s}$.

We then control the size of $\|f\|_{A^{1+s}}$ through the Gronwall lemma. □

Step 3 : Equation (26)

We are now going to prove theorem 1, by approximating the solution v by the inductively defined f_n (equation (25)) and $g_{n,k}$ (equation (26)). We shall prove by induction that we can find a time T such that for all n and k we have

$$\sup_{0<t<T} \|f_n\|_{A^{1+\sigma}} \le 4C_0\|v_0\|_{A^{1+\sigma}} \text{ and } \sup_{0<t<T} \|g_{n,k}\|_{A^{1+\sigma}} \le 4C_0\|v_0\|_{A^{1+\sigma}} \tag{38}$$

where C_0 is the constant $C_{1+\sigma,\sigma}$ in Lemma 4. Recall that we defined inductively $g_{n,k}$ as $g_{n,0} = v_0$ and

$$\left\{\begin{array}{l}\partial_t g_{n,k+1} + f_n.\nabla g_{n,k+1} = \sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] g_{n,k} \\ \qquad g_{n,k+1\,|t=0} = v_0\end{array}\right. \tag{39}$$

We assume that f_n is divergence free and that $\sup_{0<t<T} \|f_n\|_{A^{1+\sigma}} \leq 4C_0 \|v_0\|_{A^{1+\sigma}}$ and $\sup_{0<t<T} \|g_{n,k}\|_{A^{1+\sigma}} \leq 4C_0 \|v_0\|_{A^{1+\sigma}}$. Now, using $\tau \mapsto X_{t,x}^{(n)}(\tau)$ the characteristic curves associated to the vector field f_n, we have the following expression for $g_{n,k+1}$:

$$g_{n,k+1} = v_0(X_{t,x}^{(n)}(0)) + \int_0^t \big(\sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] g_{n,k}\big)(\tau, X_{t,x}^{(n)}(\tau))\, d\tau \tag{40}$$

We write $\delta_0 = C_0 \|v_0\|_{A^{1+\sigma}}$. Using Lemmas 3 and 4, we find that, for some constant D_0 which depends neither on v_0, nor on n or k, nor on T,

$$\sup_{0<t<T} \|g_{n,k+1}\|_{A^{1+\sigma}} \leq \delta_0 e^{4C_0 T \delta_0} + C_0 D_0 T e^{4C_0 T \delta_0} (4\delta_0)^2 \tag{41}$$

so that the induction is valid if T is small enough to ensure that

$$e^{4C_0 T \delta_0}(1 + 16 C_0 D_0 \delta_0 T) < 4. \tag{42}$$

Step 4 : Equation (25)

If we consider the operator $\mathscr{L}_n$ defined by $\mathscr{L}_n g = h$ with

$$h(t,x) = \int_0^t \big(\sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] g\big)(\tau, X_{t,x}^{(n)}(\tau))\, d\tau \tag{43}$$

we have

$$\sup_{0<t<T} \|\mathscr{L}_n g\|_{A^{1+\sigma}} \leq 4C_0 \delta_0 D_0 T e^{4C_0 T \delta_0} \sup_{0<t<T} \|g\|_{A^{1+\sigma}} \tag{44}$$

so that $\mathscr{L}_n$ is a contraction on $L^\infty((0,T),(A^{1+\sigma})^d)$ (under condition (42)). Thus, $g_{n,k}$ converges to the fixed point $f_{n+1} = v_0(X_{t,x}^{(n)}(0)) + \mathscr{L}_n f_{n+1}$. We find that f_{n+1} is a solution of (25) (so that f_{n+1} is divergence free) and that $\sup_{0<t<T} \|f_{n+1}\|_{A^{1+\sigma}} \leq 4C_0 \|v_0\|_{A^{1+\sigma}}$.

Step 5 : Equation (8)

The last step in the proof of Theorem 1 is to check the convergence of f_n to a solution v of equation (8). Let $k_n = f_{n+1} - f_n$. We have

$$\begin{aligned}\partial_t k_{n+1} + f_{n+1}.\nabla k_{n+1} &= -k_n.\nabla f_{n+1} + \textstyle\sum_{i=1}^d [f_{n+1,i}, \mathbb{P}\partial_i] k_{n+1} \\ &\qquad + \textstyle\sum_{i=1}^d [k_{n,i}, \mathbb{P}\partial_i] f_{n+1}\end{aligned} \tag{45}$$

with

$$\begin{gathered}\textstyle\sum_{i=1}^d [\partial_l k_{n,i}, P_{j,k}\partial_i] h = \\ \textstyle\sum_{i=1}^d \partial_l k_{n,i}, S_0 P_{j,k} \partial_i h - \sum_{i=1}^d \partial_i S_0 P_{j,k}(\partial_l k_{n,i} h) \\ + \textstyle\sum_{i=1}^d \partial_l k_{n,i}(Id - S_0), P_{j,k}\partial_i h - \sum_{i=1}^d (Id - S_0) P_{j,k}(\partial_l k_{n,i} \partial_i h)\end{gathered} \tag{46}$$

This gives

$$
(47)\quad \begin{cases} k_{n+1} = \int_0^t G_n(\tau, X_{t,x}^{(n+1)}(\tau))\, d\tau \\ G_n(t,x) = -k_n.\nabla f_{n+1} + \sum_{i=1}^d [f_{n+1,i}, \mathbb{P}\partial_i] k_{n+1} + \sum_{i=1}^d [k_{n,i}, \mathbb{P}\partial_i] f_{n+1} \end{cases}
$$

hence (by Lemmas 3 and 4, and hypotheses (H5), (H8) and (H9)) we find that, for some constant D_1 which depends neither on v_0, nor on n or T, we have

$$
(48)\quad \sup_{0<t<T} \|k_{n+1}\|_{A^\sigma} \leq D_1 e^{D_1 4\delta_0 T} T (4\delta_0 \sup_{0<t<T} \|k_n\|_{A^\sigma} + 4\delta_0 \sup_{0<t<T} \|k_{n+1}\|_{A^\sigma})
$$

If T is small enough to grant that

$$
(49)\quad 4\delta_0 D_1 e^{D_1 4\delta_0 T} T < 1/4
$$

we find that

$$
(50)\quad \sup_{0<t<T} \|k_{n+1}\|_{A^\sigma} \leq \frac{1}{3} \sup_{0<t<T} \|k_n\|_{A^\sigma}
$$

so that $\sum_{n\in\mathbb{N}} \sup_{0<t<T} \|f_{n+1} - f_n\|_{A^\sigma} < +\infty$.

Let us remark that $\partial_t f_n$ is bounded in A^σ, so that f_n belongs to $\mathscr{C}[0,T],(A^\sigma)^d)$ and converges strongly in $\mathscr{C}[0,T],(A^\sigma)^d)$ to some vector field v. This vector field is divergence-free. Moreover, due to the stability hypothesis (H4), we have that $\sup_{0<t<T} \|v\|_{A^{\sigma+1}} < +\infty$.

Now, we check that v is a solution to (8). We must prove the convergence in $\mathscr{D}'$ of $f_n.\nabla f_{n+1}$ to $v.\nabla v$ and of $\sum_{i=1}^d [f_{n,i}, \mathbb{P}\partial_i] f_{n+1}$ to $\sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v$. This is quite easy, since f_n converges strongly to v in L^∞ and $\partial_i f_n$ converges *-weakly to $\partial_i v$ in L^∞. This gives by interpolation strong convergence in $B_\infty^{\alpha,\infty}$ for all $\alpha \in (1/2, 1)$, from which we get the required convergence. □

4. The scale of Besov spaces.

We may apply quite directly Theorem 1 to the case of an intitial value v_0 in a Besov space :

Theorem 2. *Let $v_0 \in B_{p,q}^{1+\sigma}$ be a divergence free vector field. Assume that $1 \leq p \leq +\infty$, and that $\sigma > d/p$ and $1 \leq q \leq +\infty$, or that $\sigma = d/p$ and $q = 1$. Then there exists a positive T such that the Cauchy problem*

$$
(51)\quad \begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ v_{|t=0} = v_0 \\ \nabla.v = 0 \end{cases}
$$

has a unique solution $v \in \mathscr{C}([0,T], B_{p,q}^\sigma)$ such that $\sup_{0\leq t\leq T} \|v\|_{B_{p,q}^{1+\sigma}} < +\infty$.

Proof. We introduce the scale of Besov spaces $B^s_{p,q}$ for $0 < s \leq 1+\sigma$ and we check that this scale satisfies hypotheses (H1) to (H9) :

◇ **Hypothesis (H1) : integrability** : for $s > 0$, $B^s_{p,q} \subset L^p \subset L^1_{\rm loc}(\mathbb{R}^d)$

◇ **Hypothesis (H2) : monotony** : For $s_1 < s_2$, $B^{s_2}_{p,q} \subset B^{s_1}_{p,q}$

◇ **Hypothesis (H3) : regularity** : $f \in B^{1+s}_{p,q} \Leftrightarrow f \in B^s_{p,q}$ and $\nabla f \in B^s_{p,q}$

◇ **Hypothesis (H4) : stability** : If a sequence $(f_n)_{n\in\mathbf{N}}$ is bounded in $B^s_{p,q}$ and converges in $\mathscr{D}'(\mathbb{R}^d)$ then the limit belongs to $B^s_{p,q}$ and we have $\|\lim_{n\to+\infty} f_n\|_{B^s_{p,q}} \leq \liminf_{n\to+\infty} \|f_n\|_{B^s_{p,q}}$. ($B^s_{p,q}$ is the dual space of the closure of $\mathscr{D}$ in $B^{-s}_{p/(p-1),q/(q-1)}$).

◇ **Hypothesis (H5) : invariance** : for all $x_0 \in \mathbb{R}^d$ and all $f \in B^s_{p,q}$ we have the equality $\|f(x-x_0)\|_{B^s_{p,q}} = \|f\|_{B^s_{p,q}}$.

◇ **Hypothesis (H6) : interpolation**

To prove that (H6) is fullfilled, we may use the real interpolation functor, as we have, for $s_1 < s < s_2 \in \mathbb{R}$, that $B^s_{p,q} = [B^{s_1}_{p,q}, B^{s_2}_{p,q}]_{\theta,q}$ with $\theta = \frac{s-s_1}{s_2-s_1}$ [3].

◇ **Hypothesis (H7) : transport by Lipschitz flows**

Let $u \in L^1((0,T),\mathbf{Lip})$ be a divergence-free vector field and let $S(t)$ be the operator that maps $f_0 \in L^p$ to the solution $f \in \mathscr{C}([0,T], L^1_{loc})$ $(f(t,x) = \big(S(t)f_0\big)(x))$of the transport equation

$$\text{(52)} \qquad \begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases}$$

We have $\|S(t)f_0\|_p = \|f_0\|_p$. Moreover, we have, when $f_0 \in W^{1,p}$, $\partial_j S(t) f_0 = \sum_{k=1}^d S(t)\partial_k f_0 \ \partial_j X_{k,t,x}(0)$. so that $\sup_{0\leq t\leq T} \|f(t,.)\|_{W^{1,p}} \leq C e^{C\int_0^T \|u\|_{\mathbf{Lip}}\, dt} \|f_0\|_{W^{1,p}}$. The case of the $B^s_{p,q}$ norm follows by interpolation, since, for $0 < s < 1$, we have $B^s_{p,q} = [L^p, W^{1,p}]_{\theta,q}$ with $\theta = s$.

◇ **Hypothesis (H8) : singular integrals**

Let T be a bounded linear operator from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) which satisfies the following conditions

- T is bounded on L^2 : $\|T(f)\|_2 \leq C_0 \|f\|_2$
- outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \leq C_0 \frac{1}{|x-y|^d(1+|x-y|)}$
- outside from the diagonal, K satisfies $|\nabla_x K(x,y)| \leq C_0 |x-y|^{-d-1}$ and $|\nabla_y K(x,y)| \leq C_0 |x-y|^{-d-1}$
- $T(1) = T^*(1) = 0$ in BMO

Then, T is bounded from $B^s_{p,q}$ to $B^s_{p,q}$ for all $0 < s < 1$ and $\|T\|_{\mathscr{L}(B^s_{p,q},B^s_{p,q})} \leq C_s C_0$ [15].

$\diamond$ **Hypothesis (H9) : pointwise products with $B^\sigma_{p,q}$**

It is well known that, for any positive s, $B^s_{p,q}\cap L^\infty$ is a Banach algebra [3][16]. For $\sigma > n/p$ and $1\le q\le +\infty$, or for for $\sigma = n/p$ and $q=1$, we have $B^\sigma_{p,q}\subset L^\infty$ (continuous embedding). Thus, the pointwise product $(f,g)\mapsto fg$ is a bounded bilinear operator from $B^\sigma_{p,q}\times E$ to E when $E=B^\sigma_{p,q}$ and when $E=L^p$, hence, by interpolation, when $E=B^s_{p,q}$ for any $s\in(0,\sigma]$ (since, for $0<s<\sigma$, $B^s_{p,q}=[L^p,B^\sigma_{p,q}]_{\theta,q}$ with $\theta = s/\sigma$).

Thus, we find that Theorem 2 is only a corollary of Theorem 1. $\square$

5. The scale of Triebel-Lizorkin spaces.

We may as well apply quite directly Theorem 1 to the case of an intitial value v_0 in a Triebel–Lizorkin space. Let us recall that Besov spaces may be defined through the Littlewood–Paley decomposition as

$$f\in B^s_{p,q}\Leftrightarrow f\in\mathscr{S}', S_0f\in L^p \text{ and } (2^{js}\|\Delta_j f\|_p)_{j\in\mathbf{N}}\in l^q \tag{53}$$

Similarly, for $1\le p,q<+\infty$, the Triebel–Lizorkin space $F^s_{p,q}$ [3] may be defined as :

$$f\in F^s_{p,q}\Leftrightarrow f\in\mathscr{S}', S_0f\in L^p \text{ and } \Big(\sum_{j\in\mathbf{N}} 2^{jsq}|\Delta_j f|^q\Big)^{1/q}\in L^p \tag{54}$$

We may prove easily the following Theorem (announced in [5] and fully proved in [7] for $p>1$) :

Theorem 3. *Let $v_0\in F^{1+\sigma}_{p,q}$ be a divergence free vector field. Assume that $1\le p,q<+\infty$, and that $\sigma>d/p$. Then there exists a positive T such that the Cauchy problem*

$$\begin{cases} \partial_t v+v.\nabla v=\sum_{i=1}^d[v_i,\mathbb{P}\partial_i]v \\ v_{|t=0}=v_0 \\ \nabla.v=0\end{cases} \tag{55}$$

has a unique solution $v\in\mathscr{C}([0,T],F^\sigma_{p,q})$ such that $\sup_{0\le t\le T}\|v\|_{F^{1+\sigma}_{p,q}}<+\infty$.

Proof. We introduce the scale of Triebel–Lizorkin spaces $F^s_{p,q}$ for $0<s\le 1+\sigma$ and we check that this scale satisfies hypotheses (H1) to (H9) :

$\diamond$ **Hypothesis (H1) : integrability** : for $s>0$, $F^s_{p,q}\subset L^p\subset L^1_{\text{loc}}(\mathbb{R}^d)$

$\diamond$ **Hypothesis (H2) : monotony** : For $s_1<s_2$, $F^{s_2}_{p,q}\subset F^{s_1}_{p,q}$

$\diamond$ **Hypothesis (H3) : regularity** : $f\in F^{1+s}_{p,q}\Leftrightarrow f\in F^s_{p,q}$ and $\nabla f\in F^s_{p,q}$

$\diamond$ **Hypothesis (H4) : stability** : If a sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $F^s_{p,q}$ and converges in $\mathscr{D}'(\mathbb{R}^d)$ then the limit belongs to $F^s_{p,q}$ and we have $\|\lim_{n\to+\infty} f_n\|_{F^s_{p,q}} \leq \liminf_{n\to+\infty} \|f_n\|_{F^s_{p,q}}$: it is enough to check that we have the pointwise convergence of $\Delta_j f_n$ to $\Delta_j f$ (where f is the limit of f_n) and then to conclude by applying twice Fatou's lemma.

$\diamond$ **Hypothesis (H5) : invariance** : for all $x_0 \in \mathbb{R}^d$ and all $f \in F^s_{p,q}$ we have the equality $\|f(x-x_0)\|_{F^s_{p,q}} = \|f\|_{F^s_{p,q}}$.

$\diamond$ **Hypothesis (H6) : interpolation**

To prove that (H6) is fullfilled, we may use the complex interpolation functor, as we have, for $s_1 < s < s_2 \in \mathbb{R}$, that $F^s_{p,q} = [F^{s_1}_{p,q}, F^{s_2}_{p,q}]_\theta$ with $\theta = \frac{s-s_1}{s_2-s_1}$ [3].

$\diamond$ **Hypothesis (H7) : transport by Lipschitz flows**

Let $u \in L^1((0,T),\mathbf{Lip})$ be a divergence-free vector field and let $S(t)$ be the operator that maps $f_0 \in L^p$ to the solution $f \in \mathscr{C}([0,T], L^1_{loc})$ ($f(t,x) = \big(S(t)f_0\big)(x)$)of the transport equation

$$\text{(56)} \qquad \begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases}$$

Indeed, we write again $f(t,x) = f_0(X_{t,x}(0))$; $x \mapsto X_{t,x}(0)$ is a bi-Lipschitzian homeomorphism and the partial derivatives $\partial_j(X_{t,x}(0)$ are controlled in L^∞ norm by $Ce^{C\int_0^t \|u\|_{\mathbf{Lip}}\, d\tau}$. Thus, we must prove that $F^s_{p,q}$ is stable under composition with a bi-Lipschitzian homeomorphism X when $0 < s < 1$. This is easy to check, using the characterization of $F^s_{p,q}$ through finite differences [27] : for $1 \leq p,q < +\infty$ and for $0 < s < 1$, we have :

$$\text{(57)} \quad f \in F^s_{p,q} \Leftrightarrow f \in L^p \text{ and } (\int_0^1 \int_{|h|<t} t^{-d-sq} |f(x) - f(x+h)|^q \, dh \, dt)^{1/q} \in L^p$$

(with equivalence of norms). Let J be the Jacobian matrix of X, $K(x) = \|J(x)\|_{\mathrm{op}} = \sup_{|y|\leq 1} |J(x)y|$. We have

$$\text{(58)} \qquad \|f\circ X\|_p \leq \|\det J^{-1}\|_\infty^{\frac{1}{p}} \|f\|_p$$

whereas

$$\text{(59)} \qquad \begin{array}{c} \int_{|h|<t} |f(X(x)) - f(X(x+h))|^q \, dh \\ \leq \|\det J^{-1}\|_\infty \int_{|k|<\|K\|_\infty t} |f(X(x)) - f(X(x)+k)|^q \, dk. \end{array}$$

We make a change of variable $k = \|K\|_\infty h$ and we write $g(x) = f(\|K\|_\infty x)$, we then get

$$\text{(60)} \qquad \begin{array}{c} \int_{|h|<t} |f(X(x)) - f(X(x+h))|^q \, dh \\ \leq \|\det J^{-1}\|_\infty \|K\|_\infty^d \int_{|h|<t} |g(\|K\|_\infty^{-1} X(x)) - g(\|K\|_\infty^{-1} X(x) + h)|^q \, dh \end{array}$$

A further change of variable $y = \|K\|_\infty^{-1} X(x))$ gives us that the norm of $f \circ X$ in $F^s_{p,q}$ is controlled by $\|f\|_p + \|g\|_{F^s_{p,q}}$. And we easily control the norm of g by the norm of f in $F^s_{p,q}$, so that we may conclude.

◇ **Hypothesis (H8) : singular integrals**

Let T be a bounded linear operator from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) which satisfies the following conditions

• T is bounded on L^2 : $\|T(f)\|_2 \leq C_0 \|f\|_2$

• outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \leq C_0 \frac{1}{|x-y|^d(1+|x-y|)}$

• outside from the diagonal, K satisfies $|\nabla_x K(x,y)| \leq C_0 |x-y|^{-d-1}$ and $|\nabla_y K(x,y)| \leq C_0 |x-y|^{-d-1}$

• $T(1) = T^*(1) = 0$ in BMO

Then, T is bounded from $F^s_{p,q}$ to $F^s_{p,q}$ for all $0 < s < 1$ and $\|T\|_{\mathscr{L}(F^s_{p,q}, F^s_{p,q})} \leq C_s C_0$.

Indeed, the boundedness of such an operator T on the homogeneous space $\dot{F}^s_{p,q}$ has been proved by several authors (for $p > 1$, we may quote [10] [**?**]; for $p = 1$, see [9]). Now, the norm of $F^s_{p,q}$ is equivalent (for $s > 0$) to the sum of the norm in $\dot{F}^s_{p,q}$ and the norm of $B^{s/2}_{p,q}$, so that boundedness on $\dot{F}^s_{p,q}$ and on $B^{s/2}_{p,q}$ gives boundedness on $F^s_{p,q}$.

◇ **Hypothesis (H9) : pointwise products with $F^\sigma_{p,q}$**

It is well known that, for any positive s, $F^s_{p,q} \cap L^\infty$ is a Banach algebra [3]. Moreover, if $0 < s < \varepsilon < 1$, then the pointwise product $(f,g) \mapsto fg$ is a bounded bilinear operator from $B^\varepsilon_{\infty,\infty} \times F^s_{p,q}$ to $F^s_{p,q}$ [24]. For $\sigma > n/p$, we have $F^\sigma_{p,q} \subset L^\infty$ (continuous embedding), and more precisely $F^\sigma_{p,q} \subset B^{\sigma-d/p}_{\infty,\infty}$. Thus, the pointwise product $(f,g) \mapsto fg$ is a bounded bilinear operator from $F^\sigma_{p,q} \times E$ to E when $E = F^\sigma_{p,q}$ and when $E = F^s_{p,q}$ with $0 < s < \min(1, \sigma - d/p)$, hence, by interpolation, when $E = F^s_{p,q}$ for any $s \in (0, \sigma]$.

Thus, we find that Theorem 3 is only a corollary of Theorem 1. □

6. Atoms and molecules.

The continuity of singular integrals on Triebel–Lizorkin spaces can be proved in an “elementary” way by proving that this class of operators preserve the localization and the scale of so-called “molecules” (see in particular [9] and[12]). The preservation of molecules is the basis for the construction of an algebra of singular integral operators introduced by Y. Meyer [20] and the author [14].

We define $\mathscr{A}_\varepsilon$ $(0 < \varepsilon \leq 1)$ as the following class of Calderòn–Zygmund operators : a bounded linear operator T from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) belongs to $\mathscr{A}_\varepsilon$ if it fullfills the following conditions :

• T is bounded on L^2 : $\|T(f)\|_2 \leq C_0 \|f\|_2$

• outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \leq C_0 \frac{1}{|x-y|^d(1+|x-y|)}$

• outside from the diagonal, K satisfies $|K(x,y)-K(z,y)| \leq C_0|x-z|^\varepsilon(\frac{1}{|x-y|^{d+\varepsilon}}+\frac{1}{|z-y|^{d+\varepsilon}})$

• outside from the diagonal, K satisfies $|K(x,y)-K(x,z)| \leq C_0|y-z|^\varepsilon(\frac{1}{|x-y|^{d+\varepsilon}}+\frac{1}{|x-z|^{d+\varepsilon}})$

• $T(1)=T^*(1)=0$ in BMO

We shall define a norm on $\mathscr{A}_\varepsilon$ by taking $\|f\|_{\mathscr{A}_\varepsilon}$ as the infimum of the constants C_0 which satisfies the above four inequalities.

Now, we define an α-molecule f centered at $x=x_0$ at scale r (what we shall write as $f \in \mathscr{M}^\alpha(x_0,r)$) by the following requirements : $f \in \mathscr{M}^\alpha(x_0,r)$ if it fullfills the following conditions :

• $|f(x)| \leq \frac{r^\alpha}{(r+|x-x_0|)^{d+\alpha}}$

• $|f(x)-f(y)| \leq (\frac{|x-y|}{r})^\alpha(\frac{r^\alpha}{(r+|x-x_0|)^{d+\alpha}}+\frac{r^\alpha}{(r+|y-x_0|)^{d+\alpha}})$

• $\int_{\mathbb{R}^d} f(x)\,dx=0$

We shall use the following result of [14] :

Theorem 4. *A) If $0<\beta<\alpha\leq\varepsilon\leq 1$ and if $T\in\mathscr{A}_\varepsilon$, then there exists a positive $\lambda>0$ such that for every $x_0\in\mathbb{R}^d$ and every $r>0$ we have for every $f\in\mathscr{M}^\alpha(x_0,r)$ that $\lambda T(f)\in\mathscr{M}^\beta(x_0,r)$.*
B) Let $0<\varepsilon<\beta\leq\alpha\leq 1$. If T is a bounded linear operator on L^2 and if there exists a positive $\lambda>0$ such that for every $x_0\in\mathbb{R}^d$ and every $r>0$ we have for every $f\in\mathscr{M}^\alpha(x_0,r)$ that $\lambda T(f)\in\mathscr{M}^\beta(x_0,r)$, then $T\in\mathscr{A}_\varepsilon$.
C) The set $\mathscr{A}^\varepsilon=\cup_{\eta<\varepsilon}\mathscr{A}_\eta$ is an algebra of Calderón–Zygmund operators.

Using this theory of molecules, or using the characterization of $\mathscr{A}^\varepsilon$ by the matrix of $T\in\mathscr{A}^\varepsilon$ in a wavelet basis, we have the following theorem of Meyer [21] :

Theorem 5. *If $0<\varepsilon\leq 1$ and if $T\in\mathscr{A}_\varepsilon$, then, for $0<\alpha<\varepsilon$, the operator $(-\Delta)^{\alpha/2}\circ T\circ(-\Delta)^{-\alpha/2}$ belongs to $\mathscr{A}^{\varepsilon-\alpha}$. Moreover, if $\alpha<\beta<\varepsilon$ and $0<\gamma<\beta-\alpha$, $\|(-\Delta)^{\alpha/2}\circ T\circ(-\Delta)^{-\alpha/2}\|_{\mathscr{A}_\gamma}\leq C_{\alpha,\beta,\gamma}\|T\|_{\mathscr{A}_\beta}$*

7. Sobolev spaces over the Morrey–Campanato spaces and Lorentz spaces.

Theorem 5 will give us a new way of establishing well-posedness of the Euler equations. Indeed, we introduce a class $\mathscr{B}_{CZ}$ of Banach spaces by the following conditions : we will say that a Banach space B of functions defined on $\mathbb{R}^d$ belongs to $\mathscr{B}_{CZ}$ if it fullfills the following requirements :

◇ **Hypothesis (K1) : integrability**

$B\subset L^1_{\text{loc}}(\mathbb{R}^d)$ (continuous embedding)

◇ **Hypothesis (K2) : stability**

If a sequence $(f_n)_{n\in\mathbf{N}}$ is bounded in B and converges in $\mathscr{D}'(\mathbb{R}^d)$ then the limit belongs to B and we have $\|\lim_{n\to+\infty} f_n\|_B\leq C_s \liminf_{n\to+\infty}\|f_n\|_B$.

⋄ **Hypothesis (K3) : invariance**
The map $(f,g) \in \mathscr{D} \times B \mapsto f * g$ extends to a bounded bilinear operator from $L^1 \times B$ to B.

⋄ **Hypothesis (K4) : pointwise product**
The map $(f,g) \mapsto fg$ is a bounded bilinear operator from $L^\infty \times B$ to B.

⋄ **Hypothesis (K5) : bi-Lipschitzian homeomorphisms**
If X is a bi-Lipschitzian measure-preserving homeomorphism, if J is its Jacobian matrix, then for every $f \in B$ we have $f \circ X \in B$ and moreover, for two positive constants C and D which don't depend neither on X nor on f, we have $\|f \circ X\|_B \le C(1+\|J\|_\infty)^D \|f\|_B$.

⋄ **Hypothesis (K6) : singular integrals**
For every $\varepsilon \in (0,1]$ and every $T \in \mathscr{A}_\varepsilon$, T is bounded from B to B and $\|T\|_{\mathscr{L}(B,B)} \le C\|T\|_{\mathscr{A}_\varepsilon}$

⋄ **Hypothesis (K7) : high frequencies control**
there exists some $\kappa \in \mathbb{R}$ such that $B \subset B^\kappa_{\infty,\infty}$.

We shall define the Sobolev space $W^{k,B}$ for $k \in \mathbb{N}$ as the space of the functions $f \in B$ such that, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$, we have $\partial^\alpha f \in B$. We may prove a variant of Theorem 1 :

Theorem 6. *Let $B \in \mathscr{B}_{CZ}$ such that $B \subset B^\kappa_{\infty,\infty}$. Let $N \in \mathbb{N}$ such that $N+\kappa > 0$. Let $v_0 \in W^{N+1,B}$ be a divergence free vector field. Then there exists a positive T such that the Cauchy problem*

$$
\begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ v_{|t=0} = v_0 \\ \nabla . v = 0 \end{cases} \tag{61}
$$

has a unique solution $v \in \mathscr{C}([0,T], W^{N,B})$ such that $\sup_{0\le t\le T} \|v\|_{W^{N+1,B}} < +\infty$.

Proof. Let us first remark that , for $f \in \mathscr{S}'$, we have $f \in W^{k,B} \Leftrightarrow (Id-\Delta)^{k/2} f \in B$, due to hypothesis (K6). We thus may introduce the scale of Banach spaces $B^s = (Id-\Delta)^{s/2} B$ for $0 \le s \le 1+N$ and we check that this scale satisfies hypotheses (H1) to (H9) :

⋄ **Hypothesis (H1) : integrability** : for $s > 0$, $B^s \subset B^0 = B \subset L^1_{\text{loc}}(\mathbb{R}^d)$

⋄ **Hypothesis (H2) : monotony** : For $s_1 < s_2$, $B^{s_2} \subset B^{s_1}$ (since $(Id-\Delta)^{\frac{s_1-s_2}{2}}$ is a convolution operator with a kernel in L^1

⋄ **Hypothesis (H3) : regularity** : $f \in B^{1+s} \Leftrightarrow f \in B^s$ and $\nabla f \in B^s$ (owing to (K6))

◇ **Hypothesis (H4) : stability** : If a sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in B^s and converges in $\mathscr{D}'(\mathbb{R}^d)$ then the limit belongs to B^s and we have $\|\lim_{n\to+\infty} f_n\|_{B^s} \leq \liminf_{n\to+\infty} \|f_n\|_{B^s}$. (Just check that $(Id-\Delta)^{s/2} f_n$ converges in $\mathscr{S}'$ to $(Id-\Delta)^{s/2} f$, where $f = \lim_{n\to+\infty} f_n$, and then apply (K2)).

◇ **Hypothesis (H5) : invariance** : it is obvious since we can commute convolution operators.

◇ **Hypothesis (H6) : interpolation**

To prove that (H6) is fullfilled, we may use the complex interpolation functor, as it is easy to check that we have, for $0 \leq s_1 < s < s_2$, that $B^s = [B^{s_1}, B^{s_2}]_\theta$ with $\theta = \frac{s-s_1}{s_2-s_1}$.

◇ **Hypothesis (H7) : transport by Lipschitz flows**

Let $u \in L^1((0,T),\mathbf{Lip})$ be a divergence-free vector field and let $S(t)$ be the operator that maps $f_0 \in B$ to the solution $f \in \mathscr{C}([0,T], L^1_{loc})$ ($f(t,x) = \big(S(t)f_0\big)(x)$)of the transport equation

$$\begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases} \tag{62}$$

Due to (K5), we have $\|S(t)f_0\|_B \leq Ce^{C\int_0^T \|u\|_{\mathbf{Lip}}\,dt}\|f_0\|_B$. Moreover, we have, when $f_0 \in W^{1,B}$, $\partial_j S(t) f_0 = \sum_{k=1}^d S(t)\partial_k f_0\ \partial_j X_{k,t,x}(0)$. so that (using (K4) and (K5)), we get

$$\sup_{0\leq t\leq T} \|f(t,.)\|_{W^{1,B}} \leq Ce^{C\int_0^T \|u\|_{\mathbf{Lip}}\,dt}\|f_0\|_{W^{1,B}} \tag{63}$$

The case of the B^s norm follows by interpolation, for $0 < s < 1$.

◇ **Hypothesis (H8) : singular integrals**

Let T be a bounded linear operator from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) which satisfies the following conditions

• T is bounded on L^2 : $\|T(f)\|_2 \leq C_0\|f\|_2$

• outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \leq C_0 \frac{1}{|x-y|^d(1+|x-y|)}$

• outside from the diagonal, K satisfies $|\nabla_x K(x,y)| \leq C_0|x-y|^{-d-1}$ and $|\nabla_y K(x,y)| \leq C_0|x-y|^{-d-1}$

• $T(1) = T^*(1) = 0$ in BMO

Then, T is bounded from B^s to B^s for all $0 < s < 1$ and $\|T\|_{\mathscr{L}(B^s,B^s)} \leq C_s C_0$: indeed, it is easy to check that, for positive s, $(-\Delta^{s/2}$ is well defined on B and that $f \in B^s \Leftrightarrow f \in B$ and $(-\Delta)^{s/2} f \in B$ (with equivalence of norms $\|(Id-\Delta)^{s/2} f\|_B$ and $\|f\|_B + \|(-\Delta)^{s/2} f\|_B$). Now, if $T \in \mathscr{A}_1$ and $0 < s < 1$, we find that $\|Tf\|_B \leq C\|f\|_B$ (due to (K6)) and that $\|(-\Delta)^{s/2} Tf\|_B = \|\big((-\Delta)^{s/2} \circ T \circ (-\Delta)^{-s/2}\big)(-\Delta)^{s/2} f\|_B \leq C\|(-\Delta)^{s/2} f\|_B$ (due to Theorem 5 and (K6)).

◇ **Hypothesis (H9) : pointwise products with B^N**

From (K6) and (K4), we find that, for f and g in $B^N \subset L^\infty$, we control the size of $(-\Delta)^{Nz/2} f\ (-\Delta)^{N(1-z)/2} g$ in B^0 when $\mathrm{Re}\, z = 0$ or $\mathrm{Re}\, z = 1$. By complex interpolation, we find that we control $(-\Delta)^{z/2} f\ (-\Delta)^{(1-z)/2} g$ in B^0 when $0 \leq \mathrm{Re}\, z \leq 1$. In particular, we find that, for f and g in B^N and α and β in $\mathbb{N}^d$ with $|\alpha| + |\beta| = N$, we control $\partial^\alpha f \partial^\beta g$ in B^0. This proves that the pointwise product $(f, g) \mapsto fg$ is bounded from $B^N \times B^N$ to B^N. On the other hand, we have (from (K4)) that the pointwise product is bounded from $B^N \times B^0$ to B^0. By interpolation, it is bounded from $B^N \times B^s$ to B^s for $0 \leq s \leq N$.

Thus, we find that Theorem 6 is only a corollary of Theorem 1. □

Example 1 : Lebesgue spaces.

For $1 < p < +\infty$, $L^p \in \mathscr{B}_{CZ}$. Thus, theorem 6 gives again Theorem 3 in the case of $W^{N+1,p}$ with $N \in \mathbb{N}$ and $N > d/p$. (Recall that $W^{N+1,p} = F^{N+1}_{p,2}$).

Example 2 : Lorentz spaces.

For $1 < p < +\infty$ and $1 \leq q \leq +\infty$, the Lorentz space $L^{p,q}$ belongs to $\mathscr{B}_{CZ}$. Hypotheses (K1) to (K7) are easy to check, since, for $1 < p_1 < p < p_2 < +\infty$, we have $L^{p,q} = [L^{p_1}, L^{p_2}]_{\theta,q}$ with $\theta = \frac{p-p_1}{p_2-p_1}$. Theorem 6 gives the existence of a solution to the Euler equations, when the initial value belongs to $W^{N+1,L^{p,q}}$ with $1 < p < +\infty$, $1 \leq q \leq +\infty$, $N \in \mathbb{N}$ and $N > d/p$.

Example 3 : homogeneous Morrey–Campanato spaces.

For a ball $B = B(x_0, r)$, we define 1_B the characteristic function of B and $|B|$ the Lebesgue measure of B. The homogeneous Morrey–Campanato space $\dot{M}^{p,q}$ is then defined, for $1 < p < +\infty$ and $p \leq q \leq +\infty$ by $f \in \dot{M}^{p,q} \Leftrightarrow \sup_B |B|^{1/q-1/p} \|1_B f\|_p < +\infty$ (with norm $\|f\|_{\dot{M}^{p,q}} = \sup_B |B|^{1/q-1/p} \|1_B f\|_p$). It is easy to check that, for $1 < p \leq q < +\infty$, we have $\dot{M}^{p,q} \in \mathscr{B}_{CZ}$. Theorem 6 gives the existence of a solution to the Euler equations, when the initial value belongs to $W^{N+1,\dot{M}^{p,q}}$ with $1 < p \leq q < +\infty$, $N \in \mathbb{N}$ and $N > d/q$.

Example 4 : homogeneous Lorentz–Morrey–Campanato spaces.

The homogeneous Lorentz-Morrey–Campanato space $\dot{M}^{p,q,r}$ is then defined, for $1 < p < +\infty$, $p \leq q \leq +\infty$ and $1 \leq r \leq +\infty$, by $f \in \dot{M}^{p,q,r} \Leftrightarrow \sup_B |B|^{1/q-1/p} \|1_B f\|_{L^{p,r}} < +\infty$ (with norm $\|f\|_{\dot{M}^{p,q,r}} = \sup_B |B|^{1/q-1/p} \|1_B f\|_{L^{p,r}}$). It is easy to check that, for $1 < p \leq q < +\infty$ and $1 \leq r \leq +\infty$, we have $\dot{M}^{p,q,r} \in \mathscr{B}_{CZ}$. Theorem 6 gives the existence of a solution to the Euler equations, when the initial value belongs to $W^{N+1,\dot{M}^{p,q,r}}$ with $1 < p \leq q < +\infty$, $1 \leq r \leq +\infty$, $N \in \mathbb{N}$ and $N > d/q$.

Example 5 : multiplier spaces $\dot{X}^r$.

For $0 < r < d/2$, the homogeneousSobolev space $\dot{H}^r$ is defined, by $f \in \dot{H}^r \Leftrightarrow f \in L^{\frac{2d}{d-2r}}$ and $(-\Delta)^{r/2} f \in L^2$. Then the space $\dot{X}^r$ is defined as the space of pointwise multipliers from $\dot{H}^r$ to L^2 [16] : $\|f\|_{\dot{X}^r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_2$. Those spaces were first studied by Maz'ya [17] [18]. It is easy to check that, for $0 < r < 1$, we have $\dot{X}^r \in \mathscr{B}_{CZ}$. Hypotheses (K1) to (K4) are quite obvious. For (K5), we may write the norm in $\dot{H}^r$ (for $0 < r < 1$) as $\|f\|_{\dot{H}^r} = \left(\int\int \frac{|f(x)-f(y)|^2}{|x-y|^{n+2r}}\, dx\, dy\right)^{1/2}$ and thus check easily that $\dot{H}^r$ (as well as L^2) is stable under bi-Lipschitzian changes of variable; thus, $\dot{X}^r$ is stable as well under bi-Lipschitzian changes of variable and (K5) is fullfilled. The stability of $\dot{X}^r$ under the action of a Calderón–Zygmund operator has been established by Verbitsky in [**?**] and thus (K6) is fullfilled. Moreover, (K7) is obvious, since $\dot{X}^r \subset B^{-r}_{\infty,\infty}$. Theorem 6 then gives the existence of a solution to the Euler equations, when the initial value belongs to $W^{N+1,\dot{X}^r}$ with $0 < r < 1$, $N \in \mathbb{N}$ and $N \geq 1$.

8. Besov spaces over the Lorentz spaces or the Morrey–Campanato spaces.

In [16], we developed a theory of Besov spaces over shift-invariant Banach spaces of local measures. A shift-invariant Banach space of local measures is a space E which is the dual of a space E^* such that :
i) $\mathscr{D}$ is dense in E^*
ii) the norm of E^* is invariant through space translation : $\|f(x-x_0)\|_{E^*} = \|f\|_{E^*}$
iii) E^* is stable through space dilation : for all $\lambda > 0$, $\sup_{\|f\|_{E^*}\leq 1} \|f(\lambda x)\|_{E^*} < +\infty$
iv) the pointwise product $(f,g) \mapsto fg$ is a bounded map from $\mathscr{C}_b \times E^*$ to E^*.

Then, for $s \in \mathbb{R}$ and $1 \leq q \leq +\infty$, the Besov space $B^s_{E,q}$ is defined as the interpolation space $B^s_{E,q} = [(Id-\Delta)^{-s_1/2}E, (Id-\Delta)^{s_2/2}E]_{\theta,q}$ for $s_1 < s < s_2$ and $\theta = \frac{s-s_1}{s_2-s_1}$. It does not depend on s_1 nor s_2 and can be characterized through the Littlewood–Paley decomposition as

$$f \in B^s_{E,q} \Leftrightarrow f \in \mathscr{S}', S_0 f \in E \text{ and } (2^{js}\|\Delta_j f\|_E)_{j\in\mathbb{N}} \in l^q \tag{64}$$

One more time, we may easily apply Theorem 1 to solve the Euler equations in some generalized Besov spaces :

Theorem 7. *Let E be a shift–invariant Banach space of local measures and assume moreover that $E \in \mathscr{B}_{CZ}$. Let $\sigma > 0$ and $1 \leq q \leq +\infty$ be such that $B^\sigma_{E,q} \subset L^\infty$. Let $v_0 \in B^{1+\sigma}_{E,q}$ be a divergence free vector field. Then there exists a positive T such that the Cauchy problem*

$$\begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ v_{|t=0} = v_0 \\ \nabla . v = 0 \end{cases} \tag{65}$$

has a unique solution $v \in \mathscr{C}([0,T], B^\sigma_{E,q})$ such that $\sup_{0\leq t\leq T} \|v\|_{B^{1+\sigma}_{E,q}} < +\infty$.

Proof. We introduce the scale of Banach spaces $B^s_{E,q}$ for $0 < s \leq 1+\sigma$ and we check that this scale satisfies hypotheses (H1) to (H9). Hypotheses (H1) to (H5) are obvious (integrability, monotony, regularity, stability and invariance).

◇ **Hypothesis (H6) : interpolation**

To prove that (H6) is fullfilled, we may use the real interpolation functor, as it is easy to check that we have, for $0 \leq s_1 < s < s_2$, that $B^s_{E,q} = [B^{s_1}_{E,q}, B^{s_2}_{E,q}]_{\theta,q}$ with $\theta = \frac{s-s_1}{s_2-s_1}$.

◇ **Hypothesis (H7) : transport by Lipschitz flows**

This is a direct consequence of the same property for the scale $B^s = (Id - \Delta)^{-s/2}E$, since for $0 < s_1 < s < s_2 < 1$ we have $B^s_{E,q} = [(Id-\Delta)^{-s_1/2}E, (Id-\Delta)^{s_2/2}E]_{\theta,q}$.

◇ **Hypothesis (H8) : singular integrals**

This is again a direct consequence of the same property for the scale $B^s = (Id-\Delta)^{-s/2}E$.

◇ **Hypothesis (H9) : pointwise products with $B^\sigma_{E,q}$**

In [16] we have shown that, for any positive s, $B^s_{E,q} \cap L^\infty$ is a Banach algebra. Thus, the pointwise product $(f,g) \mapsto fg$ is a bounded bilinear operator from $B^\sigma_{E,q} \times F$ to F when $F = B^\sigma_{E,q}$ and when $F = E$, hence, by interpolation, when $F = B^s_{E,q}$ for any $s \in (0,\sigma]$ (since, for $0 < s < \sigma$, $B^s_{E,q} = [E, B^\sigma_{E,q}]_{\theta,q}$ with $\theta = s/\sigma$).

Thus, we find that Theorem 7 is only a corollary of Theorem 1. □

Example 1 : Lorentz spaces.

Theorem 7 gives the existence of a solution to the Euler equations, when the initial value belongs to $B^{\sigma+1}_{L^{p,q},r}$ with $1 < p < +\infty$, $1 \leq q \leq +\infty$, $\sigma > d/p$ and $1 \leq r \leq +\infty$ (or $\sigma = d/p$ and $r = 1$). The case $r = +\infty$ was discussed in [25].

Example 2 : homogeneous Morrey–Campanato spaces.

While the Sobolev spaces built on $\dot{M}^{p,q}$ are known as Q-spaces [29], the Besov spaces are known as Kozono-Yamazaki spaces [13]. Theorem 7 gives the existence of a solution to the Euler equations, when the initial value belongs to $B^{\sigma+1}_{\dot{M}^{p,q},r}$ with $1 < p \leq q < +\infty$, $\sigma > d/q$ and $1 \leq r \leq +\infty$ (or $\sigma = d/q$ and $r = 1$). Such a result was announced in [26].

Example 3 : homogeneous Lorentz–Morrey–Campanato spaces.

Similarly, Theorem 7 gives the existence of a solution to the Euler equations, when the initial value belongs to $B^{\sigma+1}_{\dot{M}^{p,q,r},t}$ with $1 < p \leq q < +\infty$, $1 \leq r \leq +\infty$, $\sigma > d/q$ and $1 \leq t \leq +\infty$ (or $\sigma = d/q$ and $t = 1$).

9. Related equations.

Theorem 1 can be adapted to deal with other equations that are quite close to the Euler equations.

Example 1 : the ideal MHD equations. The ideal MHD equations introduce a new variable b : now, we consider two divergence-free vector fields $v_0 = (v_{0,1}, \dots, v_{0,d})$

and b_0 on $\mathbb{R}^d$ and we try to solve the following Cauchy problem :

$$(66)\qquad \begin{cases} \partial_t v + v.\nabla v = \nabla p - \frac{1}{2}\nabla|b|^2 + b.\nabla b \\ \partial_t b + v.\nabla b = \ b.\nabla v \\ \mathbf{div}\ v = 0, \quad \mathbf{div}\ b = 0 \\ v_{|t=0} = v_0, \quad b_{|t=0} = b_0 \end{cases}$$

One more time, we consider only solutions for which we can get rid of the pressure term (here, $\nabla(p - \frac{1}{2}|b|^2)$) by use of the Leray projection operator $\mathbb{P}$, and we write

$$(67)\qquad \begin{cases} \partial_t v + \sum_{i=1^d} \mathbb{P}\partial_i(v_i v - b_i b) = 0 \\ \partial_t b + \sum_{i=1}^d \mathbb{P}\partial_i(v_i b - b_i v) = 0 \\ v_{|t=0} = v_0, \quad b_{|t=0} = b_0 \\ \mathbf{div}\ v = 0, \quad \mathbf{div}\ b = 0 \end{cases}$$

Following [7], we introduce the new unknown quantities $\alpha = v + b$ and $\beta = v - b$ and we find that

$$(68)\qquad \begin{cases} \partial_t \alpha + \sum_{i=1^d} \mathbb{P}\partial_i(\beta_i \alpha) = 0 \\ \partial_t \beta + \sum_{i=1}^d \mathbb{P}\partial_i(\alpha_i \beta) = 0 \\ \alpha_{|t=0} = v_0 + b_0, \quad \beta_{|t=0} = v_0 - b_0 \\ \mathbf{div}\ \alpha = 0, \quad \mathbf{div}\ \beta = 0 \end{cases}$$

and finally

$$(69)\qquad \begin{cases} \partial_t \alpha + \beta.\nabla \alpha = \sum_{i=1^d}[\beta_i, \mathbb{P}\partial_i]\alpha \\ \partial_t \beta + \alpha.\nabla.\beta = \sum_{i=1}^d[\alpha_i, \mathbb{P}\partial_i]\beta \\ \alpha_{|t=0} = v_0 + b_0, \quad \beta_{|t=0} = v_0 - b_0 \\ \mathbf{div}\ \alpha = 0, \quad \mathbf{div}\ \beta = 0 \end{cases}$$

The resolution of (69) follows exactly the same lines as the resolution of the Euler equations and we find easily the following theorem :

Theorem 8. *Let A^s be a scale of spaces satisfying hypotheses (H1) to (H8) and let $\sigma > 0$ satisfy hypothesis (H9). Let $v_0 \in A^{1+\sigma}$ and $b_0 \in A^{1+\sigma}$ be two divergence* *free vector fields. Then there exists a positive T such that the Cauchy problem*

$$(70)\qquad \begin{cases} \partial_t v + \sum_{i=1^d} \mathbb{P}\partial_i(v_i v - b_i b) = 0 \\ \partial_t b + \sum_{i=1}^d \mathbb{P}\partial_i(v_i b - b_i v) = 0 \\ v_{|t=0} = v_0, \quad b_{|t=0} = b_0 \\ \mathbf{div}\, v = 0, \quad \mathbf{div}\, b = 0 \end{cases}$$

has a unique solution (v,b) in $\mathscr{C}([0,T],A^\sigma)$ such that $\sup_{0\leq t\leq T} \|v\|_{A^{\sigma+1}} + \|b\|_{A^{1+\sigma}} < +\infty$.

Examples :

Theorem 8 gives existence of solutions in the following cases :

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{p,q}$, $A^\sigma = B^\sigma_{p,q}$, $1 \leq p \leq +\infty$, $\sigma > d/p$, $1 \leq q \leq +\infty$ (Theorem 2)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{p,q}$, $A^\sigma = B^\sigma_{p,q}$, $1 \leq p < +\infty$, $\sigma = d/p$, $q = 1$ (Theorem 2)

$\diamond$ $A^{1+\sigma} = F^{1+\sigma}_{p,q}$, $A^\sigma = F^\sigma_{p,q}$, $1 \leq p < +\infty$, $\sigma > d/p$, $1 \leq q < +\infty$ (Theorem 3)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma,L^{p,q}}$, $A^\sigma = W^{\sigma,L^{p,q}}$, $1 < p < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > d/p$, $1 \leq q \leq +\infty$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma,\dot{M}^{p,q}}$, $A^\sigma = W^{\sigma,\dot{M}^{p,q}}$, $1 < p \leq q < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > d/q$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma,\dot{M}^{p,q,r}}$, $A^\sigma = W^{\sigma,\dot{M}^{p,q,r}}$, $1 < p \leq q < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > d/q$, $1 \leq r \leq +\infty$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{L^{p,q},r}$, $A^\sigma = B^\sigma_{L^{p,q},r}$, $1 < p < +\infty$, $\sigma > d/p$, $1 \leq q \leq +\infty$, $1 \leq r \leq +\infty$ (Theorem 7)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{\dot{M}^{p,q},r}$, $A^\sigma = B^\sigma_{\dot{M}^{p,q},r}$, $1 < p \leq q < +\infty$, $\sigma > d/q$, $1 \leq r \leq +\infty$ (Theorem 7)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{\dot{M}^{p,q,r},t}$, $A^\sigma = B^\sigma_{\dot{M}^{p,q,r},t}$, $1 < p \leq q < +\infty$, $\sigma > d/q$, $1 \leq r \leq +\infty$, $1 \leq t \leq +\infty$ (Theorem 7)

Example 2 : the quasi-geostrophic equation.

The quasi-geostrophic equation (QG) is related to fluid mechanics [23] ; its mathematical study was initiated by Constantin, Majda and Tabak [8] in 1994. The quasi-geostrophic equation (QG) describes the evolution of a function $\theta(t,x)$,

$t > 0$, $x \in \mathbb{R}^2$ as

$$\left\{\begin{array}{c} \partial_t \theta + u.\nabla\theta = 0 \\ u = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{array}\right. \tag{71}$$

where R_i is the Riesz transform $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ (so that the vector field u is divergence-free : **div** $u = 0$).

The same formalism as for Euler equations will provide solutions, except that we don't need hypothesis (H8) any longer (since there is no right-hand term in equations (71)), but that we need $A^{1+\sigma}$ to be stable under the Riesz transforms, in order to ensure that u is still Lipschitzian. Thus, we get the following theorem :

Theorem 9. *Let A^s be a scale of spaces satisfying hypotheses (H1) to (H7) and let $\sigma > 0$ satisfy hypothesis (H9). Assume moreover that the Riesz transforms are bounded on $A^{1+\sigma}$. Let $\theta_0 \in A^{1+\sigma}$. Then there exists a positive T such that the Cauchy problem*

$$\left\{\begin{array}{c} \partial_t \theta + u.\nabla\theta = 0 \\ u = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{array}\right. \tag{72}$$

has a unique solution θ in $\mathscr{C}([0,T], A^\sigma)$ such that $\sup_{0\le t\le T} \|\theta\|_{A^{1+\sigma}} < +\infty$.

Examples :

Theorem 9 gives existence of solutions in the following cases :

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{p,q}$, $A^\sigma = B^\sigma_{p,q}$, $1 < p < +\infty$, $\sigma > 2/p$, $1 \le q \le +\infty$ (Theorem 2)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{p,q}$, $A^\sigma = B^\sigma_{p,q}$, $1 < p < +\infty$, $\sigma = 2/p$, $q = 1$ (Theorem 2)

$\diamond$ $A^{1+\sigma} = F^{1+\sigma}_{p,q}$, $A^\sigma = F^\sigma_{p,q}$, $1 < p < +\infty$, $\sigma > 2/p$, $1 \le q < +\infty$ (Theorem 3)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma, L^{p,q}}$, $A^\sigma = W^{\sigma, L^{p,q}}$, $1 < p < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > 2/p$, $1 \le q \le +\infty$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma, \dot{M}^{p,q}}$, $A^\sigma = W^{\sigma, \dot{M}^{p,q}}$, $1 < p \le q < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > 2/q$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = W^{1+\sigma, \dot{M}^{p,q,r}}$, $A^\sigma = W^{\sigma, \dot{M}^{p,q,r}}$, $1 < p \le q < +\infty$, $\sigma \in \mathbb{N}$, $\sigma > 2/q$, $1 \le r \le +\infty$ (Theorem 6)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{L^{p,q},r}$, $A^\sigma = B^\sigma_{L^{p,q},r}$, $1 < p < +\infty$, $\sigma > 2/p$, $1 \le q \le +\infty$, $1 \le r \le +\infty$ (Theorem 7)

$\diamond$ $A^{1+\sigma} = B^{1+\sigma}_{\dot{M}^{p,q},r}$, $A^\sigma = B^\sigma_{\dot{M}^{p,q},r}$, $1 < p \le q < +\infty$, $\sigma > 2/q$, $1 \le r \le +\infty$ (Theorem 7)

$\diamond\, A^{1+\sigma} = B^{1+\sigma}_{\dot M^{p,q,r},t}$, $A^\sigma = B^\sigma_{\dot M^{p,q,r},t}$, $1 < p \le q < +\infty$, $\sigma > 2/q$, $1 \le r \le +\infty$, $1 \le t \le +\infty$ (Theorem 7)

10. The critical case.

Thus far, there are two hypotheses we did not really use. In all our examples, our spaces A^s for $0 < s < 1$ were stable under transportation by a vector field in $L^1\mathbf{Lip}$ (even if the vector field was not divergence-free in hypothesis (H7)) and were stable as well under the action of a Calderón–Zygmund operator T satisfying $T(1) = 0$ (even if $T^*(1) \neq 0$ in hypothesis (H8)). (Even for Theorem 5, $T^*(1) = 0$ is not required, as we shall see in the following section.) Those conditions are crucial only in the critical case $\sigma = 0$ (initial value in $B^1_{\infty,1}$ [22].

The main lemma is then the following one :

Lemma 5. *If $f \in B^0_{\infty,1}$ is a divergence-free vector field and if $g \in B^1_{\infty,1}$, then $f.\nabla g \in B^0_{\infty,1}$.*

Proof. This is easily proved by paradifferential calculus. Using the Littlewood–Paley decomposition of f and of g, we write

$$f.\nabla g = S_0 f.\nabla g + (f - S_0 f).\nabla S_0 g + \sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{N},|j-k|\ge 3} \Delta_j f.\nabla \Delta_k g + \sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{N},|j-k|\le 2} \sum_{i=1}^d \partial_i(\Delta_j f_i \Delta_k g) \tag{73}$$

and we easily estimate each of the four terms in the right-hand side of (73) : we use the well-known fact that if $h = \sum_{j=0}^\infty h_j$ where the Fourier transform of h_j is supported in an annulus $a2^j \le |\xi| \le b2^j$ (or a ball if $a = 0$) and if $s \in \mathbb{R}$, then $\|h\|_{B^s_{p,q}}$ is controlled by $C_{a,b,s,p,q}\|2^j s\|h_j\|_p\|_{l^q}$ if $a > 0$ or if $s > 0$ and $a = 0$; $\Delta_j f.\nabla\Delta_k g$ has its Fourier transform supported in an annulus (with radius of order $2^{\max(j,k)}$) if $|k - j| \ge 3$; if $|j - k| \le 2$, we can only say that the Fourier transform is supported in a ball with radius of order 2^j. Thus, we cannot estimate the term $\sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{N},|j-k|\le 2} \Delta_j f.\nabla\Delta_k g$ directly in $B^0_{\infty,1}$ (this is a serious obstruction : as a matter of fact, $B^0_{1,\infty}$ is not an algebra) and we have to use the fact that f is divergence free to rewrite this term as $\mathbf{div}\,(\sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{N},|j-k|\le 2} \Delta_k g \Delta_j f)$ and estimate $\sum_{j\in\mathbf{N}} \sum_{k\in\mathbf{N},|j-k|\le 2} \Delta_k g\Delta_j f$ in $B^1_{\infty,1}$. □

We shall get generalizations of Lemma 3 and Lemma 4 as easy consequences of Lemma 5.

Lemma 6. *Let $u \in B^1_{\infty,1}$ with $\mathbf{div}\, u = 0$. Then the operator $\sum_{i=1}^d [u_i, P_{j,k}\partial_i]$ is bounded on $B^s_{\infty,1}$ for every $s \in [0,1]$ and we have $\|\sum_{i=1}^d [u_i, P_{j,k}\partial_i] f\|_{B^s_{\infty,1}} \le C_{s,\sigma}\|f\|_{B^s_{\infty,1}}\|u\|_{B^1_{\infty,1}}$.*

Proof. We already know that the operator $T_{j,k} = \sum_{i=1}^d [u_i, P_{j,k}\partial_i]$ is bounded on $B^s_{p,q}$ for $0 < s < 1$, $1 \le p \le +\infty$ and $1 \le q \le +\infty$. Since $T^*_{j,k} = -T_{j,k}$, we get by duality that $T_{j,k}$ is bounded on $B^s_{p,q}$ for $-1 < s < 0$, $1 \le p \le +\infty$ and $1 \le q \le +\infty$. By interpolation, it is true as well for $s = 0$.

Thus, $T_{j,k}$ is bounded on $B^s_{\infty,1}$ for $0 \leq s < 1$. We take $f \in B^1_{\infty,1}$ and try to estimate $g = \sum_{i=1}^d [u_i, P_{j,k}\partial_i] f$ in $B^1_{\infty,1}$. We must equivalently estimate $\|g\|_{B^0_{\infty,1}}$ and, for $l = 1, \dots, d$, $\|\partial_l g\|_{B^0_{\infty,1}}$. We just write

$$\partial_l g = \sum_{i=1}^d [u_i, P_{j,k}\partial_i]\partial_l f + \sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f \tag{74}$$

so that we find

$$\lg\|_{B^1_{\infty,1}} \leq C\big(\|T_{j,k}\|_{\mathscr{L}(B^0_{\infty,1}, B^0_{\infty,1})}\|f\|_{B^1_{\infty,1}} + \sum_{l=1}^d \|\sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f\|_{B^0_{\infty,1}}\big). \tag{75}$$

We thus need to estimate $\|\sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f\|_{B^0_{\infty,1}}$. We write

$$\begin{cases} \sum_{i=1}^d [\partial_l u_i, P_{j,k}\partial_i] f = & A + B + C + D \\ A = & \partial_l u . P_{j,k} \nabla S_0 f \\ B = & -\sum_{i=1}^d \partial_i S_0 P_{j,k}(\partial_l u_i f) \\ C = & \partial_l u . \nabla (Id - S_0) P_{j,k} f \\ D = & -\sum_{i=1}^d (Id - S_0) P_{j,k}(\partial_l u . \nabla f) \end{cases} \tag{76}$$

A and B are obviously controlled in $B^0_{\infty,1}$ norm. On the other hand, $(Id - S_0)P_{j,k}$ is bounded on $B^1_{\infty,1}$ and on $B^0_{\infty,1}$, so that Lemma 5 gives the control of C and D. □

Lemma 7. *Let $u \in L^1([0,T], B^1_{\infty,1})$ with* **div** $u = 0$. *Let $f_0 \in B^s_{\infty,1}$ for some $s \in [0,1]$. Then the solution f of the transport equation*

$$\begin{cases} \partial_t f + u.\nabla f = 0 \\ \\ f_{|t=0} = f_0 \end{cases} \tag{77}$$

satisfies $\sup_{0 \leq t \leq T} \|f(t,.)\|_{B^s_{\infty,1}} \leq C_s e^{C_s \int_0^T \|u(t,.)\|_{B^1_{\infty,1}} dt} \|f_0\|_{B^s_{\infty,1}}$

Proof. Let $\tau \mapsto X_{t,x}(\tau)$ be the characteristic curves associated to the vector field u. The solution of (77) is given by $f(t,x) = f_0(X_{t,x}(0))$. We already that, for $0 \leq t \leq T$, the mapping $f_0 \mapsto f_0(X_{t,x}(0))$ is an isomorphism on $B^s_{p,q}$ for $0 < s < 1$, $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. But writing for $f_0 \in B^{-s}_{\infty,1}$ and $g_0 \in B^1_{1,\infty}$

$$\frac{d}{dt}\int f_0(X_{t,x}(0)) g_0(X_{t,x}(0)) dx = \int gu.\nabla f + fu.\nabla g \, dx = 0 \tag{78}$$

we find by a duality argument that the mapping $f_0 \mapsto f_0(X_{t,x}(0))$ is as well an isomorphism on $B^{-s}_{\infty,1}$ for $0 < s < 1$. The case $s = 0$ follows by interpolation.

Now, let us assume that $f_0 \in B^1_{\infty,1} \subset$ **Lip**. We write that its derivatives $(\partial_1 f, \dots, \partial_d f)$ are solutions of the system

$$\text{for } j = 1, \dots, d, \quad \partial_t \partial_j f + u.\nabla \partial_j f = -\partial_j u.\nabla f \tag{79}$$

Thus, , we find that $H(t,x) = \begin{pmatrix} \partial_1 f \\ \vdots \\ \partial_d f \end{pmatrix}$ is solution of the fixed-point problem

$$H(t,x) = H(0,X_{t,x}(0)) + \int_0^t \big((\nabla \otimes u).S_0 H\big)(\tau, X_{t,x}(\tau))\, d\tau + \int_0^t \big((\nabla \otimes u).\nabla(Id - S_0)\frac{1}{\Delta}\mathbf{div}\, H\big)(\tau, X_{t,x}(\tau))\, d\tau \tag{80}$$

This problem has a unique solution in $L^\infty((0,T),(B^0_{\infty,1})^d)$ and we finally get that $f \in L^\infty_t B^1_{\infty,1}$. We then control the size of $\|f\|_{B^1_{\infty,1}}$ through the Gronwall lemma. □

Owing to Lemmas 6 and 7, we get easily the following theorem of [22] :

Theorem 10. *Let $v_0 \in B^1_{\infty,1}$ be a divergence free vector field. Then there exists a positive T such that the Cauchy problem*

$$\begin{cases} \partial_t v + v.\nabla v = \sum_{i=1}^d [v_i, \mathbb{P}\partial_i] v \\ v_{|t=0} = v_0 \\ \nabla.v = 0 \end{cases} \tag{81}$$

has a unique solution $v \in \mathscr{C}([0,T], B^0_{\infty,1})$ such that $\sup_{0\le t\le T} \|v\|_{B^1_{\infty,1}} < +\infty$

Proof. We can follow the same lines as for Theorem 1 (or Theorem 2). Now, the only thing we have to check is the convergence of f_n to v. Recall the identity satisfied by $k_n = f_{n+1} - f_n$:

$$\begin{cases} k_{n+1} = \int_0^t G_n(\tau, X^{(n+1)}_{t,x}(\tau))\, d\tau \\ G_n(t,x) = -k_n.\nabla f_{n+1} + \sum_{i=1}^d [f_{n+1,i}, \mathbb{P}\partial_i] k_{n+1} + \sum_{i=1}^d [k_{n,i}, \mathbb{P}\partial_i] f_{n+1} \end{cases} \tag{82}$$

We see that we have to control the term $\| \sum_{i=1}^d [k_{n,i}, \mathbb{P}\partial_i] f_{n+1}\|_{B^0_{\infty,1}}$ by $\|k_n\|_{B^0_{\infty,1}} \|f_{n+1}\|_{B^1_{\infty,1}}$. We have no problem for $|\sum_{i=1}^d k_{n,i}\mathbb{P}\partial_i S_0 f_{n+1}$ nor for $\sum_{i=1}^d S_0 \mathbb{P}\partial_i (k_{n,i} f_{n+1})$. Lemma 5 gives an easy control for $k_n.\nabla(Id - S_0)\mathbb{P} f_{n+1}$ as well as for $(Id - S_0)\mathbb{P}(k_n.\nabla f_{n+1})$. □

The case of the MHD equations is similar to the Euler equations :

Theorem 11. *Let $v_0 \in B^1_{\infty,1}$ and $b_0 \in B^1_{\infty,1}$ be two divergence–free vector fields. Then there exists a positive T such that the Cauchy problem*

$$\begin{cases} \partial_t v + \sum_{i=1^d} \mathbb{P}\partial_i (v_i v - b_i b) = 0 \\ \partial_t b + \sum_{i=1}^d \mathbb{P}\partial_i (v_i b - b_i v) = 0 \\ v_{|t=0} = v_0, \quad b_{|t=0} = b_0 \\ \mathbf{div}\, v = 0, \quad \mathbf{div}\, b = 0 \end{cases} \tag{83}$$

has a unique solution (v,b) in $\mathscr{C}([0,T], B^0_{\infty,1})$ such that $\sup_{0\le t\le T} \|v\|_{B^1_{\infty,1}} + \|b\|_{B^1_{\infty,1}} < +\infty$.

We cannot hope to solve the quasi–geostrophic equation in the critical space, since it is not stable under the Riesz transforms. But we may just add a slight further requirement to get a solution :

Theorem 12. *Let* $\theta_0 \in B^1_{\infty,1} \cap L^p$ *with* $1 < p < +\infty$. *Then there exists a positive* T *such that the Cauchy problem*

$$\begin{cases} \partial_t \theta + u.\nabla\theta = 0 \\ u = (-R_2\theta, R_1\theta) \\ \theta(0,.) = \theta_0 \end{cases} \tag{84}$$

has a unique solution θ *in* $\mathscr{C}([0,T], B^0_{\infty,1})$ *such that* $\sup_{0\le t\le T} \|\theta\|_{B^1_{\infty,1}} + \|\theta\|_p < +\infty$.

11. Relaxing unnecessary hypotheses.

As a matter of fact, the spaces A^s $(0 < s < 1)$ considered in Theorems 2, 3, 6 and 7 were stable under more general singular integral operators : they satisfy more precisely the following hypothesis

◇ **Hypothesis (H10) : singular integrals**

Let T be a bounded linear operator from $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$ (with distribution kernel $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$) which satisfies the following conditions

• T is bounded on L^2 : $\|T(f)\|_2 \le C_0\|f\|_2$

• outside from the diagonal $x = y$, K is a continuous function such that $|K(x,y)| \le C_0 \frac{1}{|x-y|^d(1+|x-y|)}$

• outside from the diagonal, K satisfies $|\nabla_x K(x,y)| \le C_0|x-y|^{-d-1}$ and $|\nabla_y K(x,y)| \le C_0|x-y|^{-d-1}$

• $T(1) = 0$ in BMO

Then, T is bounded from A^s to A^s for all $0 < s < 1$ and $\|T\|_{\mathscr{L}(A^s,A^s)} \le C_s C_0$

For $A^s = B^s_{p,q}$, see [15]. For $A^s = F^s_{p,q}$, see [9]. For $A^s = (Id-\Delta)^{-s/2}E$ with $E = L^{p,q}$, $E = \dot{M}^{p,q}$ or $E = \dot{M}^{p,q,r}$, we shall use a variant of Theorem 5 (see Lemma 8 below). For $A^s = B^s_{E,t}$ with $E = L^{p,q}$, $E = \dot{M}^{p,q}$ or $E = \dot{M}^{p,q,r}$, this is a consequence of the case of $(Id-\Delta)^{-s/2}E$ (by interpolation).

Lemma 8. *Let* T *be a bounded linear operator from* $\mathscr{D}(\mathbb{R}^d)$ *to* $\mathscr{D}'(\mathbb{R}^d)$ *(with distribution kernel* $K(x,y) \in \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$*) which satisfies the following conditions*

• T *is bounded on* L^2 *:* $\|T(f)\|_2 \le C_0\|f\|_2$

• *outside from the diagonal* $x = y$, K *is a continuous function such that* $|K(x,y)| \le C_0 \frac{1}{|x-y|^d}$

• *outside from the diagonal,* K *satisfies* $|\nabla_x K(x,y)| \le C_0|x-y|^{-d-1}$ *and* $|\nabla_y K(x,y)| \le C_0|x-y|^{-d-1}$

• $T(1) = 0$ *in* BMO

Then, for $0 < \alpha < 1$, *the operator* $(-\Delta)^{\alpha/2} \circ T \circ (-\Delta)^{-\alpha/2}$ *belongs to* $\mathscr{A}^{1-\alpha}$.

Proof. Let $T_\alpha = (-\Delta)^{\alpha/2} \circ T \circ (-\Delta)^{-\alpha/2}$. We know from [14] that T_α is bounded on L^2. The problem is to estimate its kernel. This could be done through a molecular approach : if $(\psi_{\varepsilon,j,k})_{1\leq\varepsilon\leq 2^d-1, j\in\mathbb{Z}, k\in\mathbb{Z}^d}$ is an Hilbertian wavelet basis of L^2, then the kernel of T_α is given in $\mathscr{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ by

$$K_\alpha(x,y) = \sum_{\varepsilon=1}^{2^d-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} T_\alpha(\psi_{\varepsilon,j,k})(x)\psi_{\varepsilon,j,k}(y) \tag{86}$$

However, we will prove Lemma 8 by using Theorem 5. We have $b = T^*(1) \in BMO$. Using the homogeneous Littlewood–Paley decomposition, we introduce the operator $\pi_b : f \mapsto \sum_{j\in\mathbb{Z}} S_{j-2}(f\Delta_j b)$. π_b is a Calderón–Zygmund operator such that $\pi_b(1) = 0$ and $\pi_b^*(1) = b$. Thus, we may write $T = \pi_b + S$ with $S(1) = S^*(1) = 0$. We know, by Theorem 5, that $(-\Delta)^{\alpha/2} \circ S \circ (-\Delta)^{-\alpha/2}$ belongs to $\mathscr{A}^{1-\alpha}$. We must estimate the kernel L_α of $(-\Delta)^{\alpha/2} \circ \pi_b \circ (-\Delta)^{-\alpha/2}$. If S_j is the convolution operator with $\mathscr{F}^{-1}\varphi(2^{-j}\xi)$, Δ_j the convolution operator with $\mathscr{F}^{-1}(\psi(2^{-j}\xi))$, and if $\omega = \mathscr{F}^{-1}(|\xi|^\alpha\varphi)$ and $\Omega = \mathscr{F}^{-1}(|\xi|^{-\alpha}\sum_{k=-3}^{3}\psi(2^k\xi))$, then we have

$$L_\alpha(x,y) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} 2^{jd}\omega(2^j(x-z))\Delta_j b(z) 2^{jd}\Omega(2^j(z-y))\, dz. \tag{87}$$

It is then a classical computation to estimate the size and the regularity of L_α. □

12. Maximal solutions.

Due to uniqueness of solutions in Theorem 1, we may define $T_\sigma(v_0)$ the maximal existence time for a solution in $A^{1+\sigma}$:

$$T_\sigma(v_0) = \sup\{T > 0 \;/\; \exists v \in (L^\infty((0,T), A^{1+\sigma}))^d \text{ solution of (8)}\}. \tag{88}$$

If we have $v_0 \in (A^{1+\sigma})^d$ (under the hypotheses of Theorem 1), then we have

$$T_\sigma(v_0) < +\infty \Rightarrow \sup_{0<t<T_\sigma(v_0)} \|v\|_{A^{1+\sigma}} = +\infty \tag{90}$$

Under very slight further assumptions, it is easy to check that $T_\sigma(v_0)$ does not actually depend on σ.

Theorem 13. *Let A^s be a scale of spaces satisfying hypotheses (H1) to (H8). Assume that there exists a Banach space E and a $\sigma_0 > 0$ such that, for all $\sigma > \sigma_0$, σ satisfies hypothesis (H9) and the following hypothesis :*
⋄ **Hypothesis (H11) :** *$A^\sigma \subset E$ and*

$$\|fg\|_{A^\sigma} \leq C_\sigma(\|f\|_E\|g|_{A^\sigma} + \|g\|_E\|g\|_{A^\sigma}) \tag{91}$$

Then, for $\sigma_0 < \sigma < \tau$ and $v_0 \in A^{1+\tau}$, we have $T_\sigma(v_0) = T_\tau(v_0)$.

Proof. By induction on τ. We prove that if it is true for $\tau = \sigma + k$ (for some $k \in \mathbb{N}$), then it is true for $\sigma + k < \tau \leq \sigma + k + 1$. We estimate $\|v\|_{A^{1+\tau}}$ as $\|v\|_{A^\tau} + \sum_{i=1}^d \|\partial_i v\|_{A^\tau}$. We write

$$\partial_t \partial_i v + v.\nabla.\partial_i v = \sum_{j=1}^d [v_j, \mathbb{P}\partial_j]\partial_i v - S_0 \mathbb{P}\mathrm{div}\,(\partial_i v \otimes v) - \mathbb{P}(Id - S_0)(\partial_i v.\nabla v) \tag{92}$$

We then get

$$\begin{aligned} \|\partial_i v(t,.)\|_{A^\tau} \leq \quad & Ce^{D\int_0^\tau \|v(s,.)\|_{A^{\sigma+k}}\, ds}(\|\partial_i v_0\|_{A^\tau} \\ & + \int_0^t \|\partial_i v(s,.) \otimes v(s,.)\|_{A^\tau} + \|\partial_i v(s,.).\nabla v(s,.)\|_{A^\tau}\, ds) \end{aligned} \tag{93}$$

and finally

$$\begin{aligned} & \|v(t,.)\|_{A^{1+\tau}} \leq \\ & Ce^{D\int_0^\tau \|v(s,.)\|_{A^{\sigma+k}}\, ds}(\|v_0\|_{A^{1+\tau}} + \int_0^t \|v(s,.)\|_E \|v(s,.)\|_{A^{1+\tau}}\, ds) \end{aligned} \tag{94}$$

and we conclude with Gronwall's lemma. □

13. Conclusion.

Except for Lemma 5, we made no use of the paradifferential calculus. Of course, our tools are deeply related to the paradifferential calculus. However, we avoid the rigidity of the Littlewood–Paley decomposition and in a way replaced it by a molecular approach. Indeed, a Littlewood-Paley decomposition is stable neither through a transport equation nor under the action of a singular integral operator. On the other hand, a molecular decompostion will be stable, since a molecuke is preserved under a transport equation (moving the center along the characteristic curve and deforming the profile of the molecule, but without altering too much its scale), or through the action of a singular integral operator (with roughly speaking the same center and the same scale, but with a deformation of the profile). Similarly, a wavelet decomposition is not preserved, but transformed into a vaguelette decomposition [16]. In a way, it means that the equations we have studied in this paper could be numerically approximated by the method of travelling wavelets proposed in [2].

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