Transport Equations with Low Regularity Vector Fields

by

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MA4K9 Dissertation

Submitted to The University of Warwick

Mathematics Institute

April, 2017

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Acknowledgements

I owe my most sincere gratitude and appreciation to my supervisor, Dr. Xue-Mei Li for her guidance, patience and encouragement throughout the development of this dissertation.

1 Introduction

In this dissertation we would like to give a brief excursus on the problem of the existence and uniqueness for a solution of a deterministic transport equation drifted by a time-dependent vector field. We will focus especially on showing how different regularity conditions on the drift field $\mathbf{b}$ require diverse concepts of solutions.
The literature about this argument is really wide and complex. Remarkable results have been obtained for example by R.J. Di Perna and P.L. Lions in [1] where a Sobolev regularity on the field is considered, or by L. Ambrosio in [9] where the previous DiPerna-Lions theory is extended to a bounded variation regularity. It seems however that all this papers assume at some point in their treatise a prerequisite knowledge on the smooth or quasi-smooth case. On the other side, we have not found any paper considering these more classical settings in an explicit way.
The main purpose of this dissertation has been hence to try to fill the gap between the easiest case of a transport equation with a constant field, sometimes considered as an introductive example in general PDE theory book (for example in [3]) and the more advanced topics studied in the research papers.

Given a vector field $\mathbf{b}\colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and a function $u_0\colon \mathbb{R}^d \to \mathbb{R}$, we are therefore interested in the well-posedness of a differential equation of the form:

$$\begin{cases} \partial_t U + \mathbf{b} \cdot \nabla U = 0 & \text{on } (0,+\infty) \times \mathbb{R}^d \\ U = u_0 & \text{on } \{0\} \times \mathbb{R}^d. \end{cases} \tag{1}$$

2 Constant Drift Field

We begin presenting in this section a transport equation drifted by a constant vector field. The simple setting will allow us to show clearly how to solve the equation and also to exhibit an explicit formula for its solutions.
Despite its simplicity, this example will give us a first view on the method of work that will be used also later, in more general frameworks.

Defined the context, our first step is to state clearly what it means for a function to solve equation (1).

Definition 1. Let $\mathbf{b}\colon (0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ be a vector field and $u_0\colon \mathbb{R}^d \to \mathbb{R}$ a function. A (classical) **solution** of the transport equation (1) is a function $u\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ such that

- u is in $C^1((0,\infty)\times\mathbb{R}^d)$;
- $\partial_t u(t,x) + b\cdot\nabla u(t,x) = 0$ for every point (t,x) in $(0,\infty)\times\mathbb{R}^d$;
- $u(0,x) = u_0(x)$ for every x in $\mathbb{R}^d$.

Next result shows an important property of the solution of the equation and explains, even if only in this simple case, why $\mathbf{b}$ is called the drift field of the equation.

Lemma 2. *Let $b\in\mathbb{R}^d$ be a vector, (t_0,x_0) a point in $[0,\infty)\times\mathbb{R}^d$ and $u\colon [0,+\infty)\times\mathbb{R}^d\to\mathbb{R}$ a solution of the transport equation* (1)*. Then, the curve $z\colon [-t_0,\infty)\to\mathbb{R}$ defined by*

$$z(s) := u(s+t_0, x_0+sb)$$

is constant.

Proof. To prove this, we have just to show that the derivative of z is equal to 0. Indeed,

$$z'(s) \;=\; b\cdot\nabla u(s+t_0,x_0+sb) + \partial_t u(s+t_0,x_0+sb) \;=\; \big[b\cdot\nabla u + \partial_t u\big](s+t_0,x_0+sb) \;=0$$

for every s in $(-t_0,\infty)$. Hence, z is a constant on $(-t,\infty)$.
To extend the result to $-t_0$, just use the fact that z is continuous there. □

Roughly speaking, the previous result shows that a function u, to be a solution, has to be constant on every line (s, x_0+sb).
In reality this should not surprise us. Indeed, it is possible to rewrite the transport equation (1) in this equivalent form:

$$(1,b)\cdot(\partial_t,\partial_1,\ldots,\partial_d)U \;=\; 0$$

from which it is clear that a particular directional derivative of a solution vanishes, the one along the vector $(1,b)$ on $\mathbb{R}^{d+1}$.

Thanks to the last lemma, we can now find an explicit formula for a solution of problem (1) with a constant field.

Theorem 3. *Let u_0 be a function in $C^1(\mathbb{R}^d)$ and b a vector in $\mathbb{R}^d$. Then, the function $u\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ defined by*

$$u(t,x) := u_0(x - tb)$$

is the unique solution of the transport equation (1) *with initial value u_0.*

Proof. Existence). It is necessary just to calculate $u_0(x-tb)$ inside the equation since the initial value is trivially satisfied. Then,

$$\partial_t \big[u_0(x-tb)\big] + b \cdot \nabla\big[u_0(x-tb)\big] \;=\; -b \cdot \nabla u_0(x-tb) + b \cdot \nabla u_0(x-tb) \;=\; 0.$$

Uniqueness). We firstly assume that another solution $v \in C^1([0,+\infty) \times \mathbb{R}^d)$ of the same problem exists. Then, fixed a point (t,x) in $(0,+\infty) \times \mathbb{R}^d$, we can consider the function $z(s) = v(s+t, x+sb)$. From lemma 2, we already know that $z(s)$ is a constant on $[-t,+\infty)$ and thus, that

$$v(t,x) \;=\; z(0) \;=\; z(-t) \;=\; v(0, x-tb) \;=\; u_0(x-tb) \;=\; u(t,x)$$

for every point (t,x) in $(0,+\infty) \times \mathbb{R}^d$. Moreover, $v(0,x) = u_0(x) = u(0,x)$ on $\mathbb{R}^d$ and hence, $v = u$. □

Remark. In conclusion, we want to summarize the reasonings that have allowed us to show the existence and uniqueness of a solution in this particular setting.
We have firstly discovered that for every point (t,x) there is a line $(s+t, x+sb)$ over which a solution u of the transport equation (1) has to be constant. Since this line passes to the point $x-tb$ at time 0, the value of u in (t,x) has to coincide with that in $(0, x-tb)$. Hence, a solution is uniquely determined by its values at time 0.
Since on the other side, the function u coincides also with the given u_0 at the time 0 by definition, it follows that the only possible candidate to be solution was the function $u(t,x) = u_0(x-tb)$. To finish, we have just showed that the candidate really solves the problem, by direct calculation inside the equation.

3 Time-Dependent Drift Field: Smooth Regularity

We switch now to a more general setting considering a transport equation (1) with a smooth time-dependent vector field. As we will see, this case will exhibit a lot of similarities with the previous constant one. In fact, the method of reasoning to find a solution and to prove its uniqueness will follow, even if with the slightly additional difficulty of a variable drift field, the way of proving we have done before.

We start recalling a classical result in the ODE theory that can be found in almost every book on the argument.
Under the following assumptions on the drift field
Assumption 1. $\mathbf{b}$ is a continuous vector field from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}^d$ such that

- the function $x\mapsto \mathbf{b}(t,x)$ is in $C^1(\mathbb{R}^d)$ for every t in $[0,\infty)$;
- there exists a continuous function $l\colon[0,\infty)\to[0,\infty)$ such that

$$|\mathbf{b}(t,x)|\le l(t)\big(1+|x|\big)\quad \text{on } [0,\infty)\times\mathbb{R}^d;$$

it is well known that the Cauchy problem

$$\begin{cases} X'(t)=\mathbf{b}(t,X(t)) & \text{on } (0,+\infty)\\ X(0)=x \end{cases}$$

has a unique C^1-flow $\mathbf{\Phi}\colon[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ of solutions such that the function $x\mapsto\mathbf{\Phi}(t,x)$ is a C^1-diffeomorphism for every fixed t in $[0,\infty)$.

We want now to find an explicit formula for a solution in this context. To do so, we will emulate the way of reasoning used in the constant field case and summarized at the end of the last section.
Lemma 4. *Let u be a solution of the transport equation* (1) *and* (t_0,x_0) *a point in* $[0,\infty)\times\mathbb{R}^d$*. Under assumption 1, the curve $z\colon[-t_0,\infty)\to\mathbb{R}$ defined by*

$$z(s):=u(t_0+s,\mathbf{\Phi}(s+t_0,x_0))$$

is constant.

Proof. Analogously to the constant vector field case, we prove the result showing that the derivative of z is equal to 0. Indeed,

$$\begin{aligned} z'(s) &= \partial_t u\big(t_0+s,\mathbf{\Phi}(s+t_0,x_0)\big)+\mathbf{\Phi}'(s+t_0,x_0)\cdot\nabla u\big(s+t_0,\mathbf{\Phi}(s+t_0,x_0)\big) = \\ &= \Big[\partial_t u+\mathbf{b}\cdot\nabla u\Big]\big(s+t_0,\mathbf{\Phi}(s+t_0,x)\big) = 0. \end{aligned}$$

□

It is not difficult now to show the existence and the uniqueness of a solution in a similar way to how we have done in the proof of theorem 3. This is exactly what the next result shows.

Theorem 5 (Existence and Uniqueness of Solution). *Let u_0 be a function in $C^1(\mathbb{R}^d)$. Under assumption 1, the function $u\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ defined by*

$$u(t,y) := u_0(\mathbf{\Phi}^{-1}(t,y))$$

is the unique solution of the transport equation (1) *with initial value u_0. Furthermore, if u_0 is compactly supported, then $x \mapsto u(t,x)$ is in $C^1_c(\mathbb{R}^d)$ for every t in $[0,\infty)$.*

Proof. Existence). First of all, notice that u is in $C^1([0,\infty)\times\mathbb{R}^d)$ since it is a composition of C^1-functions. Then, we can consider $u(t,\mathbf{\Phi}(t,x)) = u_0(x)$ since $\mathbf{\Phi}$ is a diffeomorphism, fixed t. Hence,

$$\begin{aligned}0 = \partial_t u_0(x) \;&=\; \partial_t\big[u(t,\mathbf{\Phi}(t,x))\big] \;= \\ &=\; \partial_t u(t,\mathbf{\Phi}(t,x)) + \nabla u(t,\mathbf{\Phi}(t,x))\cdot\partial_t\mathbf{\Phi}(t,x) \;=\; \big(\partial_t u + \nabla u\cdot\mathbf{b}\big)(t,\mathbf{\Phi}(t,x))\end{aligned}$$

where in the last step we used $\partial_t\mathbf{\Phi}(t,x) = \mathbf{b}(t,\mathbf{\Phi}(t,x))$. Since for any $(t,y)\in(0,+\infty)\times\mathbb{R}^d$, there exists $x\in\mathbb{R}^d$ such that $\mathbf{\Phi}(t,x) = y$, we have proven that u is a solution.
Uniqueness). Let v be another solution of the transport equation (1) with initial value u_0. Analogously to the constant vector field case, we fix a point (t,x) in $[0,\infty)\times\mathbb{R}^d$ and consider the function $z(s) := v\big(t+s,\mathbf{\Phi}(s+t,x)\big)$. From the previous lemma, we already know that the function z is constant. Moreover, since $\mathbf{\Phi}$ is a diffeomorphism for every fixed t, there exists a point x in $\mathbb{R}^d$ such that $\mathbf{\Phi}(t,x) = y$. Hence,

$$\begin{aligned}v(t,y) \;=\; v\big(t,\mathbf{\Phi}(t,x)\big) \;=\; z(0) \;=\; z(-t) \;=\; v\big(0,\mathbf{\Phi}(0,x)\big) \;=\; u_0(x) \;= \\ =\; u_0\big(\mathbf{\Phi}^{-1}(t,y)\big) \;=\; u(t,y).\end{aligned}$$

Compact support). To show that $u(t,\cdot)$ is compactly supported, we just to prove that $\operatorname{supp}(u(t,\cdot))$ is bounded, since u is a continuous function.
Firstly, notice from the definition of the flow $\mathbf{\Phi}$

$$\mathbf{\Phi}(t,x) \;=\; x + \int_0^t \mathbf{b}(s,\mathbf{\Phi}(s,x))\,ds,$$

that it is bounded by

$$|\mathbf{\Phi}(t,x) - x| \;\le\; \int_0^t \big|\mathbf{b}\big(s,\mathbf{\Phi}(s,x)\big)\big|\,ds \le t\,\|b(s,\mathbf{\Phi}(s,x))\|_{L^\infty([0,t])} \quad (*).$$

Then, using $(*)$, we obtain that

$$\begin{aligned}\operatorname{supp}\big(u(t,\cdot)\big) \;&=\; \{y\in\mathbb{R}^d\colon u(t,y)\neq 0\} \;=\; \{y\in\mathbb{R}^d\colon u_0(\mathbf{\Phi}^{-1}(t,y))\neq 0\} \;= \\ &=\; \{y\in\mathbb{R}^d\colon \mathbf{\Phi}^{-1}(t,y)\in\operatorname{supp}(u_0)\} \;\subseteq\; \operatorname{supp}(u_0) + B(0,tC(t))\end{aligned}$$

where $C(t) := \|b(s, \phi(s,x))\|$ on $L^\infty([0,t])$. Hence, $\text{supp}(u(t,\cdot))$ is bounded for every fixed t in $[0,\infty)$. □

4 Time-Dependent Drift Field: Quasi-smooth Regularity

Here we start discussing the problem in a non-smooth case. This section indeed will focus on the analysis of a transport equation drifted by a vector field which is differentiable in x but only integrable over time. To avoid all the problems at infinity, we will limit our study to a finite time horizon.
For this purpose, let us assume from this point further a fixed time $T > 0$.

Let $C_b^1(\mathbb{R}^d)$ denote the space of all the C^1-function $f\colon \mathbb{R}^d \to \mathbb{R}$ such that f and all its partial derivatives are bounded, equipped with the norm

$$\|f\|_{C_b^1} := \|f\|_\infty + \sum_{j=1}^{d} \|\partial_j f\|_\infty$$

where $\|f\|_\infty := \sup |f(x)|$ is the standard supremum norm.
In particular, $\left(C_b^1, \|\cdot\|_{C_b^1}\right)$ is a Banach space and hence it makes sense to consider the Bochner space $L^1(0,T;C_b^1(\mathbb{R}^d))$. For a rigorous definition of this space and a view of the related results we will use later, we suggest the reader to go at the end of the paper and see the relative Appendix.

As we have seen in the previous section, the transport equation and the associated Cauchy problem are strictly linked together and it is possible a priori to find an explicit formula for solutions of the former if an existence result on the latter is already known. For this reason, we firstly consider a Cauchy problem with the same characteristics:

$$\begin{cases} X'(t) = \mathbf{b}(t, X(t)) & \text{on } (0,T); \\ X(t_0) = x_0; \end{cases} \tag{2}$$

where (t_0, x_0) is a point in $[0,T] \times \mathbb{R}^d$ and $\mathbf{b}$ is a function on $[0,T] \times \mathbb{R}^d$ not necessarily continuous.

The problem now is that we can not hope to find a classic solution in this setting, since the continuity of the field $\mathbf{b}$ guaranteed a solution to be continuously differentiable.
On the other hand, we remember that in the continuous case the Cauchy problem (2) was equivalent to the integral equation

$$X(t) = x_0 + \int_{t_0}^{t} \mathbf{b}\big(s, X(s)\big)\, ds \quad \text{on } [0,T]. \tag{3}$$

But the above integral expression is defined for many other functions $\mathbf{b}$ that are not

necessarily continuous. Indeed, it required only a local integrability for $\mathbf{b}\big(s, X(s)\big)$ to make sense.
We could then define as solution of the Cauchy problem (2) a function satisfying the integral equation (3), even when the drift field is not continuous. We remark, however, that in this more general setting, a solution doesn't need to be continuously differentiable anymore and, in reality, neither to be differentiable in every point of the interval $(0, T)$.

This reasoning gives us the idea to extend the definition of a solution in the following way:
Definition 6. Let (t_0, x_0) be a point in $[0, T] \times \mathbb{R}^d$ and $\mathbf{b}$ a function on $(0, T) \times \mathbb{R}^d$.
An **extended solution** of the Cauchy problem (2) is a function $x\colon [0, T] \to \mathbb{R}^d$ such that

- x is absolutely continuous on $[0, T]$;
- $x(t_0) = x_0$;
- $x'(t) = \mathbf{b}(t, x(t))$ a.e. on $(0, T)$.

Remark. Notice that the definition is well-posed since an absolute continuous function on $[0, T]$ is almost everywhere differentiable, thanks to the fundamental theorem of Lebesgue calculus.

In a similar way, we can also extend the definition of a solution for a transport equation:
Definition 7. Let $\mathbf{b}\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and $u_0\colon \mathbb{R}^d \to \mathbb{R}$ a function. An **extended solution** of the transport equation (1) with initial value u_0 is a function $u\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that

- the function $x \to u(t, x)$ is in $C^1(\mathbb{R}^d)$ for every fixed t in $[0, T]$;
- the function $t \to u(t, x)$ is absolutely continuous on $[0, T]$ for every fixed x in $\mathbb{R}^d$;
- $u(0, x) = u_0(x)$ on $\mathbb{R}^d$;
- $\partial_t u + \mathbf{b} \cdot \nabla u = 0$ on $\mathbb{R}^d$ and a.e. on $[0, T]$.

Notation. Given a functional normed space X over $\mathbb{R}^d$, we will say that a vector field $\mathbf{b}\colon \mathbb{R}^d \to \mathbb{R}$ is in X if $\mathbf{b}^i$ is in X for every $i = 1, \dots, d$.
In this case, we define the norm of $\mathbf{b}$ in X as

$$\|\mathbf{b}\| := \Big(\sum_{i=1}^{d} \|\mathbf{b}^i\|^2\Big)^{1/2}.$$

Defined a solution for the transport equation that is suitable in this context, we turn now on investigate which conditions allows us to solve it.
Our key result will be the existence and uniqueness of a extended solution of the transport equation (1) under the following conditions:
Assumption 2.
- u_0 is a function in $C^1_b(\mathbb{R}^d)$;
- $\mathbf{b}$ is a vector field in $L^1(0, T; C^1_b(\mathbb{R}^d))$.

As already done before, we firstly analyze the associated Cauchy problem (2) showing the existence of a flow of solutions with suitable properties. After that, we will use the flow to construct an explicit formula for the extended solutions, proving so the existence and uniqueness result.

Theorem 8. *Let* (t_0, x_0) *be a point in* $[0,T]\times\mathbb{R}^d$ *and* $\mathbf{b}$ *a vector field in* $L^1(0,T;C^1_b(\mathbb{R}^d))$. *Then there exists a unique extended solution of the Cauchy problem* (2) *on* $[0,T]$.

Proof. Firstly, notice that finding an extended solution of the Cauchy problem (2) is equivalent to solving the following integral equation:

$$X(t) := x_0 + \int_{t_0}^{t} \mathbf{b}\big(s, X(s)\big)\, ds \quad \text{ on } [0,T].$$

Then, denote for simplicity

$$L(t) := \int_{t_0}^{t} \|\mathbf{b}(s)\|_{C^1_b}\, ds$$

for every t in $[0,T]$ and notice that $L'(t) = \|\mathbf{b}(t)\|_{C^1_b}$ a.e. on $[0,T]$.
Consider now the space $C([0,T])$ of all the continuous function $x\colon [0,T]\to\mathbb{R}^d$ with the following norm

$$\|x\|_0 := \sup_{[0,T]}\big|e^{-2L(t)}x(t)\big|$$

and notice that this is equivalent to the standard supremum norm on $C([0,T])$, since

$$e^{-2L(T)}\|x\|_\infty \le \|x\|_0 \le \|x\|_\infty.$$

Hence, $\Big(C([0,T]), \|\cdot\|_0\Big)$ is a Banach Space.
Furthermore, define an operator $F\colon \Big(C([0,T]), \|\cdot\|_0\Big) \to \Big(C([0,T]), \|\cdot\|_0\Big)$ such that

$$F\big(x\big)(t) := x_0 + \int_{t_0}^{t} \mathbf{b}\big(s, x(s)\big)\, ds \qquad \forall\, x \in C([0,T])$$

To finish, we need to show that T has a fixed point. To do that, we want to apply the Banach fixed point theorem on $\Big(C([0,T]), \|\cdot\|_0\Big)$ and hence, we need F to be a contraction. Indeed, for every $t \ge t_0$ (the case $t < t_0$ is similar),

$$\begin{aligned}
\Big|e^{-2L(t)}\Big(F\big(x\big)(t) - F\big(y\big)(t)\Big)\Big| &\le e^{-2L(t)}\int_{t_0}^{t}\big|\mathbf{b}\big(s,x(s)\big) - \mathbf{b}\big(s,y(s)\big)\big|\, ds \le \\
&\le e^{-2L(t)}\int_{t_0}^{t}\|\mathbf{b}(s)\|_{C^1_b} e^{2L(s)}e^{-2L(s)}\big|x(s)-y(s)\big|\, ds \le \\
&\le e^{-2L(t)}\|x-y\|_0\int_{t_0}^{t}\|\mathbf{b}(s)\|_{C^1_b} e^{2L(s)}\, ds \le e^{-2L(t)}\frac{1}{2}\Big(e^{2L(t)}-1\Big)\|x-y\|_0 = \\
&= \frac{1}{2}\Big(1-e^{-2L(t)}\Big)\|x-y\|_0.
\end{aligned}$$

Since $\sup_{[0,T]} \frac{1}{2}\Big(1-\exp\{-2L(t)\}\Big) < 1$, we can finally apply the Banach fixed point theorem that guarantees the existence of a function x in $C([0,T])$ such that $F(x) = x$, i.e.

$$x(t) := x_0 + \int_{t_0}^{t} \mathbf{b}\big(s, x(s)\big)\, ds$$

and we can conclude. □

Lemma 9 (Gronwall's Lemma for a.e. inequalities)**.** *Let $R\colon [0,T] \to \mathbb{R}$ an absolutely continuous function and L,f integrable function on $[0,T]$.*
If $R'(t) \le L(t)R(t) + f(t)$ a.e. on $[0,T]$, then

$$R(t) \;\le\; e^{\int_{t_0}^{t} L(s)\,ds} R(t_0) + \int_{t_0}^{t} e^{\int_{s}^{t} L(p)\,dp} f(s)\,ds \quad \forall\, t \in [0,T].$$

Proof. We start noticing that since R is absolutely continuous on $[0,T]$, it is almost everywhere differentiable there thanks to the fundamental theorem of Lebesgue calculus. Moreover, for almost every t in $[0,T]$,

$$\begin{aligned}\partial_t\big(R(t)e^{-\int_{t_0}^{t} L(s)\,ds}\big) \;&=\; R'(t)e^{-\int_{t_0}^{t} L(s)\,ds} - R(t)e^{-\int_{t_0}^{t} L(s)\,ds} L(t) \;= \\ &=\; e^{-\int_{t_0}^{t} L(s)\,ds}\big(R'(t) - L(t)R(t)\big) \;\le\; f(t)e^{-\int_{t_0}^{t} L(s)\,ds}.\end{aligned}$$

and, integrating both sides of the equation, we find that

$$R(t)e^{-\int_{t_0}^{t} L(s)\,ds} - R(t_0) \;\le\; \int_{t_0}^{t} f(s)\, e^{-\int_{t_0}^{s} L(p)\,dp}\, ds.$$

Finally, adding $R(t_0)$ and dividing by $\exp\Big\{-\int_{t_0}^{t} L(s)\,ds\Big\}$ both sides, we conclude that

$$R(t) \le\; R(t_0) + e^{\int_{t_0}^{t} L(s)\,ds} \int_{t_0}^{t} f(t)e^{-\int_{t_0}^{s} L(p)\,dp}\, ds \;=\; R(t_0) + \int_{t_0}^{t} f(t)e^{\int_{t_0}^{t} L(p)\,dp - \int_{t_0}^{s} L(p)\,dp}\, ds$$

for almost every t in $[0,T]$. To extend the result to every t, just notice that both sides of the equation are formed by continuous functions on $[0,T]$. □

Corollary 10. *Let $\mathbf{b}$ be a vector field in $L^1(0,T; C_b^1(\mathbb{R}^d))$ and $\mathbf{\Phi}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ the function that associates to every point (t,x) the extended solution at time t, of the Cauchy problem (2) with initial point (t_0, x). Then,*

1. *the function $x \to \mathbf{\Phi}(t,x)$ is Lipschitz continuous for every t in $[0,T]$ with*

$$Lip\big(\mathbf{\Phi}(t,\cdot)\big) \;\le\; \exp\Big\{\int_{t_0}^{t} \|\mathbf{b}(s)\|_{C_b^1}\, ds\Big\};$$

2. *the function $x \mapsto \mathbf{\Phi}(t,x)$ is a C^1-diffeomorphism for every fixed t in $[0,T]$.*

Proof. 1). We start fixing two points x, y in $\mathbb{R}^d$ and noticing that $|\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)|^2$ is absolutely continuous and hence differentiable a.e. on $[0,T]$. Hence, we can write for almost every t, that

$$\partial_t|\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)|^2 = 2\big(\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big) \cdot \partial_t\big(\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big) =$$
$$2\big(\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big) \cdot \big(\mathbf{b}(t,\mathbf{\Phi}(t,x)) - \mathbf{b}(t,\mathbf{\Phi}(t,y))\big) \leq 2\|\mathbf{b}(t)\|_{C^1_b}\big|\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big|^2$$

where in the last inequality we have used the Lagrange theorem on $\mathbb{R}^d$.
Then, applying Gronwell's lemma 9 to the previous inequality, we find that

$$\big|\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big|^2 \leq \exp\Big\{2\int_{t_0}^t \|\mathbf{b}(s)\|_{C^1_b}\,ds\Big\}\,|x-y|^2$$

or, equivalently, that

$$\big|\mathbf{\Phi}(t,x) - \mathbf{\Phi}(t,y)\big| \leq \exp\Big\{\int_{t_0}^t \|\mathbf{b}(s)\|_{C^1_b}\,ds\Big\}|x-y|.$$

2). At this point, we need to take into account the changes in the initial time t_0. For this reason, we will denote by ϕ the function that associates for every point (t, t_0, x) in $[0,T] \times [0,T] \times \mathbb{R}^d$ the extended solution at time t of the Cauchy problem (2) with initial point (t_0, x).
Firstly, we show that ϕ satisfies the following semigroup property:

$$\phi(t,t_0,x) = \phi\big(t,s,\phi(s,t_0,x)\big) \qquad \forall\, s,t,t_0 \in [0,T] \quad \forall\, x \in \mathbb{R}^d \quad (*).$$

But this is clear, indeed the LHS and the RHS of the equation are, as functions on the variable t, solution of the Cauchy problem (2) with initial point (s, x). It follows from the uniqueness of an extended solution that the two sides are hence equal.
Fixed t_0, t in $[0,T]$, notice that $x \mapsto \mathbf{\Phi}(t,x) = \phi(t,t_0,x)$. Using $(*)$, we can show now that $\mathbf{\Phi}$ is invertible. Indeed,

$$x = \phi(t_0,t_0,x) = \phi(t_0,t,\phi(t,t_0,x)) = \phi(t_0,t,\mathbf{\Phi}(t,x))$$

for every x in $\mathbb{R}^d$ and

$$y = \phi(t,t,y) = \phi(t,t_0,\phi(t_0,t,y)) = \mathbf{\Phi}(t,\phi(t_0,t,y))$$

for every y in $\mathbb{R}^d$. From the last two equalities, we have just showed that the function $x \mapsto \mathbf{\Phi}(t,x)$ has inverse $\mathbf{\Phi}^{-1}(t,y) := \phi(0,t,y)$ and hence that it is invertible.
Moreover, $\mathbf{\Phi}^{-1}$ and $\mathbf{\Phi}$ are Lipschitz continuous thanks to point 1). □

Theorem 11 (Existence and Uniqueness Theorem)**.** *Under assumption 2, the function*

$u\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}$ defined by

$$u(t,y) := u_0(\mathbf{\Phi}^{-1}(t,y))$$

is the unique extended solution of the transport equation (1) *with initial value u_0. Moreover,*

- *u is in $C(0,T;C^1_b(\mathbb{R}^d))$;*
- *if u_0 is compactly supported, then the function $x\mapsto u(t,x)$ has a compact support.*

Proof. The proof of this result is essentially a copy of that in theorem 5. The only additional difficulty in this case is to pay attention to the equations that involves time-derivatives of u. Indeed, now we know only that the solution is absolutely continuous with respect to t, and hence, all of this equations will have a sense only a.e. on $[0,T]$. □

We conclude the analysis of the transport equation in the quasi-smooth case proving an L^p-estimates for an extended solution under the additional assumption of a compactly supported initial value.
This theorem will play a crucial role later in the next section, when we will use it for showing the uniform boundedness of a family of mollified solutions in the proof of theorem 18. In the writing of this part we have followed an idea found in [6].

Firstly, a particular class of functions is defined:
Definition 12. A function $\beta\colon\mathbb{R}\to\mathbb{R}$ is a **renormalizing map** if

- $\beta(0)=0$;
- $\beta\geq 0$;
- β is in $C^1_b(\mathbb{R})$.

For the proof of our main result, we also need two lemmas concerning the renormalizing maps just defined. The first one will show the existence of a sequence of such maps with some good properties of convergence.
In the second lemma, we will see instead that a renormalized solution, i.e. the composition of an extended solution with a renormalizing map, is again a solution of the transport equation (1) but respect to the renormalized initial value.
Lemma 13. *Let $p\in[1,\infty)$. There exists a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of renormalizing maps such that, for every function u in $L^p(\mathbb{R}^d)$,*

- *$\beta_n(u)$ is a smooth function on $\mathbb{R}^d$;*
- *$\|\beta_n(u)\|_{L^1(\mathbb{R}^d)}\to\|u\|^p_{L^p(\mathbb{R}^d)}$.*

Proof. We just consider a mollifier $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ on $\mathbb{R}$ and define, for every n in $\mathbb{N}$, the function

$$\beta_n(s) := (n \wedge |s|^p) * \rho_{\frac{1}{n}}.$$

The result then follows easily by the mollification properties showed in theorem 35. □

Lemma 14. *Let u_0 be a function in $C^1_b(\mathbb{R}^d)$, $\mathbf{b}$ a vector field in $L^1(0,T;C^1_b(\mathbb{R}^d))$ and β a renormalizing map. If $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is an extended solution of the transport equation* (1) *with initial value u_0, then $\beta \circ u$ is an extended solution of the same equation with initial value $\beta \circ u_0$.*
Moreover, if $x \mapsto u(t,x)$ is compactly supported, then $x \mapsto \big(\beta\circ u\big)(t,x)$ is in $C^1_c(\mathbb{R}^d)$ for every fixed t in $[0,T]$.

Proof. We start noticing that $\beta\circ u$ is absolutely continuous over time since u is and β has bounded derivatives, and that similarly, $\beta\circ u$ is in $C^1_b(\mathbb{R}^d)$.
To conclude, we just need to check that $\beta\circ u$ solves the transport equation (1) with initial value $\beta\circ u_0$. But this is true, since

$$\big(\beta\circ u\big)(0,x) \;=\; \beta\big(u(0,x)\big) \;=\; \beta\big(u_0(x)\big) \;=\; \big(\beta\circ u_0\big)(x)$$

for every x in $\mathbb{R}^d$ and

$$\partial_t\big(\beta\circ u\big) + \mathbf{b}\cdot\nabla(\beta\circ u) \;=\; \beta'(u)\partial_t u + \beta'(u)\mathbf{b}\cdot\nabla u \;=\; \beta'(u)\big[\partial_t u + \mathbf{b}\cdot\nabla u\big] \;=\; 0$$

on $\mathbb{R}^d$, a.e. on $[0,T]$.
To show that $x\mapsto \big(\beta\circ u\big)(t,x)$ has a compact support, we just observe that

$$\operatorname{supp}(\beta\circ u) \subseteq \operatorname{supp}(u)$$

since $\beta(0)=0$. □

We are now ready to prove our estimate for an extended solution.
Theorem 15. *Let $p\in[1,+\infty]$ and assume that*

- *u_0 is in $C^1_c(\mathbb{R}^d)$;*
- *$\mathbf{b}$ is a vector field in $L^1(0,T;C^1_b(\mathbb{R}^d))$.*

Then, any extended solution u of the transport equation (1) *with initial value u_0 is in $L^\infty(0,T;L^p(\mathbb{R}^d))$. Moreover, u satisfies the following a priori estimate*

$$\|u\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \le e^{C/p}\|u_0\|_{L^p(\mathbb{R}^d)}$$

with C a finite constant.

Proof. We start considering the case $p = \infty$. Remembering the characterization of a solution u given in theorem 11, we can easily bound it by

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \;=\; \|u_0(\mathbf{\Phi}^{-1}(t,x))\|_{L^\infty(\mathbb{R}^d)} \;\leq\; \|u_0\|_{L^\infty(\mathbb{R}^d)}$$

for every t in $[0,T]$.
Let us assume now $p \in [1,\infty)$. First of all, we consider a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of renormalizing maps such that $\|\beta_n \circ u\|_{L^1(\mathbb{R}^d)} \to \|u\|^p_{L^p(\mathbb{R}^d)}$, whose existence is assured by lemma 13. Then, we know from lemma 14 that for every fixed n, $\beta_n \circ u$ is an extended solution of the transport equation (1) with initial value $\beta_n \circ u_0$. It follows that the function $\beta_n \circ u$ is a.e. differentiable over time and that it satisfies

$$\partial_t\big(\beta_n \circ u\big) \;=\; \beta_n'(u)\,\partial_t u \;=\; -\beta_n'(u)\,\mathbf{b}\cdot\nabla u \;=\; -\mathbf{b}\cdot\nabla(\beta_n \circ u)$$

a.e. on $[0,T]$. By dominated convergence theorem we can now show that the function

$$t \in (0,T) \mapsto \int_{\mathbb{R}^d} \beta_n(u(t,\cdot))\,dx \in \mathbb{R}$$

is absolutely continuous. On the other side,

$$\begin{aligned}\int \partial_t(\beta_n \circ u)\,dx \;&=\; \int \beta_n'(u)\,\partial_t u\,dx \;=\; -\int \beta_n'(u)\,\mathbf{b}\cdot\nabla u\,dx \\ &=\; -\int \mathbf{b}\cdot\nabla(\beta_n \circ u)\,dx \;\overset{*}{=}\; \int \operatorname{div}(\mathbf{b})\,(\beta_n \circ u)\,dx\end{aligned}$$

where in $(*)$ we have used the integration by parts formula.
Then, we notice that $\partial_t(\beta_n \circ u)$ is in $L^1(\mathbb{R}^d)$ a.e. on $[0,T]$ and thus, that it is possible to commute derivative and integral in the first term of the last expression.
Since $C^1_b(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ and

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{C^1_b(\mathbb{R}^d)} \quad \forall\, f \in C^1_b(\mathbb{R}^d),$$

it follows by lemma 40 that $\operatorname{div}(\mathbf{b})$ in $L^1(0,T;L^\infty(\mathbb{R}^d))$ and hence that

$$\partial_t \int \beta_n \circ u\,dx \;=\; \int \operatorname{div}(\mathbf{b})\,(\beta_n \circ u)\,dx \;\leq\; \|\operatorname{div}\big(\mathbf{b}\big)\|_{L^\infty}(t) \int \beta_n \circ u\,dx$$

a.e. on $[0,T]$. Applying Gronwall's lemma 9 to the last inequality, we finally obtain the estimate

$$\int \beta_n \circ u\,dx \;\leq\; \exp\Big\{\int_0^t \|\operatorname{div}\big(\mathbf{b}\big)(s)\|\,ds\Big\} \int \beta_n \circ u_0\,dx \;\leq\; e^C \int \beta_n \circ u_0\,dx$$

with $C := \|\operatorname{div}(\mathbf{b})\|$ on $L^1(0,T;L^\infty(\mathbb{R}^d))$.

We can now let n goes to $+\infty$ and using lemma 13, find that

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \le e^{C/p}\|u_0\|_{L^p(\mathbb{R}^d)}.$$

Recombining the two cases together, we end the proof showing that

$$\|u\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} = \sup_{[0,T]} \text{ess}\|u(t)\|_{L^p(\mathbb{R}^d)} \le e^{C/p}\|u_0\|_{L^p(\mathbb{R}^d)}$$

for every p in $[1,\infty]$. □

5 Time-Dependent Drift Field: Locally Integrable Regularity

In this section, we will discuss a transport equation in the more general context we consider here, a drift field integrable over time and only locally integrable over space.
We want to remark that in the writing of this part we have followed essentially [1], except for lemma 22 that comes instead from [5].

Given a vector field $\mathbf{b}$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, we define the weak divergence of $\mathbf{b}$ in a distributional sense, i.e. as a function $\text{div}(\mathbf{b})$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \mathbf{b}\cdot\nabla\phi\,dx = -\int_{\mathbb{R}^d} \text{div}(\mathbf{b})\phi\,dx$$

for every test function ϕ on $\mathbb{R}^d$.

From this point further, we will fix q to be the exponential conjugate of p, i.e. q is the number in $[1,\infty]$ such that:

$$\frac{1}{p}+\frac{1}{q}=1.$$

Assumption 3.

- u_0 is a function in $L^p(\mathbb{R}^d)$;
- $\mathbf{b}$ is a vector field in $L^1(0,T;L^q_{\text{loc}}(\mathbb{R}^d))$;
- the weak divergence $\text{div}(\mathbf{b})$ of $\mathbf{b}$ exists and it is in $L^1(0,T;L^q_{\text{loc}}(\mathbb{R}^d))$.

Definition 16. Let $p\in[1,+\infty]$. Under assumption 3, a function u in $L^\infty(0,T;L^p(\mathbb{R}^d))$ is a **weak solution** of the transport equation (1) with initial value u_0 if it satisfies

$$\int_0^T\int_{\mathbb{R}^d} u\big[\partial_t\phi+\text{div}(\phi\mathbf{b})\big]\,dx\,dt + \int_{\mathbb{R}^d} u_0(x)\,\phi(0,x)\,dx = 0 \tag{4}$$

for every test function ϕ on $([0,T)\times\mathbb{R}^d)$.
Remark. We just recall that under these assumptions on $u_0, \mathbf{b}^i$ and $\text{div}(\mathbf{b})$, the integral (4) is well-defined for a function $u\in L^\infty(0,T;L^p(\mathbb{R}^d))$.
On one side, the second term on the LHS makes sense since $\phi(0,x)$ is in $C_c^\infty(\mathbb{R}^d)$, u_0 is in

$L^p(\mathbb{R}^d)$ and hence the product $\phi(0,\cdot)\, u_0$ is integrable.
On the other side, notice that $\operatorname{div}(\phi\, \mathbf{b})$ can be rewritten as $\phi \operatorname{div}(\mathbf{b}) + \mathbf{b} \cdot \nabla\phi$ and that $\phi, \nabla\phi$ and $\partial_t \phi$ are test functions on $[0,T) \times \mathbb{R}^d$. Then, $\operatorname{div}(\phi\mathbf{b})$ is in $L^1(0,T; L^q(\mathbb{R}^d))$ and hence $u\big[\partial_t\phi + \operatorname{div}(\phi\mathbf{b})\big]$ is integrable by Holder's theorem 39 for Bochner Spaces.

To make sense, we would like that the new concept of solution just defined could be seen as an extension of the previous one in a less regular context. In practise, an extended solution should coincide with a weak one when the conditions are good enough for either to be defined. This is exactly what the next theorem shows.

Theorem 17. *Let $p \in [1,\infty]$, u_0 in $C_b^1(\mathbb{R}^d)$ and $\mathbf{b}$ a vector field in $L^1(0,T; C_b^1(\mathbb{R}^d))$. Then, any extended solution of the transport equation (1) with initial value u_0 is also a weak solution of the same problem.*

Proof. Let u be an extended solution for the transport equation we are considering.
Firstly, notice that we already know that u is in $L^\infty(0,T; L^p(\mathbb{R}^d))$ thanks to corollary 15.
To conclude, we have just to show that u satisfies the equation (4). Consider a test function ϕ on $[0,T) \times \mathbb{R}^d$ and notice that

$$\begin{aligned} 0 \;=\; -\int_0^T \int \big[\partial_t u + \mathbf{b}\cdot\nabla u\big]\phi\, dx\, dt \;&=\; -\int_0^T \int \phi\, \partial_t u\, dx\, dt - \int_0^T \int \phi\, \mathbf{b}\cdot\nabla u\, dx\, dt \\ &= \int_0^T \int u\, \partial_t\phi\, dx\, dt + \int u_0\, \phi(0,x)\, dx + \int_0^T \int u \, \operatorname{div}(\phi\mathbf{b})\, dx\, dt \end{aligned}$$

where in the last equality we have applied the integration by parts formula. □

It is important to highlight the main difference of the proving method used here respect to those we have shown in the previous sections.
Indeed, we will not prove the existence of a (weak) solution showing an explicit formula for it, neither explaining a practice way to find it. Instead, we will show the result through a "renormalisation" method. The fundamental idea in it will be to lead our case back to the previous quasi-smooth case and to use the already proven result 11 to show the existence of some "renormalized" solution. After that, we will just go back to our original setting in a way to make the solutions founded convergent to a weak one.
In practise, the "renormalization" will consist in mollifying the components of the equation, i.e to convolute them with a mollifier, and in truncating them in order to make them smooth and bounded in x.

Theorem 18 (Existence of a Weak Solution)**.** *Let $p \in (1,+\infty)$. Under assumption 3 and with the additional condition $\operatorname{div}(\mathbf{b}) \in L^1(0,T; L^\infty(\mathbb{R}^d))$, there exists a weak solution of the transport equation (1) with initial value u_0.*

Proof. We start considering a mollifier $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ on $\mathbb{R}^d$ and defining for every fixed time t,

$$u_{0,\epsilon} := \big(u_0 B(0, \frac{1}{\epsilon})\big) * \rho_\epsilon \qquad \mathbf{b}^i_\epsilon := (\mathbf{b}^i \wedge \frac{1}{\epsilon}) * \rho_\epsilon.$$

In particular, we know that $u_{0,\epsilon} \in C^1_c(\mathbb{R}^d)$, $\mathbf{b}_\epsilon \in L^1(0,T; C^1_b(\mathbb{R}^d))$ and

$$\begin{aligned} u_{0,\epsilon} &\to u_0 && \text{on } L^p(\mathbb{R}^d) \\ \mathbf{b}^i_\epsilon &\to \mathbf{b}^i && \text{on } L^1(0,T; L^q_{\text{loc}}(\mathbb{R}^d)) \qquad (*) \\ \operatorname{div}(\mathbf{b}_\epsilon) &\to \operatorname{div}(\mathbf{b}) && \text{on } L^1(0,T; L^q_{\text{loc}}(\mathbb{R}^d)) \end{aligned}$$

thanks to theorem 41 and the fact that $\operatorname{div}(\mathbf{b}_\epsilon) = \big(\operatorname{div}(\mathbf{b})\big) * \rho_\epsilon$.
Fixed ϵ in $(0,1]$ we have already proven that there exists an extended solution u_ϵ in $C(0,T; C^1_b(\mathbb{R}^d))$ of the transport equation

$$\begin{cases} \partial_t u_\epsilon + \mathbf{b}_\epsilon \cdot \nabla u_\epsilon = 0 \\ u_\epsilon = u_{0,\epsilon} \end{cases}$$

and, thanks to theorem 17, also that u_ϵ satisfies

$$0 = \int_0^T \int_{\mathbb{R}^d} u_\epsilon \left[\partial_t \phi + \operatorname{div}(\phi \mathbf{b}_\epsilon)\right] dx\, dt + \int_{\mathbb{R}^d} u_{0,\epsilon}\, \phi(0,x)\, dx.$$

To end the proof, we want to pass the limit inside the integrals in the previous equality.
Firstly, it is clear that

$$\lim_\epsilon \int_{\mathbb{R}^d} u_{0,\epsilon}(x)\phi(0,x)\, dx \;=\; \int_{\mathbb{R}^d} u_0(x)\phi(0,x)\, dx$$

since $u_{0,\epsilon} \to u_0$ on $L^p(\mathbb{R}^d)$.
Then, we notice that the families $C_\epsilon := \|\operatorname{div}(\mathbf{b}_\epsilon)\|_{L^1(0,T;L^\infty(\mathbb{R}^d))}$ and $\|u_{0,\epsilon}\|_{L^p}$ are bounded on $\mathbb{R}$. Hence, $\{u_\epsilon \colon \epsilon \in (0,1]\}$ is also bounded in $L^\infty(0,T; L^p(\mathbb{R}^d))$ since by theorem 15, it satisfies

$$\|u_\epsilon\|_{L^\infty(0,T;L^p(\mathbb{R}^d))} \;\leq\; e^{C_\epsilon/p} \|u_{0,\epsilon}\|_{L^p(\mathbb{R}^d)}$$

for every ϵ in $(0,1]$. In particular, this yields that $\{u_\epsilon \colon \epsilon \in (0,1]\}$ is a family of bounded functionals in $L^1(0,T; L^q(\mathbb{R}^d))'$, thanks to theorem 42.
Furthermore, we know that $L^1(0,T; L^q(\mathbb{R}^d))$ is separable from theorem 43. Thus, we can apply the sequential Banach-Alaoglu theorem and find a subsequence $\{u_n\}_{n\in\mathbb{N}}$ of $\{u_\epsilon\}$ that w^*-converges to a function u in $L^\infty(0,T; L^p(\mathbb{R}^d))$.
Remembering that

$$\partial_t \phi + \operatorname{div}(\phi \mathbf{b}_\epsilon) \to \partial_t \phi + \operatorname{div}(\phi \mathbf{b}) \quad \text{ on } L^1(0,T; L^q(\mathbb{R}^d)),$$

we can finally show that

$$\lim_n \langle u_n, \partial_t \phi + \operatorname{div}(\phi \mathbf{b}_n)\rangle \;=\; \langle u, \partial_t \phi + \operatorname{div}(\phi \mathbf{b})\rangle$$

or, equivalently, that

$$\lim_n \int_0^T \int_{\mathbb{R}^d} u_n \big(\partial_t \phi + \operatorname{div}(\phi \mathbf{b}_n)\big)\, dx\, dt \;=\; \int_0^T \int_{\mathbb{R}^d} u \big(\partial_t \phi + \operatorname{div}(\phi \mathbf{b})\big)\, dx\, dt$$

and we have concluded. □

We turn now on proving a stability property of the weak solutions of a transport equation. Indeed, we will see in the next theorem that smooth approximations in x of a solution will be again solution of the equation with small error terms, under the following hypothesis:

Assumption 4.

- u is a function in $L^\infty(0, T; L^p_{\text{loc}}(\mathbb{R}^d)$;
- $\mathbf{b}$ is a vector field in $L^1(0, T; W^{1,\alpha}_{\text{loc}}(\mathbb{R}^d))$;
- $\{\rho_\epsilon : \epsilon \in (0, 1]\}$ is a mollifier on $\mathbb{R}^d$.

Theorem 19 (Behaviour under smooth approximations)**.** *Let $p \in [1, \infty)$, $\alpha \geq q$, $\beta \in [1, \infty)$ such that $\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}$ and u a weak solution of the transport equation* (1)*.*
Under assumption 4, the distribution r_ϵ defined by

$$r_\epsilon := \partial_t(u * \rho_\epsilon) + \mathbf{b} \cdot \nabla(u * \rho_\epsilon)$$

is a function in $L^1(0, T; L^\beta_{\text{loc}}(\mathbb{R}^d))$. Furthermore,

$$r_\epsilon \xrightarrow{\epsilon} 0 \quad \text{in } L^1(0, T; L^\beta_{loc}(\mathbb{R}^d)).$$

The proof of this result will be quite long as we will involve some technical tools and knowledge from distribution theory and on Bochner spaces. We suggest the reader to go to the Appendix for a better explanation of the results used next and, if necessary, to the references there included.
For a better reading of it, we have divided the proof of theorem 19 into three lemmas.

Proof of Theorem 19. Assuming the following lemmas to be already proven, it is easy to show that theorem 19 holds.
Indeed, seeing all the following equations in a distributional sense, we can write that

$$r_\epsilon \;=\; \partial_t(u * \rho_\epsilon) + \mathbf{b} \cdot \nabla(u * \rho_\epsilon) - \Big[(\mathbf{b} \cdot \nabla u) * \rho_\epsilon + (\mathbf{b} \cdot \nabla u) * \rho_\epsilon\Big].$$

Then, applying lemma 21 and rearranging the equation, we find that

$$r_\epsilon \;=\; (\partial_t u + \mathbf{b} \cdot \nabla u) * \rho_\epsilon + \mathbf{b} \cdot \nabla(u * \rho_\epsilon) - (\mathbf{b} \cdot \nabla u) * \rho_\epsilon \;=\; \mathbf{b} \cdot \nabla(u * \rho_\epsilon) - (\mathbf{b} \cdot \nabla u) * \rho_\epsilon$$

where in the last equality we have used that $(\partial_t u + \mathbf{b}\cdot\nabla u) * \rho_\epsilon = 0$, since u is a weak solution.
It follows that $r_\epsilon = \mathbf{b}\cdot\nabla(u*\rho_\epsilon) - (\mathbf{b}\cdot\nabla u)*\rho_\epsilon$ as distributions and thus, that r_ϵ is a function in $L^1(0,T;L^\beta_{loc}(\mathbb{R}^d))$ such that

$$r_\epsilon \to 0 \quad \text{on } L^1(0,T;L^\beta_{loc}(\mathbb{R}^d))$$

thanks to lemma 23.

□

Theorem 20 (Fundamental Theorem of calculus for distributions)**.** *Let f be a function in $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ and y a point in $\mathbb{R}^d$. Then,*

$$f(x+y) - f(y) \;=\; \int_0^1 y\cdot\nabla f(x+ty)\,dt \qquad \text{a.e. on } \mathbb{R}^d.$$

Proof. For a proof of this result, see [2], page 145. □

Lemma 21. *Let u be a function in $L^1(0,T,L^1_{\text{loc}}(\mathbb{R}^d))$ and $\{\rho_\epsilon\colon \epsilon\in(0,1]\}$ a mollifier on $\mathbb{R}^d$. Then,*

$$\partial_t(u*\rho_\epsilon) = (\partial_t u)*\rho_\epsilon$$

as distributions on $[0,T]\times\mathbb{R}^d$.

Proof. Fixed a test function ϕ on $(0,T)\times\mathbb{R}^d$, we notice that

$$\begin{aligned}
&-\langle(\partial_t u)*\rho_\epsilon,\phi\rangle \;=\; -\langle\partial_t u, \int \phi(t,y)\rho_\epsilon(y-x)\,dy\rangle \\
&\quad=\; \langle u, \partial_t \int_{\mathbb{R}^d}\phi(t,y)\rho_\epsilon(y-x)\,dy\rangle \;=\; \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(t,x)\partial_t\phi(t,y)\rho_\epsilon(y-x)\,dy\,dx\,dt \\
&\int_0^T\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} u(t,x)\rho_\epsilon(y-x)\,dx\Big]\partial_t\phi(t,y)\,dy\,dt \;=\; \int_0^T\int_{\mathbb{R}^d}\big(u*\rho_\epsilon\big)(y)\,\partial_t\phi(t,y)\,dy\,dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;=\; -\langle\partial_t(u*\rho_\epsilon),\phi\rangle.
\end{aligned}$$

Hence, $(\partial_t u)*\rho_\epsilon = \partial_t(u*\rho_\epsilon)$ and we have finished.

□

Lemma 22. *Let $p\in[1,+\infty]$, $\alpha\geq q$ and $\beta\in[1,\infty)$ such that $\frac{1}{\beta}=\frac{1}{\alpha}+\frac{1}{p}$.*

- *u is a function in $L^p_{\text{loc}}(\mathbb{R}^d)$;*
- *$\mathbf{b}$ is a time-independent vector field in $W^{1,\alpha}_{\text{loc}}(\mathbb{R}^d)$;*
- *$\{\rho_\epsilon\colon \epsilon\in(0,1]\}$ is a mollifier on $\mathbb{R}^d$.*

Then, for every compact K in $\mathbb{R}^d$,

$$\|(\mathbf{b}\cdot\nabla u)*\rho_\epsilon - \mathbf{b}\cdot\nabla(u*\rho_\epsilon)\|_{L^\beta(K)} \leq C\|u\|_{L^p(K')}\,\|\mathbf{b}\|_{W^{1,\alpha}(K')}$$

with $K' := K + B(0,1)$ and C a constant independent from ϵ.

Proof. Firstly, we check the well-posedness of the LHS of the inequality. $\mathbf{b}\cdot\nabla(u*\rho_\epsilon)$ is well-defined in the classical sense, since the function $u*\rho_\epsilon$ is smooth. The first term instead is defined only in weak sense. Indeed, it can be rewritten as

$$(\mathbf{b}\cdot\nabla u)*\rho_\epsilon := \int_{B(x,\epsilon)} \mathbf{b}(y)\cdot\nabla u(y)\rho_\epsilon(x-y)\,dy \;=\; -\int_{B(x,\epsilon)} \operatorname{div}\big(\mathbf{b}(y)\rho_\epsilon(x-y)\big)u(y)\,dy$$

where the expression is well-posed since $\mathbf{b}$ is in $W^{1,\alpha}_{loc}(\mathbb{R}^d)$ and ρ_ϵ is smooth.
Now, notice that

$$\begin{aligned}(\mathbf{b}\cdot\nabla u)*\rho_\epsilon - \mathbf{b}\cdot\nabla(u*\rho_\epsilon) &\overset{\star}{=} -\int \operatorname{div}\big(\mathbf{b}(y)\rho_\epsilon(x-y)\big)u(y)\,dy \;-\; \mathbf{b}\cdot(u*\nabla\rho_\epsilon)\\ = -\int u(y)\operatorname{div}\big(\mathbf{b}\big)(y)\rho_\epsilon(x-y) &- u(y)\mathbf{b}(y)\cdot\nabla\rho_\epsilon(x-y) + u(y)\mathbf{b}(x)\cdot\nabla\rho_\epsilon(x-y)\,dy\\ &= -\big[\big(\operatorname{div}(\mathbf{b})u\big)*\rho_\epsilon\big](x) + \int u(y)\nabla\rho_\epsilon(x-y)\cdot\big[\mathbf{b}(y)-\mathbf{b}(x)\big]\,dy\end{aligned}$$

where in $\star$ we have used property 2) of theorem 32.
On one side, the first term in the last expression can be uniformly bounded using property 3) of theorem 32 by

$$\|\big(\operatorname{div}(\mathbf{b})u\big)*\rho_\epsilon\|_{L^\beta(K)} \;\leq\; \|\operatorname{div}(\mathbf{b})u\|_{L^\beta(K')} \;\leq\; \|\operatorname{div}(\mathbf{b})\|_{L^\alpha(K')}\,\|u\|_{L^p(K')}$$

where $K' := K + B(0,1)$. On the other side, the second term can be rewritten as

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} u(y)\nabla\rho_\epsilon(x-y)\cdot\big[\mathbf{b}(y)-\mathbf{b}(x)\big]\,dy\Big| &\leq \int_{\mathbb{R}^d} |u(y)|\,|\nabla\rho_\epsilon(x-y)|\,|\mathbf{b}(y)-\mathbf{b}(x)|\,dy\\ \leq \frac{C}{\epsilon}\fint_{B(x,\epsilon)} |u(y)|\,|\nabla\rho(\frac{x-y}{\epsilon})|\,|\mathbf{b}(y)-\mathbf{b}(x)|\,dy &\leq \frac{C}{\epsilon}\fint_{B(x,\epsilon)} |u(y)|\,|\mathbf{b}(y)-\mathbf{b}(x)|\,dy\\ &\leq C\Big[\fint_{B(x,\epsilon)} |u(y)|^p\,dy\Big]^{1/p}\Big[\fint_{B(x,\epsilon)} \big(\frac{1}{\epsilon}|\mathbf{b}(y)-\mathbf{b}(x)|\big)^\alpha\,dy\Big]^{1/\alpha} \qquad (*)\end{aligned}$$

with C representing different constants, each of them independent from ϵ.
Again, the first term of $(*)$ can be easily bounded by

$$\begin{aligned}\Big\|\big(\fint_{B(x,\epsilon)} |u(y)|^p\,dy\big)^{1/p}\Big\|_{L^p(K)} &\leq \Big[\int_K\int_{K'} \frac{1}{m_\epsilon}|u(y)|^p\,\mathbb{1}_{B(0,\epsilon)}(y-x)\,dy\,dx\Big]^{1/p}\\ = \Big[\int_{K'}\Big(\int_K \frac{1}{m_\epsilon}\mathbb{1}_{B(y,\epsilon)}(x)\,dx\Big)\,|u(y)|^p\,dy\Big]^{1/p} &= \Big(\int_{K'} |u(y)|^p\,dy\Big)^{1/p} = \|u\|_{L^p(K')}\end{aligned}$$

with m_ϵ representing the volume of a ball of radius ϵ.
Now, the second term of $(*)$ can be rewritten, using theorem 20, as

$$\fint_{B(x,\epsilon)} \big(\frac{1}{\epsilon}|\mathbf{b}(y)-\mathbf{b}(x)|\big)^\alpha\,dy = \fint_{B(x,\epsilon)} \Big|\int_0^1 \frac{y-x}{\epsilon}\cdot D\mathbf{b}(x+t(y-x))\,dt\Big|^\alpha dy$$
$$\le \fint_{B(x,\epsilon)} \int_0^1 |D\mathbf{b}(x+t(y-x))|^\alpha \Big|\frac{y-x}{\epsilon}\Big|^\alpha dt\,dy$$

and, applying the change of variable $z = ty + (1-t)x$, as

$$\fint_{B(x,\epsilon)} \int_0^1 |D\mathbf{b}(x+t(y-x))|^\alpha \big|\frac{y-x}{\epsilon}\big|^\alpha dt\,dy = \int_0^1 \fint_{B(x,t\epsilon)} |D\mathbf{b}(z)|^\alpha \frac{dz}{t^d}\,dt$$
$$= C\int_{\mathbb{R}^d} |D\mathbf{b}(z)|^\alpha \Big(\int_0^1 \mathbb{1}_{B(0,t\epsilon)}(x-z)\frac{1}{(\epsilon t)^d}\,dt\Big)\,dt = C|D\mathbf{b}|^\alpha * \chi_\epsilon$$

where $\chi_\epsilon(x) := \int_0^1 \mathbb{1}_{B(0,t\epsilon)}(x)(\epsilon t)^{-d}\,dt$. In particular, notice that χ_ϵ is equal to

$$\frac{1}{(d-1)\epsilon^d}\Big[\Big(\frac{\epsilon}{|x|}\Big)^{d-1} - 1\Big]\mathbb{1}_{B(0,\epsilon)}(x)$$

and hence that it is in $C_c^\infty(\mathbb{R}^d)$. Thanks to property 3) of theorem 32, we can now bound the second term of $(*)$ by

$$\big\|\big(|D\mathbf{b}|^\alpha * \chi_\epsilon\big)^{1/\alpha}\big\|_{L^\alpha(K)} = \||D\mathbf{b}|^\alpha * \chi_\epsilon\|_{L^1(K)}^{1/\alpha} = \|D\mathbf{b}\|_{L^\alpha(K')}\|\chi_\epsilon\|_{L^1(\mathbb{R}^d)}^{1/\alpha}.$$

To conclude, we need χ_ϵ to be bounded in norm. But this is true, since

$$\|\chi_\epsilon\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{1}{(d-1)\epsilon^d}\Big[\Big(\frac{\epsilon}{|x|}\Big)^{d-1} - 1\Big]\mathbb{1}_{B(0,\epsilon)}(x)\,dx$$
$$= \frac{C}{\epsilon^d}\int_{B(0,\epsilon)} \Big(\frac{\epsilon}{|x|}\Big)^{d-1} - 1\,dx \le \frac{C}{\epsilon}\int_{B(0,\epsilon)} |x|^{-d+1}\,dx \le C.$$

Finally, we can combine all the bounds in one and show that

$$\|(\mathbf{b}\cdot\nabla u)*\rho_\epsilon - \mathbf{b}\cdot\nabla(u*\rho_\epsilon)\|_{L^\beta(K)} = \|(u\,\mathrm{div}(\mathbf{b}))*\rho_\epsilon\|_{L^\beta(K)}$$
$$+ C\,\Big\|\Big(\fint_{B(x,\epsilon)} |u(y)|^p\,dy\Big)^{1/p}\Big\|_{L^p(K)} \||D\mathbf{b}|^\alpha * \chi_\epsilon\|_{L^\alpha(K)} \le C\|\mathrm{div}(\mathbf{b})\|_{L^\alpha(K')}\,\|u\|_{L^p(K')}$$
$$+ C\|u\|_{L^p(K')}\|D\mathbf{b}\|_{L^\alpha(K')}\Big] \le C\|\mathbf{b}\|_{W^{1,\alpha}(K')}\|u\|_{L^p(K')}.$$

□

Remark. We want just to remark that hypothesis in lemma 22 are essentially the assumption 4 where a time-independent case is considered, or equivalently, when we take fixed a time t in $[0,T]$ in the assumption 4.

Lemma 23. *Let $p \in [1,+\infty)$, $\alpha \geq q$ and $\beta \in [1,\infty)$ such that $\frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{p}$. Then, under assumption 4,*

$$(\mathbf{b} \cdot \nabla u) * \rho_\epsilon - \mathbf{b} \cdot \nabla(u * \rho_\epsilon) \to 0 \quad \text{in } L^1(0,T; L^\beta_{loc}(\mathbb{R}^d)).$$

Proof. We start fixing a time t in $(0,T)$ and a compact K in $\mathbb{R}^d$. Then, we can apply the previous lemma 22 to find a first bound

$$\|(\mathbf{b} \cdot \nabla u) * \rho_\epsilon - \mathbf{b} \cdot \nabla(u * \rho_\epsilon)\|_{L^\beta(K)} \leq C\|u\|_{L^p(K)} \|\mathbf{b}\|_{W^{1,\alpha}(K)}.$$

Moreover, integrating over the time and applying Holder's inequality, we obtain that

$$\|(\mathbf{b} \cdot \nabla u) * \rho_\epsilon - \mathbf{b} \cdot \nabla(u * \rho_\epsilon)\|_{L^1(0,T;L^\beta(K))} \leq C\|u\|_{L^\infty(0,T;L^p(K))} \|\mathbf{b}\|_{L^1(0,T;W^{1,\alpha}(K))}$$

and hence that $(\mathbf{b} \cdot \nabla u) * \rho_\epsilon - \mathbf{b} \cdot \nabla(u * \rho_\epsilon)$ is bounded on $L^1(0,T; L^\beta(K))$ uniformly with respect to ϵ.
Thanks to the above uniform bound, now we just need to prove the required convergence only in the smooth case. Indeed, the general case would follow easily from the smooth one by a density argument. Assuming that u and $\mathbf{b}$ are smooth,

$$\begin{aligned}
(\mathbf{b} \cdot \nabla u) * \rho_\epsilon - \mathbf{b} \cdot \nabla(u * \rho_\epsilon) &= \int u(y)\big(\mathbf{b}(y) - \mathbf{b}(x)\big) \cdot \nabla\rho_\epsilon(x-y)\, dy - \big(\mathrm{div}(\mathbf{b})u * \rho_\epsilon\big)(x) \\
&\overset{(\star)}{=} -\int \mathrm{div}_x\big(u(y)\mathbf{b}(y) - u(y)\mathbf{b}(x)\big)\rho_\epsilon(x-y)\, dy - \big[\mathrm{div}(\mathbf{b})u * \rho_\epsilon\big](x) \\
= \mathrm{div}\big(\mathbf{b}\big)(x) \int u(y)\rho_\epsilon(x-y)\, dy &- \Big(\mathrm{div}(\mathbf{b})u * \rho_\epsilon\Big)(x) = \mathrm{div}(\mathbf{b})\big(u * \rho_\epsilon\big) - \big(\mathrm{div}(\mathbf{b})u\big) * \rho_\epsilon
\end{aligned}$$

where in $(\star)$ we have used the integration by parts formula.
Finally, we can end the proof using property 2) of theorem 41 to show that

$$\mathrm{div}(\mathbf{b})\big(u * \rho_\epsilon\big) - \big(\mathrm{div}(\mathbf{b})u\big) * \rho_\epsilon \to \mathrm{div}(\mathbf{b})u - \mathrm{div}(\mathbf{b})u = 0$$

on $L^1(0,T; L^\beta(\mathbb{R}^d))$. □

We want now to show that it is possible to prove the uniqueness of a solution also in this more general context, under the following hypothesis:
Assumption 5.

- u_0 is a function in $L^p(\mathbb{R}^d)$;
- $\mathbf{b}$ is a vector field in $L^1(0,T; W^{1,q}_{loc}(\mathbb{R}^d))$ such that

$$\frac{\mathbf{b}}{1+|x|} \in L^1(0,T; L^1(\mathbb{R}^d)) + L^1(0,T; L^\infty(\mathbb{R}^d));$$

- the weak divergence $\mathrm{div}(\mathbf{b})$ exists and it is in $L^1(0,T; L^\infty(\mathbb{R}^d))$.

Lemma 24 (Gronwall's Lemma for Distribution)**.** *Let f, R be two integrable functions on $(0,T)$ such that*

$$\partial_t f \le R(t) f$$

as distributions on $(0,T)$. Then, $f = 0$ a.e. on $[0,T]$.

Proof. For a proof of this result see [2]. □

Theorem 25 (Uniqueness of a Weak Solution)**.** *Let $p \in [1, +\infty)$. Under assumption 5, there exists at most one weak solution u in $L^\infty(0,T; L^p(\mathbb{R}^d))$ of the transport equation (1) with initial condition u_0.*

Proof. First of all, notice that it is sufficient to prove that $u = 0$ if u is a weak solution of the equation with the initial value $u_0 = 0$.
Indeed, assuming that u, v are two weak solution of the equation with the same initial value u_0, the function $u - v$ would be a solution with the initial point $u_0 = 0$, by linearity of the equation.
We start considering a mollifier $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ on $\mathbb{R}^d$ and defining for shortness, $u_\epsilon := u * \rho_\epsilon$. By definition of r_ϵ in theorem 19 with $\alpha = q$ and $\beta = 1$, we already know that

$$\partial_t u_\epsilon \overset{d}{=} r_\epsilon - \mathbf{b} \cdot \nabla u_\epsilon$$

as distributions and that the RHS is a function in $L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^d))$. It follows that also $\partial_t u_\epsilon$ is a function in $L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^d))$ and moreover, that

$$t \mapsto \partial_t u_\epsilon(t,x)$$

is in $L^1(0,T)$ a.e. on $\mathbb{R}^d$. Hence, u_ϵ is absolutely continuous on $[0,T]$ for almost every x.
Fixed $M > 0$, we consider now a sequence $\{\beta_n\}_{n \in \mathbb{N}}$ of renormalizing maps such that $\beta_n(u) \to |u \wedge M|^p$ on $L^1(0,T; L^1(\mathbb{R}^d))$. Then, for every n in $\mathbb{N}$,

$$\partial_t\big(\beta_n(u_\epsilon)\big) + \mathbf{b} \cdot \nabla\big(\beta_n(u_\epsilon)\big) \;=\; \beta_n'(u_\epsilon)\partial_t u_\epsilon + \beta_n'(u_\epsilon)\mathbf{b} \cdot \nabla u_\epsilon \;=\; \beta_n'(u_\epsilon) r_\epsilon$$

and, letting ϵ goes to 0,

$$\partial_t \beta_n(u) + \mathbf{b} \cdot \nabla \beta_n(u) \;=\; 0$$

as distributions on $(0,T) \times \mathbb{R}^d$.
We now define a cut-off function ϕ_R, by taking a positive test function ϕ on $\mathbb{R}^d$ such that $\text{supp}(\phi) \subseteq \overline{B(0,2)}$, $\phi = 1$ on $\overline{B(0,1)}$ and defining ϕ_R as

$$\phi_R(x) := \phi(\frac{x}{R}) \quad \forall\, x \in \mathbb{R}^d.$$

Then, seeing all the following equalities in a distributional sense over $(0,T)$, we find that

$$\partial_t \int_{\mathbb{R}^d} \beta_n(u)\,\phi_R\,dx \;=\; \int_{\mathbb{R}^d} \beta_n(u)\,\mathrm{div}(\phi_R\,\mathbf{b})\,dx$$

and letting n goes to ∞, that

$$\partial_t \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p \phi_R\,dx \;=\; \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p\,\mathrm{div}(\phi_R\,\mathbf{b})\,dx \quad (\star).$$

Notice now that we can rewrite the RHS of the last equality as

$$\begin{aligned}&\int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p \big(\phi_R\,\mathrm{div}(\mathbf{b}) + \mathbf{b}\cdot\nabla\phi_R\big)\,dx\\ &\qquad\qquad \le \|\,\mathrm{div}(\mathbf{b})\|_{L^\infty} \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p\phi_R\,dx + \frac{C}{R}\int_{R\le|x|\le 2R} \big(|u|\wedge M\big)^p|\mathbf{b}|\,dx\end{aligned}$$

with C a constant independent from R and ϵ. Furthermore, using our assumption on $\mathbf{b}$, we can bound it locally by

$$\frac{|\mathbf{b}|}{R}\mathbb{1}_{R\le|x|\le 2R} \;\le\; 3\frac{|\mathbf{b}|}{1+|x|}\mathbb{1}_{|x|\ge R} \;=\; 3|\mathbf{b}_1+\mathbf{b}_\infty|\mathbb{1}_{|x|\ge R}$$

where $\mathbf{b}_1, \mathbf{b}_\infty$ are functions in $L^1(0,T;L^1(\mathbb{R}^d))$ and $L^1(0,T;L^\infty(\mathbb{R}^d))$, respectively such that

$$\frac{\mathbf{b}}{1+|x|} \;=\; \mathbf{b}_1 + \mathbf{b}_\infty.$$

Hence, by $(\star)$,

$$\begin{aligned}&\partial_t \int_{\mathbb{R}^d} \big(|u|(x)\wedge M\big)^p\,dx\\ &\qquad \le \|\,\mathrm{div}(\mathbf{b})\|_{L^\infty} \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p\phi_R\,dx \,+\, 3C\int_{|x|\ge R} \big(|u|\wedge M\big)^p|\mathbf{b}_1+\mathbf{b}_\infty|\,dx\\ &\le \|\,\mathrm{div}(\mathbf{b})\|_{L^\infty} \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p\phi_R\,dx + 3C\Big[\|\mathbf{b}_\infty\|_{L^\infty}\int_{|x|\ge R}\big(|u|\wedge M\big)^p + M^p\int_{|x|\ge R}|\mathbf{b}_1|\,dx\Big].\end{aligned}$$

We can now let R goes to $+\infty$ and obtain

$$\partial_t \int_{\mathbb{R}^d} \big(|u|\wedge M\big)^p\,dx \;\overset{d}{\le}\; \|\,\mathrm{div}(\mathbf{b})\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} \big(|u\|\wedge M\big)^p\,dx.$$

Finally, lemma 24 can be applied to find that

$$\|u(t,x)\wedge M\|_{L^p(\mathbb{R}^d)} \;=\; 0 \quad \text{ on } [0,T]$$

and we can conclude letting M goes to 0. □

Corollary 26 (Existence and Uniqueness Theorem)**.** *Let* $p \in (1,+\infty)$ *and assume that*

- *u_0 is a function in $L^p(\mathbb{R}^d)$;*
- *the weak divergence* $\operatorname{div}(\mathbf{b})$ *exists and it is in $L^1(0,T;L^\infty(\mathbb{R}^d))$;*
- $\mathbf{b}$ *is a vector field in $L^1(0,T;W^{1,q}_{\mathrm{loc}}(\mathbb{R}^d))$ such that*

$$\frac{\mathbf{b}}{1+|x|} \in L^1(0,T;L^1(\mathbb{R}^d)) \,+\, L^1(0,T;L^\infty(\mathbb{R}^d))$$

Then, there exists a unique weak solution u in $L^\infty(0,T;L^p(\mathbb{R}^d))$ of the transport equation (1) *with initial condition u_0.*

Proof. The existence part follows immediately from theorem 18.
For the uniqueness of a solution, just notice that for every u, v weak solutions of the transport equation with respect to the same initial condition u_0, the function $u - v$ is a weak solution of the transport equation with initial condition $u_0 = 0$ thanks to the linearity of the equation. To conclude the proof, we just need to apply theorem 25 to $u - v$. □

6 An Application to Stochastic Transport Equations

To conclude, we want to show through an example how the results proven here for the deterministic case can be extended to the analysis of the stochastic transport equations. In particular, a transport equation perturbed by a multiplicative white noise will be considered. In this context, we will prove the existence of an L^∞-weak solution in a similar way to how we have show the existence theorem 18 in the previous section. Finally, we want to remark that the original results explained in this section was taken from [8].

Let us introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a complete and right continuous filtration $(\mathcal{F}_t)$ and a d-dimensional Brownian motion $B_t := (B_t^1, \ldots, B_t^d)$.

Given a vector field $\mathbf{b}\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and a function $u_0\colon \mathbb{R}^d \to \mathbb{R}$, we want to study a stochastic transport equation of the form

$$\begin{cases} d_t U + \mathbf{b} \cdot \nabla U(t,x) dt + \sum_{i=1}^d \partial_i U \circ dB_t^i = 0 & \text{on } (0,+\infty) \times \mathbb{R}^d \\ U = u_0 & \text{on } \{0\} \times \mathbb{R}^d. \end{cases} \tag{5}$$

where the stochastic integration is understood in the Stratonovich sense.

Before starting with our example, we need some previous background. In particular, we will use the following existence result that can be found, in a more general setting, in [7].
Definition 27. Let u_0 be a function in $C^\infty(\mathbb{R}^d)$. A process $u\colon \Omega \times [0,\infty) \times \mathbb{R}^d$ is a solution of the stochastic transport equation (5) if

- u has a continuous modification that is an adapted semi-martingale;

- it satisfies

$$u(t,x) \;=\; u_0(x) + \int_0^t \mathbf{b}(s,x)\cdot\nabla u(s,x)ds + \sum_{i=1}^d \int_0^t \partial_i u(s,x)\circ dB_s^i.$$

Theorem 28. *Let u_0 be a function in $C^\infty(\mathbb{R}^d)$ and $\mathbf{b}$ a vector field in $C^\infty(\mathbb{R}\times\mathbb{R}^d)$ and global lipschitz in x. Given $\mathbf{\Phi}(t,x)$ the stochastic flow associated to the vector field $\mathbf{b}$, the function $u\colon \Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ defined by*

$$u(t,x) := u_0(\mathbf{\Phi^{-1}}(t,x))$$

is the unique global solution of the stochastic equation (5) *with initial value u_0.*

Our aim is to show the existence of a solution, that would be suitable in a setting with a non-continuous vector field under the following hypothesis:

Assumption 6.

- u_0 is a function in $L^\infty(\mathbb{R}^d)$;
- $\mathbf{b}$ is a vector field in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$;
- the weak divergence $\text{div}(\mathbf{b})$ exists and it is in $L^1_{\text{loc}}([0,T]\times\mathbb{R}^d)$

Definition 29. Under assumption 6, a stochastic process u in $L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ is a **weak solution** of the stochastic transport equation (5) with initial value u_0 if, for every test function ϕ on $(\mathbb{R}^d)$,

- the stochastic process $\int_{\mathbb{R}^d}\phi(x)u(\omega,t,x)\,dx$ has a continuous version that is an adapted semi-martingale;
- a.e. on $\Omega\times[0,T]$, it satisfies

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\phi(x)\,dx \;=\; \int_{\mathbb{R}^d} u_0(x)\,\phi(x)\,dx \,+\, \int_0^t\int_{\mathbb{R}^d} u(s,x)\,\text{div}(\mathbf{b}\phi)(s,x)\,dx\,ds \\ +\sum_{i=1}^d\int_0^t\Big(\int_{\mathbb{R}^d} u(s,x)\partial_i\phi(x)\,dx\Big)\circ dB_s^i. \quad (6)\end{aligned}$$

In the last definition, the weak solution has been defined through the Stratonovich integration. In any case, the Stratonovich formulation can be interpreted by Ito's integrals to avoid the semi-martingale condition as follows:

Lemma 30. *Under assumption 6, a stochastic process u in $L^\infty(\Omega\times[0,T]\times L^p(\mathbb{R}^d))$ is a weak solution of the stochastic transport equation* (5) *with initial value u_0 if and only if, for every test function ϕ on* $(\mathbb{R}^d)$*,*

- *the process $\int_{\mathbb{R}^d}\phi(x)u(\omega,t,x)\,dx$ has a continuous $\mathcal{F}$-adapted version;*

- *a.e. on $\Omega \times [0,T]$, it satisfies*

$$\int_{\mathbb{R}^d} u(t,x)\phi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,\phi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\,\mathrm{div}(\mathbf{b}\phi)(s,x)\,dx\,ds$$
$$+ \int_0^t \Big(\int_{\mathbb{R}^d} u(s,x)\partial_i\phi(x)\,dx\Big) dB_s^i + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\phi(x)\,dx\,ds$$

Proof. We start remembering the relation between Ito's and Stratonovich's integral:

$$\int_0^t X \circ dB_s^i = \int_0^t X\,dB_s^i + \frac{1}{2}\langle X, B^i\rangle_t \qquad (7)$$

where $\langle\cdot,\cdot\rangle_t$ represents the joint quadratic variation, for an adapted semi-martingale X. Then, to conclude, we have just to show the following

$$\int_0^t \int_{\mathbb{R}^d} u(s,x)\partial_i^2\phi(x)\,dx\,ds = \Big\langle \int_0^{\cdot} u(\cdot,x)\partial_i\phi(x)\,dx; B_\cdot^i\Big\rangle_t.$$

From the characterization of $\int u(t,x)\partial_i\phi(x)\,dx$ in (6), it follows, from every test function ϕ on $\mathbb{R}^d$, that

$$\Big\langle \int_{\mathbb{R}^d} u(\cdot,x)\partial_i\phi(x)\,dx; B_\cdot^i\Big\rangle_t = \Big\langle \sum_{j=1}^d \int_0^{\cdot} \Big(\int_{\mathbb{R}^d} u(s,x)\partial_i\partial_j\phi(x)\,dx\Big) \circ dB_s^j; B_\cdot^i\Big\rangle_t$$
$$= \int_0^t \Big(\int_{\mathbb{R}^d} u(s,x)\partial_i^2\phi(x)\,dx\Big)\,ds.$$

□

Theorem 31. *Under assumption 6, there exists a weak solution of the transport equation* (5) *with initial value u_0.*

Proof. Let $\{\rho_\epsilon\colon \epsilon \in (0,1]\}$ be a mollifier on $\mathbb{R}\times\mathbb{R}^d$ and $\{\theta_\epsilon\colon \epsilon \in (0,1]\}$ be a cut-off on $\mathbb{R}^d$. We start extending the vector field $\mathbf{b}$ on $\mathbb{R}\times\mathbb{R}^d$ by imposing $\mathbf{b}(t,x) = 0$ for any t not in $[0,T]$. Then, we define $\mathbf{b}_\epsilon := (\mathbf{b}\theta_\epsilon) * \rho_\epsilon$ and $u_{0,\epsilon} := (u_0\theta_\epsilon) * \rho_\epsilon$. In particular, $u_{0,\epsilon}$ is in $C_c^\infty(\mathbb{R}^d)$, $\mathbf{b}_\epsilon$ in $C^\infty(\mathbb{R}\times\mathbb{R}^d)$ and

$$\begin{aligned} u_{0,\epsilon} &\to u_0 \quad \text{on } L^1_{\mathrm{loc}}(\mathbb{R}^d) \\ \mathbf{b}_\epsilon &\to \mathbf{b} \quad \text{on } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d) \qquad (*) \\ \mathrm{div}(\mathbf{b}_\epsilon) &\to \mathrm{div}(\mathbf{b}) \quad \text{on } L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d) \end{aligned}$$

thanks to theorem 35 and the fact that $\mathrm{div}(\mathbf{b}_\epsilon) = \big(\mathrm{div}(\mathbf{b})\big) * \rho_\epsilon$.
Moreover, we notice that $\mathbf{b}_\epsilon$ is globally Lipschitz continuous. For every fixed ϵ in $(0,1]$, we can then apply theorem 28 to show that there exists a solution $u_\epsilon := u_{0,\epsilon}(\mathbf{\Phi}_t^{-1})$ of the

transport equation

$$\begin{cases} d_t u_\epsilon + \mathbf{b}_\epsilon \cdot \nabla u_\epsilon dt + \nabla u \circ dB = 0 \text{ on } \mathbb{R} \times \mathbb{R}^d \\ u_\epsilon = u_{0,\epsilon} \text{ on } \mathbb{R}^d. \end{cases}$$

with $\mathbf{\Phi}(t,x)$ the stochastic flow associated to the vector field $\mathbf{b}_\epsilon$. Now, by characterization of a solution in terms of $\mathbf{\Phi}$, we find that

$$\|u_\epsilon\|_{L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$$

and thus, that u_ϵ is bounded in $L^2(\Omega \times [0,T] \times B(0,N))$ uniformly on ϵ, for every fixed N in $\mathbb{N}$. From the Eberlein-Shmulyan theorem, it follows that there exists a function u in $L^2(\Omega\times[0,T]\times B(0,N))$ and a subsequence $\{u_n\}_{n\in\mathbb{N}}$ of $\{u_\epsilon\}$ such that u_n converges weakly to u.
Fixed a test function ϕ on $\mathbb{R}^d$, define now for simplicity

$$u_n(\phi)(t) := \int_{\mathbb{R}^d} u_n(t,x)\phi(x)\, dx$$

and notice that $u_n(\phi)$ is adapted and in $L^2(\Omega \times [0,T])$. Then, it makes sense to consider

$$F_n(\phi)(t) := u_n(\phi)(t) - u_{0,n}(\phi) - \int_0^t u_n\big(\mathrm{div}(\mathbf{b}_n\phi) + \frac{1}{2}\Delta\phi\big)(s)\, ds - \int_0^t u_n(\nabla\phi)(s)\cdot dB_s$$

for every t in $[0,T]$. In particular, $F_n = 0$ for every n in $\mathbb{N}$, since a solution of the equation (5) is also a weak solution of the same problem. To finish, we want to show that $F_n(\phi)$ converges weakly to

$$u(\phi)(t) - u_0(\phi) - \int_0^t u\big(\mathrm{div}(\mathbf{b}\phi) + \frac{1}{2}\Delta\phi\big)(s)\, ds - \int_0^t u(\nabla\phi)(s)\cdot dB_s$$

on $L^2(\Omega \times [0,T])$.
Firstly, notice that $u_n(\phi)$ and $u_{0,n}(\phi)$ converges weakly in $L^2(\Omega\times[0,T])$ to $u(\phi)$ and $u_0(\phi)$, respectively (in particular $u_{0,n}(\phi)$ is a constant and thus converges since $\Omega\times[0,T]$ has finite measure). Then, it follows from $(*)$ that

$$\mathrm{div}(\mathbf{b}_n\phi) + \frac{1}{2}\Delta\phi \to \mathrm{div}(\mathbf{b}\phi) + \frac{1}{2}\Delta\phi \quad \text{on } L^1([0,T]\times\mathbb{R}^d)$$

and hence, that

$$\int_0^t u_n(\mathrm{div}(\mathbf{b}_n\phi) + \frac{1}{2}\Delta\phi)(s)\, ds \overset{w}{\to} \int_0^t u(\mathrm{div}(\mathbf{b}\phi) + \frac{1}{2}\Delta\phi)(s) \quad \text{on } L^2(\Omega\times[0,T]).$$

Moreover, the space $L^2_{AD}(\Omega \times [0,T])$ of all the adapted L^2-integrable processes is closed, and thus weakly closed, as a subset of $L^2(\Omega \times [0,T])$. Then, since $u_n(\phi) \overset{w}{\to} u(\phi)$ on

$L^2(\Omega \times [0,T])$, we find that $u(\phi)$ is adapted, too and that it makes sense to consider the Ito's integral of $u(\phi)$.
Furthermore, the Ito's integral is a continuous operator from the space $L^2_{AD}(\Omega \times [0,T])$ to $L^2(\Omega \times [0,T])$ by Ito's isometry and hence,

$$\int_0^t u_n(\partial_i \phi)\, dB_s^i \overset{w}{\to} \int_0^t u(\partial_i \phi)\, dB_s^i \quad \text{on } L^2(\Omega \times [0,T]).$$

By uniqueness of the weak limit in $L^2(\Omega \times [0,T])$, we find that

$$0 \,=\, u_(\phi) - u_0(\phi) - \int_0^t u\big(\mathrm{div}(\mathbf{b}\phi) + \frac{1}{2}\Delta\phi\big)\, ds - \sum_{i=1}^d \int_0^t u(\partial_i \phi)\, dB_s^i$$

or, equivalently, that

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\phi(x)\, dx \,=\, & \int_{\mathbb{R}^d} u_0(x)\,\phi(x)\, dx \,+\, \int_0^t \int_{\mathbb{R}^d} u(s,x)\,\mathrm{div}(\mathbf{b}\phi)(s,x)\, dx\, ds \\ & + \int_0^t \Big(\int_{\mathbb{R}^d} u(s,x)\nabla\phi(x)\, dx\Big) dB_s \,+\, \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u(s,x)\Delta\phi(x)\, dx\, ds\end{aligned}$$

a.e. on $\Omega \times [0,T]$. To show that $\int_{\mathbb{R}^d} u(t,x)\phi(x)\, dx$ has a continuous version, just use the last expression noticing that the RHS is a continuous process.
Finally, we can conclude applying lemma 30. □

7 Appendix

Mollification Theory

Theorem 32 (Smooth Approximation by Convolution)**.** *Let f be a function in $L^1_{loc}(\mathbb{R}^d)$ and ϕ a test function on $\mathbb{R}^d$. Then,*

1. *$f * \phi$ is a smooth function;*
2. *for every $i = 1, \dots, d$, $\partial_i(f * \phi) = f * \partial_i(\phi)$;*
3. *if f is in $L^p_{loc}(\mathbb{R}^d)$ for some $p \in [1, \infty)$, then $f * \phi$ is in $L^p_{loc}(\mathbb{R}^d)$ and*

$$\|f * \phi\|_{L^p(K)} \le \|f\|_{L^p(K')} \|\phi\|_{L^1(\mathbb{R}^d)}$$

where $K' = K + supp(\phi)$.

Proof. Fixed a point x_0 in $\mathbb{R}^d$, we want to show that $f * \phi$ is continuous there. For every

x in $\mathbb{R}^d$,

$$|f*\phi(x) - f*\phi(x_0)| \;=\; |\int_{\mathbb{R}^d} f(y)\big(\phi(x-y) - \phi(x_0-y)\big)\,dy| \;\leq$$
$$\leq \int_{\mathbb{R}^d} |f(y)\big(\phi(x-y) - \phi(x_0-y)\big)|\,dy.$$

Now, notice that the point-wise limit $\lim|f(y)\big(\phi(x-y) - \phi(x_0-y)\big)|$ exists for every y and that $|f(y)\big(\phi(x-y) - \psi(x_0-y)\big)| \leq |f(y)|\,|\phi(x-y) - \phi(x_0-y)|$.
Then, we can apply the dominated convergence theorem and obtain

$$\lim|f*\phi(x)f*\psi(x_0)| \;=\; \int_{\mathbb{R}^d} |f(y)\lim\big(\phi(x-y) - \psi(x_0-y)\big)|\,dy \;=\; 0.$$

We have thus proven that $f*\phi$ is continuous.
From this point further, we will just prove the equality 2). The smoothness of $f*\phi$ follows easily from that, since $\partial_i(\phi)$ is again in C_c^∞ and what we have already proven.
Notice that $\partial_i f(y)\phi(x-y)$ is well-defined for every y in $\mathbb{R}^d$ and that

$$|\partial_i f(y)\phi(x-y)| \;\leq\; |f(y)|\,|\partial_i\phi(x-y)| \;\leq\; C|f(y)|$$

since $\partial_i\phi(x-y)$ is continuous on a compact support. Then, we can apply again the dominated convergence theorem and obtain

$$\partial_i(f*\phi) \;=\; \partial_i \int_{\mathbb{R}^d} f(y)\phi(x-y)\,dy \;=\; \int_{\mathbb{R}^d} f(y)\partial_i\phi(x-y)\,dy \;=\; f*\partial_i\phi.$$

3). Assume that $p \in (1,\infty)$. For every x in $\mathbb{R}^d$,

$$\big|f*\phi(x)\big| \;=\; \Big|\int f(y)\phi(x-y)\,dy\Big| \;\leq\; \int |\phi(x-y)|^{1/q}|\phi(x-y)|^{1/p}|f(y)|\,dy \;\leq$$
$$\leq \|\phi\|_{L^1(\mathbb{R}^d)}^{1/q}\Big(\int |\phi(x-y)|\,|f(y)|^p\,dy\Big)^{1/p}$$

Hence,

$$\|f*\phi\|_{L^p(K)} \;=\; \Big[\int_K \|\phi\|_{L^1(\mathbb{R}^d)}^{p/q}\int_{K'} |\phi(x-y)|\,|f(y)|^p\,dy\,dx\Big]^{1/p} \;=$$
$$= \|\phi\|_{L^1(\mathbb{R}^d)}^{1/q}\Big[\int_{K'}\Big(\int_K |\phi(x-y)|\,dx\Big)\,|f(y)|^p\,dy\Big]^{1/p} \;\leq\; \|\phi\|_{L^1(\mathbb{R}^d)}\Big[\int_{K'} |f(y)|^p\,dy\Big]^{1/p}.$$

If $p=1$, just notice that

$$\|f*\phi\|_{L^1(K)} \;=\; \int_K\int_{K'} |\phi(x-y)|\,|f(y)|\,dy\,dx \;=$$
$$= \int_{K'}\Big(\int_K |\phi(x-y)|\,dx\Big)\,|f(y)|\,dy \;\leq\; \|\phi\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)}.$$

□

Definition 33. Let ρ be a function in $C_c^\infty(\mathbb{R}^d)$ such that

- ρ is non-negative;
- $\int \rho(x)\, dx = 1$;
- $\text{supp}(\rho) \subseteq \overline{B(0,1)}$.

A **mollifier** on $\mathbb{R}^d$ is a family of functions $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$, defined by

$$\rho_\epsilon(x) := \frac{1}{\epsilon^d}\rho(\frac{x}{\epsilon}) \qquad \forall\, x \in \mathbb{R}$$

Lemma 34 (Existence of a mollifier)**.** *There exists a mollifier $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ on $\mathbb{R}^d$ such that*

$$supp(\rho_\epsilon) = \overline{B(0,\epsilon)}.$$

Proof. We start defining the function

$$\rho(x) := \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & \text{if } |x| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

where C is a constant chosen to make the integral $\int \nu(x)\, dx = 1$.
It's easy to check that $\{\rho_\epsilon\}$ is a mollifier with the property

$$\text{supp}(\nu_\epsilon) = \overline{B(0,\epsilon)} \qquad \forall\, \epsilon > 0.$$

□

Theorem 35 (Mollification's Properties)**.** *Let $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ be a mollifier on $\mathbb{R}^d$ and f a function on $\mathbb{R}^d$. Then,*

1. *$f * \rho_\epsilon \to f$ a.e. on $\mathbb{R}^d$;*
2. *If $f \in L^p(\mathbb{R}^d)$ for some $p \in [1,\infty)$, then $f * \rho_\epsilon \to f$ on $L^p(\mathbb{R}^d)$.*

Proof. See pages $630 - 631$ of [3]. □

Bochner Spaces on Finite Time-intervals

As we have seen, the analysis of the transport equation with low regularities requires a better understanding of the concept of integrability in presence of a time dependence.
For example, we could need a function $f(t,x)$ that is continuous and bounded on the variable x and integrable on t. Even if it is possible to define a function with this properties taking them separately just fixing the variable we are not considering time by time, the

more natural way to do that is to link them extending the idea of integrability over $\mathbb{R}$ to the more general class of the Banach spaces. At that point, we could consider $f(t,x)$ as function in t valued over some functional space on x, making the integrability on t depending on the properties on x.
Extended the idea of integrability for this more general spaces, it is then easy to define a new class of L^p space, essentially in the same way the old ones are defined. These are the Bochner spaces.

In this section, we will give a brief account of the argument, restricting our attention on the finite-time case and on the results we will need further. In particular, we will focus on finding when a Bochner space owns the classical properties of an L^p-space, e.g. Holder's inequality, and when it is possible to extend the mollification results we have already proven in this more general context.

For a more in-depth analysis of the Bochner spaces in more abstract Banach spaces and for the proofs of the following results, we suggest the reader to see [4].
Definition 36. Let X be a Banach space. A function $s\colon [0,T] \to X$ is **simple** if it can be written as

$$s(t) := \sum_{i=1}^{N} x_k \mathbb{1}_{E_i}(t)$$

where $x_1, \dots, x_N$ are points in X and $E_1, \dots, E_N$ are measurable subsets of $[0,T]$.
Definition 37. Let X be a Banach space. A function $f\colon [0,T] \to X$ is **strongly measurable** on X if there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions on X such that

$$\lim_n \|s_n(t) - f(t)\| \;=\; 0 \text{ a.e. on } [0,T].$$

Definition 38. Let X be a Banach space and $p \in [0,\infty]$. The **Bochner space** $L^p(0,T;X)$ is the the family of all the functions $f\colon [0,T] \to X$ such that

- f is strongly measurable;
- the function $t \mapsto \|f(t)\|$ is in $L^p([0,T])$.

Theorem 39 (Holder's Theorem fo Bochner Spaces)**.** *Let X be a Banach space, $p \in [0,\infty]$, f a function in $L^p(0,T;X)$ and g a function in $L^q(0,T;X')$. Then, $\langle f,g\rangle$ is in $L^1(0,T,\mathbb{R})$ and*

$$\|\langle f,g\rangle\|_{L^1(0,T,\mathbb{R})} \;\le\; \|f\|_L^p(0,T;X)\,\|g\|_{L^q(0,T;X')}$$

Proof. See [4]. □

Lemma 40. *Let Y, X be two Banach spaces such that $Y \subseteq X$ and*

$$\|\cdot\|_X \le C\|\cdot\|_Y$$

for some constant $C > 0$. Then, $L^p(0,T;Y) \subseteq L^p(0,T;X)$.

Proof. Consider a function f in $L^p(0,T;Y)$. Firstly, notice that

$$\int_0^T \|f\|_X^p \, dt \le C^p \int_0^T \|f\|_Y^p \, dt < \infty.$$

Now, we have just to show that f is strongly measurable also on X. But this is clear reusing the same sequence of simple functions the measurability on Y guarantees. $\square$

Theorem 41 (Mollification's properties on Bochner spaces). *Let $p \in [1,\infty)$, f a bounded function in $L^1(0,T;L^p_{\text{loc}}(\mathbb{R}^d))$ and $\{\rho_\epsilon \colon \epsilon \in (0,1]\}$ a mollifier on $\mathbb{R}^d$. Then,*

1. *the function $f * \rho_\epsilon$ is in $L^1(0,T,C^1_b(\mathbb{R}^d))$;*
2. *$f * \rho_\epsilon \to f$ on $L^1(0,T;L^p(\mathbb{R}^d))$.*

Proof. 1). For brevity, we denote $f_\epsilon := f * \rho_\epsilon$ on $[0,T] \times \mathbb{R}^d$. We start noticing that is it possible to bound the C^1_b-norm by

$$\|f_\epsilon\|_{C^1_b(\mathbb{R}^d)} \le C\|f\|_{L^p(B(0,\epsilon))}$$

where C is a constant independent from f and t.
Since f is strong measurable on $L^p_{\text{loc}}(\mathbb{R}^d)$ for hypothesis, we can consider now a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$ on $L^p_{\text{loc}}(\mathbb{R}^d)$ such that

$$\|s_n(t) - f(t)\|_{L^p(K)} \to 0 \text{ a.e. on } [0,T]$$

for every compact K in $\mathbb{R}^d$ and define for every n in $\mathbb{N}$,

$$S_n(t,x) := s_n(t,x) * \rho_\epsilon(x) \quad \text{on } [0,T] \times \mathbb{R}^d.$$

In particular, $\{S_n\}_{n\in\mathbb{N}}$ is a sequence of simple functions on $C^1_b(\mathbb{R}^d)$ such that

$$\|S_n - f_\epsilon\|_{C^1_b}(t) = \|(s_n - f) * \rho_\epsilon\|_{C^1_b}(t) \le C\|s_n - f\|_{L^p(K)} \xrightarrow{n} 0 \text{ a.e. on } [0,T].$$

Hence, f_ϵ is strongly measurable on $C^1_b(\mathbb{R}^d)$.
To conclude, we have to show that the $L^1(0,T,C^1_b(\mathbb{R}^d))$-norm of f_ϵ is finite. But this is clear, since

$$\|f_\epsilon\|_{L^1(0,T;C^1_b(\mathbb{R})^d)} = \int_0^T \|f_\epsilon\|_{C^1_b(\mathbb{R}^d)} \, dt \le \int_0^T C\|f\|_{L^p(K)} \, dt = C\|f\|_{L^1(0,T;L^p(\mathbb{R}^d))} < \infty.$$

2). Firstly, we use lemma 40 to show that f_ϵ is in $L^1(0,T;L^p_{\text{loc}}(\mathbb{R}^d))$. Then, we notice that

$$\|f_\epsilon - f\|_{L^q(K)}(t) \le \|f_\epsilon\|_{L^q(K)} - \|f\|_{L^q(K)} \le 2\|f\|_{L^q(K)}$$

and hence, that we can apply dominated Convergence theorem to

$$\lim_{\epsilon} \int_0^T \|f_\epsilon - f\|_{L^q(K)}(t)\, dt \;=\; \int_0^T \lim_{\epsilon} \|f_\epsilon - f\|_{L^q(K)}(t)\, dt \;=\; 0$$

where the last expression is equal to 0 by theorem 35. □

Theorem 42. *Let X be a reflexive Banach Space. Then, the spaces $L^p(0,T;X)'$ and $L^q(0,T;X')$ are isometrically isomorphic through*

$$g \in L^q(0,T;X') \mapsto \Big(f \in L^p(0,T;X) \mapsto \int_0^T \langle f, g\rangle(t)\, dt\Big)$$

Proof. See pages $17 - 19$ of [4]. □

Theorem 43. *Let X be a Banach Space and $p \in [1, +\infty]$. Then, $L^p(0,T;X)$ is a Banach space.*
If moreover $p \neq \infty$ and X is separable, then $L^p(0,T;X)$ is separable.

Proof. See page 9 of [4]. □

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