

Weak discretization error techniques for singular drift SDEs

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Notations

We consider the formal SDE

$$dX_t = b(t, X_t)dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d, \quad (\text{E})$$

where

- $b \in L^r([0, T], \mathbb{B}_{p,q}^\beta(\mathbb{R}^d, \mathbb{R}^d))$ is some *singular* drift.
- Z_t is an α -stable symmetric process $\alpha \in (1, 2]$, with spectral measure equivalent to the Lebesgue measure on \mathbb{S}^{d-1} .

We denote by \mathcal{L}^α the generator of the noise, p_α its density and P_t^α the associated semi-group :

$$\forall \phi, P_t^\alpha[\phi](x) := \int_{\mathbb{R}^d} p_\alpha(t, y - x)\phi(y)dy$$

Heuristics for well-posedness

For $b \in \mathbb{B}_{\infty, \infty}^{\beta}(\mathbb{R}^d, \mathbb{R}^d) \approx \mathcal{C}^{\beta}$, we work on the PDE

$$(\partial_t + b \cdot \nabla + \mathcal{L}^{\alpha})u(t, x) = f(t, x) \text{ on } [0, T] \times \mathbb{R}^d \quad (1)$$

where f is some smooth source term.

- If $\beta > 0$, the condition $\alpha + \beta > 1$ is required to give a meaning to ∇u

Heuristics for well-posedness - $\beta > 0$

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- If $\beta > 0$, the condition $\alpha + \beta > 1$ is required to give a meaning to ∇u
 - [Por94], [PP95] : $b \in L^p \hookrightarrow \mathbb{B}_{\infty, \infty}^{-d/p}$, $\alpha \in (1, 2)$, weak WP under $\frac{d}{p} < \alpha - 1$.
 - [KR05] : $b \in L^q - L^p$, $\alpha = 2$, strong WP under $\frac{d}{p} + \frac{2}{q} < 1$.
 - [Pri12] : $b \in \mathcal{C}^{\beta}$, $\alpha \in (1, 2)$: strong WP under $\beta + \frac{\alpha}{2} > 1$.
 - [CZZ21] : $b \in \mathcal{C}^{\beta}$, $\alpha \in (0, 2)$: weak WP under $\beta + \alpha > 1$.

Heuristics for well-posedness - $\beta < 0$

For $b \in \mathbb{B}_{\infty, \infty}^{\beta}(\mathbb{R}^d, \mathbb{R}^d) \approx \mathcal{C}^{\beta}$, we work on the PDE

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where f is some smooth source term.

- If $\beta > 0$, the condition $\alpha + \beta > 1$ is required to give a meaning to ∇u
- If $\beta < 0$, in order to make sense of $b \cdot \nabla u$, we need $\beta + (\beta + \alpha - 1) > 0$

$$\text{i.e. } \beta > \frac{1 - \alpha}{2}$$

- [FIR17] : $b \in \mathcal{C}^{\beta}, \alpha = 2$, WP (virtual solutions) under $\beta > -\frac{1}{2}$.
- [ABM20] : $b \in \mathcal{C}^{\beta}, \alpha \in (1, 2)$, 1D strong WP under $\beta > \frac{1-\alpha}{2}$.
- [CdRM22] : $b \in L^r - \mathbb{B}_{p, q}^{\beta}$, weak WP under

$$2\beta + \alpha - \frac{d}{p} - \frac{\alpha}{r} > 1$$

Proposition ([DD16],[CdRM22])

Assume

$$\alpha \in \left(\frac{1 + \frac{d}{p}}{1 - \frac{1}{r}}, 2 \right) \quad \beta \in \left(\frac{1 - \alpha + \frac{2d}{p} + \frac{2\alpha}{r}}{2}, 0 \right), \quad (\text{GRD})$$

Then, the formal SDE

$$dX_t = b(t, X_t)dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d$$

rewrites

$$X_t = X_0 + \int_0^t b(s, X_s, ds) + Z_t,$$

where $\forall 0 \leq v \leq s \leq T$,

$$b(v, x, s - v) = \int_v^s \int b(r, y) p_\alpha(r - v, y - x) dy dr$$

Discretization of singular drift SDEs

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Aim : approach (E) with

$$X_{t_{k+1}}^h = X_{t_k}^h + hb(t_k, X_{t_k}^h) + (Z_{t_{k+1}} - Z_{t_k}).$$

Or, equivalently, its continuous in time version :

$$dX_t^h = b(\tau_t^h, X_{\tau_t^h}^h)dt + dZ_t$$

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Two main metrics for the error :

- Strong error :

$$\left\| \sup_{t \in (0, T)} |X_t - X_t^h| \right\|_{L^r}, \quad r > 1$$

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- Strong error :

$$\left\| \sup_{t \in (0, T)} |X_t - X_t^h| \right\|_{L^r}, \quad r > 1$$

- Weak error with a test function :

$$|\mathbb{E}_{0,x}[g(X_t^h) - g(X_t)]|$$

Weak error on densities (i.e. $g = \delta_y$) :

$$|\Gamma^h(0, x, t, y) - \Gamma(0, x, t, y)|$$

A few important results

- [BJ20] $b \in L^\infty$ total variation error : $h^{\frac{1}{2}}$.
- [LL21] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$, strong error : $h^{\frac{1}{2}} \ln(h)$.
- [DGL22] $b \in L^\infty \cap \dot{W}_m^\beta$, strong error : $h^{\frac{1+\beta}{2} -}$.
- [JM24] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$, weak error : $h^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}}$.

Sewing techniques for strong error

In a brownian noise setting, the sewing lemma ([Lê20]) allows to bound expressions of the type

$$\mathbb{E} \left[\left| \int_0^t b(s, X_s^h) - b(s, X_{\tau_s^h}^h) ds \right|^r \right], \quad (4)$$

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which appear when computing strong error rates...

...but only when b has sufficient integrability.

Weak error techniques

Let u be a solution to

$$(\partial_s + b(s, x) \cdot \nabla_x + \mathcal{L}^\alpha) u(s, x) = 0 \text{ on } [0, t) \times \mathbb{R}^d, \quad u(t, \cdot) = g \text{ on } \mathbb{R}^d,$$

By Itô's formula,

$$\begin{aligned} \mathcal{E}(g, t, x, h) &= \mathbb{E}_{0,x}[g(X_t^h) - g(X_t)] = \mathbb{E}_{0,x}[u(t, X_t^h) - u(0, x)] \\ &= \mathbb{E}_{0,x} \left[\int_0^t \left(b(r, X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h) \right) \cdot \nabla u(r, X_r^h) \right] dr. \end{aligned}$$

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Historical approach by [MP91] : for $b \in \mathcal{C}^\beta$

- Schauder estimates for u : $\|\nabla u\|_{L^\infty} < \infty$
- Use the regularity of b to control $\left| b(r, X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h) \right|$

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Historical approach by [MP91] : for $b \in \mathcal{C}^\beta$

- Schauder estimates for u : $\|\nabla u\|_{L^\infty} < \infty$
- Use the regularity of b to control $\left| b(r, X_r^h) - b(\tau_r^h, X_{\tau_r^h}^h) \right|$

$$\mathcal{E}(g, t, x, h) \leq C \|\nabla u\|_{L^\infty} \int_0^t \mathbb{E}_{0,x} \left[|X_r^h - X_{\tau_r^h}^h|^\beta \right] dr \leq C \|\nabla u\|_{L^\infty} h^{\frac{\beta}{\alpha}}. \quad (5)$$

Weak error techniques

Compare the Duhamel formulas for Γ^h and Γ :

$$\begin{aligned} & \Gamma^h(0, x, t, y) - \Gamma(0, x, t, y) \\ &= \mathbb{E}_{0,x} \left[\int_0^t b(s, X_s) \cdot \nabla_y p_\alpha(t-s, y - X_s) - b(\tau_s^h, X_{\tau_s^h}^h) \cdot \nabla_y p_\alpha(t-s, y - X_s^h) ds \right] \end{aligned}$$

When b is singular, the main difficulty lies in controlling

$$b(s, X_s) - b(s, X_{\tau_s^h}^h)$$

Weak error techniques

In the splitting of the error, we write the following

$$\begin{aligned} \mathbb{E}_{0,x} & \left[\int_0^t b(s, X_s) \cdot \nabla_y p_\alpha(t-s, y - X_s) - b(s, X_{\tau_s^h}) \cdot \nabla_y p_\alpha(t-s, y - X_{\tau_s^h}) ds \right] \\ & = \int_0^t \int [\Gamma(0, x, s, z) - \Gamma(0, x, \tau_s^h, z)] b(s, z) \cdot \nabla_y p_\alpha(t-s, y - z) dz ds \end{aligned}$$

Weak error techniques

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$$\begin{aligned} \mathbb{E}_{0,x} \left[\int_0^t [b(s, X_s) \cdot \nabla_y p_\alpha(t-s, y - X_s) - b(s, X_{\tau_s^h}) \cdot \nabla_y p_\alpha(t-s, y - X_{\tau_s^h})] ds \right] \\ = \int_0^t \int [\Gamma(0, x, s, z) - \Gamma(0, x, \tau_s^h, z)] b(s, z) \cdot \nabla_y p_\alpha(t-s, y - z) dz ds \end{aligned}$$

We can easily prove

$$|\Gamma(0, x, s, z) - \Gamma(0, x, \tau_s^h, z)| \leq C \frac{(s - \tau_s^h)^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}} p_\alpha(\tau_s^h, z - x) \leq C \frac{h^{\frac{\gamma}{\alpha}}}{(\tau_s^h)^{\frac{\gamma}{\alpha}}} p_\alpha(\tau_s^h, z - x)$$

where γ is the *gap to singularity* :

$$\gamma = \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q} > 0 \quad \text{Lebesgue}$$

$$\gamma = \alpha + \beta - 1 > 0 \quad \text{Holder}$$

$$\gamma = \beta + \frac{\alpha - 1 - \frac{d}{p} - \frac{\alpha}{r}}{2} > 0 \quad \text{Negative Besov}$$

Weak error techniques

It remains to show the boundedness of

$$\int_0^t \int p_\alpha(s, z - x) b(s, z) \nabla_y p_\alpha(t - s, y - z) dz ds.$$

If $b \in \mathfrak{H}$ in space, we have to estimate

$$\|p_\alpha(s, \cdot - x) \nabla_y p_\alpha(t - s, y - \cdot)\|_{\mathfrak{H}^*},$$

and recoup singularities in s and $t - s$, which we want to integrate against $\|b(s, \cdot)\|_{\mathfrak{H}}$.

Theorem ([F., Jourdain, Menozzi, 2024])

For $b \in L^q - L^p$, if

$$\gamma = \alpha - 1 - \frac{d}{p} - \frac{\alpha}{q} > 0,$$

then equation

$$dX_t = b(t, X_t)dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d$$

is weakly well-posed (in the sense of the martingale problem).

Theorem ([F., Jourdain, Menozzi, 2024])

For $b \in L^q - L^p$, we define the Euler scheme by

$$X_{t_{k+1}}^h = X_{t_k}^h + hb_h(U_k, X_{t_k}^h) + (Z_{t_{k+1}} - Z_{t_k}),$$

where b_h is a cut-offed b s.t. $|b_h| \leq h^{-\frac{d}{\alpha p} - \frac{1}{q}}$ and $\forall k, U_k \sim \mathcal{U}(t_k, t_{k+1})$. Then,

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \leq Ch^{\frac{\gamma}{\alpha}} p_{\alpha}(t, y - x) \quad (6)$$

- Hölder : we apply the same ideas to obtain the rate

$$|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \leq Ch^{\frac{\beta+\alpha-1}{\alpha}} p_\alpha(t, y-x)$$

under the assumption

$$\beta + \alpha - 1 > 0$$

and $[\beta < (\alpha - 1)^2$ or (Z_t) has Lebesgue spectral measure]

- Hölder : $|\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| \leq Ch^{\frac{\beta+\alpha-1}{\alpha}} p_\alpha(t, y-x)$
- Negative Besov regularity : let $b \in L^r([0, T], \mathbb{B}_{p,q}^\eta)$, $\eta < 0$ and set

$$\mathfrak{b}(s, z, h) = \frac{1}{h} \int_s^{s+h} \int b(u, y) p_\alpha(u-s, y-z) dy du.$$

Then, $\mathfrak{b}(s, \cdot, h)$ is β -Hölder with $\beta > 0$ and its Hölder modulus

$$|\mathfrak{b}(s, z, h) - \mathfrak{b}(s, z', h)| \lesssim |z - z'|^\beta h^{\frac{1}{r} - \frac{\beta}{\alpha} + \frac{\eta}{\alpha} - \frac{d}{\alpha p}}$$

We then apply the Hölder result to the SDE with drift \mathfrak{b} , and we get

$$\begin{aligned} |\Gamma(0, x, t, y) - \Gamma^h(0, x, t, y)| &\leq Ch^{\frac{\beta+\alpha-1}{\alpha}} h^{\frac{1}{r} - \frac{\beta}{\alpha} + \frac{\eta}{\alpha} - \frac{d}{\alpha p}} p_\alpha(t, y-x) \\ &\leq Ch^{\frac{\eta+\alpha - \frac{\alpha}{r} - \frac{d}{p} - 1}{\alpha}} p_\alpha(t, y-x) \end{aligned}$$

Thank you for your attention.



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