Weak well-posedness and weak discretization error for stable-driven SDEs with Lebesgue drift

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Overview

Main equation

For a fixed $T > 0$, we are interested in the weak well-posedness and the Euler-Maruyama dicretization for the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x, \qquad \forall t \in [0, T], \tag{1}$$

where $b \in L^q([0,T], L^p(\mathbb{R}^d))$ and Z_t is a d-dimensional α-stable process, whose spectral measure is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{S}^{d-1}$.

We denote by $p_\alpha(t, x)$ the density of the noise Z_t.

Assumptions

We work under the integrability condition

$$\gamma := 1 - \alpha - \frac{d}{p} - \frac{\alpha}{q} > 0, \tag{2}$$

which is the extension of the Krylov-Röckner condition for stable-driven SDEs.

State of the art

Well-posedness

- For $b \in L^q - L^p$ and SBM driving noise, [KR05] proves strong well-posedness under $\frac{d}{p} + \frac{2}{q} < 1$
- For time-homogeneous $b \in L^p$ with strictly stable noise, [Por94] and [PP95] prove weak well-posedness under $\frac{d}{p} < 1$
- With appropriate assumptions the Bessel potentials of b, [XZ20] prove strong well-posedness and give heat kernel estimates

Euler scheme

We are interested in the weak error, which is the difference between the density of the process in (1) and that of the process in (9). Denote

$$\mathcal{E}_{t,x,y}(h) := \frac{|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)|}{p_\alpha(t, y - x)}. \tag{3}$$

→ Useful when integrating against singular test functions !

Previous results for weak/strong error rates:

- If $b \in L^\infty$ and the noise is an SBM, then $d_{TV}(X^h, X) \le Ch^{\frac{1}{2}}$ ([BJ22]) and if, moreover, $b \in \dot{W}^{\beta}_{m}$, then $\| \sup_t |X^h_t - X_t| \|_{L^p} \le Ch^{\frac{\beta+1}{2}-}$ ([DGL22])
- If $b \in \mathcal{C}^\theta = \mathbb{B}^\theta_{\infty,\infty}$, we expect $\mathcal{E}_{t,x,y}(h) \le Ch^{\frac{\theta+1}{\alpha}}$ (see [Hol22], work in progress)
- If $b \in L^q - L^p$ and the noise is an SBM, then:
  - Weak error : $\mathcal{E}_{t,x,y}(h) \le Ch^{\frac{\gamma}{2}}$ with $\gamma = 1 - \frac{d}{p} - \frac{2}{q}$ ([JM23], same methodology as here)
  - Strong error : $\| \sup_t |X^h_t - X_t| \|_{L^p} \le Ch^{\frac{1}{2}} |\ln(h)|$ ([LL22], using stochastic sewing)

Threorem 1: Weak well-posedness for the diffusion

Existence

Under (2), there exists a Martingale Problem solution (X_t) to (1).
Moreover, for each $t \in (0, T]$, X_t admits a density $y \mapsto \Gamma(0, t, x, y)$ s.t. $\exists C := C(b, T) < \infty : \forall t \in (0, T], \forall (x, y) \in \mathbb{R}^d$,

$$\Gamma(0, x, t, y) \le C p_\alpha(t, y - x). \tag{4}$$

It also enjoys the following Duhamel representation : $\forall t \in (0, T], \forall (x, y) \in \mathbb{R}^d$,

$$\Gamma(0, x, t, y) = p_\alpha(t, y - x) - \int_0^t \mathbb{E}_{x,0} \left[b(r, X_r) \cdot \nabla_y p_\alpha(t - r, y - X_r) \right] \mathrm{d}r. \tag{5}$$

Uniqueness

We also obtain uniqueness of the marginal laws, i.e.

There exists a unique function Γ satisfying (4) and (5)

Heat kernel estimates for the density

Hölder regularity in the forward space variable: $\exists C := C(b, T) < \infty : \forall t \in (0, T], \forall (x, y, y') \in \mathbb{R}^d$,

$$|\Gamma(0, x, t, y) - \Gamma(0, x, t, y')| \le C \frac{|y - y'|^\gamma \wedge t^{\frac{\gamma}{\alpha}}}{t^{\frac{\gamma}{\alpha}}} \left(p_\alpha(t, y - x) + p_\alpha(t, y' - x) \right).$$

Definition of the Euler scheme

We will consider a cutoff for the drift. The possible cutoffs we consider are the following:

- If $p = q = \infty$, we take $b_h = \bar{b}_h = b$
- Otherwise, we set

$$b_h(t, y) := \frac{\min\left\{ |b(t, y)|, B h^{-\frac{d}{\alpha p} - \frac{1}{q}} \right\}}{|b(t, y)|} b(t, y) 1_{|b(t,y)|>0}, \qquad (t, y) \in [0, T] \times \mathbb{R}^d \tag{6}$$

$$\bar{b}_h(t, y) = \frac{\min\left\{ |b(t, y)|, B h^{\frac{1}{\alpha} - 1} \right\}}{|b(t, y)|} b(t, y) 1_{t \ge h, |b(t,y)|>0}, \qquad (t, y) \in [0, T] \times \mathbb{R}^d \tag{7}$$

for some constant $B > 0$ not depending on any of the parameters.

Notations for the Euler scheme:

- n time steps over $[0, T]$, with step size $h = T/n$. We denote $\forall k \in \{0, ..., n\}, t_k = kh$
- $\forall k \in \{0, ..., n-1\}$, $U_k \sim \mathcal{U}((kh, (k+1)h))$ independently of Z will be the evaluation point in time of b_h (resp. $\bar{b}_h$) for measurability concerns
- $\tau^h_s = h \lfloor \frac{s}{h} \rfloor \in (s - h, s]$, which is the last point of the time grid before s.

We then define the scheme as

$$X^h_{t_{k+1}} = X^h_{t_k} + (Z_{t_{k+1}} - Z_{t_k}) + h b_h(U_k, X^h_{t_k}), \tag{8}$$

and its time interpolation is defined as

$$\mathrm{d}X^h_t = b_h(U_{\lfloor \frac{t}{h} \rfloor}, X^h_{\tau^h_t})\mathrm{d}t + \mathrm{d}Z_t \tag{9}$$

(resp. the same dynamics with $\bar{b}_h$ in place of b_h).

Theorem 2: Weak convergence rate for the stable-driven Euler-Maruyama scheme

Density of the interpolated scheme

The solution to

$$\mathrm{d}X^h_t = b_h(U_{\lfloor \frac{t}{h} \rfloor}, X^h_{\tau^h_t})\mathrm{d}t + \mathrm{d}Z_t \tag{10}$$

(resp. the same with $\bar{b}_h$) started from x at time 0 admits at time $t \in (0, T]$ a density with respect to the Lebesgue measure on $\mathbb{R}^d$ denoted by $y \mapsto \Gamma^h(0, x, t, y)$ (resp. $y \mapsto \bar{\Gamma}^h(0, x, t, y)$) and $\forall y \in \mathbb{R}^d$,

$$\Gamma^h(0, x, t, y) = p_\alpha(t, y - x) - \int_0^t \mathbb{E}_{x,0} \left[b_h(U_{\lfloor \frac{r}{h} \rfloor}, X^h_{\tau^h_r}) \cdot \nabla_y p_\alpha(t - r, y - X^h_r) \right] \mathrm{d}r \tag{11}$$

(resp. the same equation holds with Γ^h and b_h replaced by $\bar{\Gamma}^h$ and $\bar{b}_h$).

Convergence rate

Assume that (2) holds.
Then, there exists a constant $C < \infty$ s.t. for all $h = T/n$ with $n \in \mathbb{N}^*$, and all $t \in (0, T]$, $x, y \in \mathbb{R}^d$

$$\mathcal{E}_{t,x,y}(h) \le C h^{\frac{\gamma}{\alpha}} \tag{12}$$

Remark about the rate of convergence

It would seem natural to obtain convergence with a rate $h^{\frac{\gamma+1}{\alpha}}$, using an integration by parts

$$\begin{aligned} \Gamma(0, x, t, y) &= p_\alpha(t, y - x) - \int_0^t \int b(z) \cdot \nabla_y p_\alpha(t - s, y - z) \Gamma(0, x, s, z) \mathrm{d}z\mathrm{d}s \\ &= p_\alpha(t, y - x) - \int_0^t \int \nabla_z \Gamma(0, x, s, z) \cdot b(z) p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \\ &\quad - \int_0^t \int \Gamma(0, x, s, z) \mathrm{div}(b(z)) \cdot p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \end{aligned}$$

→ Works with the expansion of (1), but not with that of (9), because good assumptions on $\mathrm{div}(b)$ do not translate into good assumptions on $\mathrm{div}(b_h)$.

→ Integrating against a $\mathcal{C}^1$ test function would work but doesn't allow to use a Grönwall lemma.

→ The full parametrix expansion solves the former but requires to compensate *multiple* gradients.

Sketch of the proof

Establish estimates for the density of the scheme

Duhamel representation:

$$\Gamma^h(t_k, x, t, y) = p_\alpha(t - t_k, y - x) - \int_{t_k}^t \mathbb{E}_{x,t_k} \left[b_h(U_{\lfloor \frac{r}{h} \rfloor}, X^h_{\tau^h_r}) \cdot \nabla_y p_\alpha(t - r, y - X^h_r) \right] \mathrm{d}r$$

Compute HK estimates: $\forall k \in \{0, ..., n-1\}, t \in (t_k, T], x, y, y' \in \mathbb{R}^d$,

$$\Gamma^h(t_k, x, t, y) \le C p_\alpha(t - t_k, y - x) \tag{13}$$

$$\begin{aligned} &|\Gamma^h(t_k, x, t, y') - \Gamma^h(t_k, x, t, y)| \\ &\qquad \le C \frac{|y - y'|^\gamma \wedge (t - t_k)^{\frac{\gamma}{\alpha}}}{(t - t_k)^{\frac{\gamma}{\alpha}}} \big(p_\alpha(t - t_k, y - x) + p_\alpha(t - t_k, y' - x) \big). \end{aligned} \tag{14}$$

For all $0 \le k < \ell < n, \ t \in [t_\ell, t_{\ell+1}], x, y \in \mathbb{R}^d$,

$$|\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t_\ell, y)| \le C \frac{(t - t_\ell)^{\frac{\gamma}{\alpha}}}{(t_\ell - t_k)^{\frac{\gamma}{\alpha}}} p_\alpha(t - t_k, y - x), \tag{15}$$

and the same estimations hold with $\bar{\Gamma}^h$ replacing Γ^h.

Well-posedness: taking the limit $h \to 0$

We obtain well-posedness through a tightness argument, and we obtain (5) thanks to the Ascoli-Arzelà theorem.

Uniqueness of the marginals is obtained using a brute-force analysis of (5).

Convergence rate: comparing the Duhamel expansions

We split the error as follows:

$$\begin{aligned} \mathcal{E}_{t,x,y}(h) = &\int_0^t \int [\Gamma(0, x, s, z) - \Gamma^h(0, x, s, z)] b(s, z) \cdot \nabla_y p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \\ &+ \int_{t_1}^{\tau^h_t - h} \int \Gamma^h(0, x, s, z)(b(s, z) - b_h(s, z)) \cdot \nabla_y p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \\ &+ \int_{t_1}^{\tau^h_t - h} \int [\Gamma^h(0, x, s, z) - \Gamma^h(0, x, \tau^h_s, z)] b_h(s, z) \cdot \nabla_y p_\alpha(t - s, y - z) \mathrm{d}z\mathrm{d}s \\ &+ \int_{t_1}^{\tau^h_t - h} \mathbb{E}_{x,0} \Big[b_h(U_{\lfloor s/h \rfloor}, X^h_{\tau^h_s}) \cdot \\ &\qquad \times (\nabla_y p_\alpha(t - U_{\lfloor s/h \rfloor}, y - X^h_{\tau^h_s}) - \nabla_y p_\alpha(t - s, y - X^h_s)) \Big] \mathrm{d}s \\ &+ \text{minor terms} \end{aligned}$$

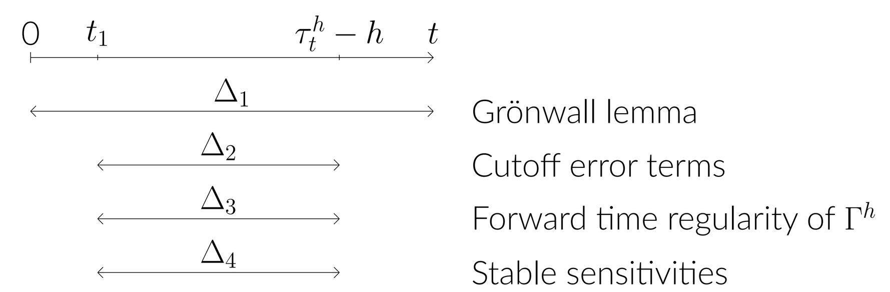

Figure 1. Splitting of the error

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