Heat kernel estimates for stable-driven SDEs with distributional drift

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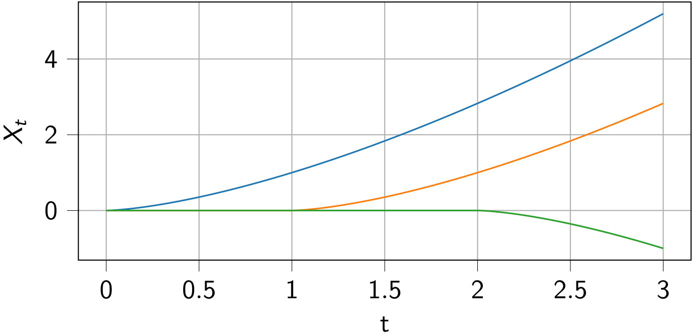
ODE

Consider

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t, \qquad t \geq 0$$

- b Lipschitz $\Rightarrow$ existence of a unique solution
- $b \in \mathcal{C}^\gamma, \gamma \in (0,1) \Rightarrow$ existence of a solution $(\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t, X_0 = 0)$



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- b Lipschitz $\Rightarrow$ existence of a unique solution
- $b \in \mathcal{C}^\gamma, \gamma \in (0,1) \Rightarrow$ existence of a solution $(\mathrm{d}X_t = \sqrt{|X_t|}\mathrm{d}t, X_0 = 0)$
- $b \in L^\infty \Rightarrow$ might not have a solution $(\mathrm{d}X_t = -\mathrm{sgn}(X_t)\mathrm{d}t, X_0 = 0, \mathrm{sgn}(0) = 1)$

Regularization by noise

Consider

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = x \in \mathbb{R}^d, t \geq 0,$$

where (W_t) is a Standard Brownian Motion (SBM).

$\rightarrow$ The impulse of the noise helps exit critical spots.

- $b \in L^\infty$: strongly well-posed [Zvonkin, 1974] [Veretennikov, 1980]
- $b \in L^q(\mathbb{R}_+, L^p(\mathbb{R}^d))$: strongly well-posed for

$$\frac{d}{p} + \frac{2}{q} < 1, \qquad p \geq 2, q > 2$$

[Krylov and Röckner, 2005]

Main SDE

We consider the *formal* SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{E}$$

where

- $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$ is a Besov drift
- Z_t is a symmetric α-stable process, $\alpha \in (1,2)$, with spectral measure equivalent to the Lebesgue measure on $\mathbb{S}^{d-1}$.

We will denote by $\mathcal{L}^\alpha$ the generator of Z, p_α its density and P^α_t the associated semi-group :

$$\forall \phi, P^\alpha_t[\phi](x) := \int_{\mathbb{R}^d} p_\alpha(t, y-x)\phi(y)\mathrm{d}y$$

We will sometimes use $\bar{p}_\alpha(v,z) = \frac{C_\alpha}{v^{\frac{d}{\alpha}}} \frac{1}{\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{d+\alpha}}, v > 0, z \in \mathbb{R}^d$, which satisfies

$$c^{-1}\bar{p}_\alpha \le p_\alpha \le c\bar{p}_\alpha$$

Expectation, heuristics

Take a time homogeneous b.

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + Z_t$$

- We expect that we can define $\int_0^t b(X_s)\mathrm{d}s$ if this term is more regular than the noise Z (which has regularity $\frac{1}{\alpha}$)
- Assume $b \in \mathcal{C}^\beta$ for some $\beta > 0$. Then (informally), $b(X_\cdot) \in \mathcal{C}^{\frac{\beta}{\alpha}}$, so we need

$$1 + \frac{\beta}{\alpha} > \frac{1}{\alpha} \iff \alpha + \beta > 1$$

- If $b \in L^p \subset \mathbb{B}^{-\frac{d}{p}}_{\infty,\infty} = \mathcal{C}^{-\frac{d}{p}}$, the same reasoning gives

$$\frac{d}{p} < \alpha - 1$$

which is consistent with the previous result.

Expectation, heuristics

For $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d,\mathbb{R}^d)) \approx \mathcal{C}^\beta$ with $\beta < 0$, this is not sufficient, because we need to work on the underlying PDE

$$(\partial_t + b \cdot \nabla + \mathcal{L}^\alpha) u(t,x) = f(t,x) \text{ on } [0,T] \times \mathbb{R}^d$$

where $\mathcal{L}^\alpha$ is the generator of the α-stable noise and f some source term.

- $\alpha + \beta > 1 \iff$ we can define ∇u
- By paraproduct rules, if we want to define the generalized functions product $b \cdot \nabla u$, we need $\beta + (\beta + \alpha - 1) > 0$

$$\text{i.e. } \beta > \frac{1-\alpha}{2} \tag{1}$$

- [Delarue and Diel, 2016] : weak 1D existence and uniqueness under $\beta > -\frac{2}{3}$ with structure on b, with $\alpha = 2$
- [Flandoli et al., 2017] : virtual solutions under (1), $\alpha = 2$
- [Athreya et al., 2020] : strong 1D existence and uniqueness under (1)
- [Kremp and Perkowski, 2022] : weak $\mathbb{R}^d$ existence and uniqueness under $\beta > \frac{2-2\alpha}{3}$ with structure on b

Definition through the solution to a PDE

If we have a smooth b, we can say that the solution to

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t$$

admits a density p, which satisfies

$$\partial_t p(t, T, x, y) + b \cdot \nabla_x p(t, T, x, y) + \mathcal{L}_x^\alpha p(t, T, x, y) = 0, \qquad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

As a solution to this PDE, p also satisfies

$$p(t, T, x, y) = p_\alpha(t, T, x, y) + \int_t^T P_{s-t}^\alpha [b \cdot \nabla p(\cdot, \cdot, T, y)](s, x)\mathrm{d}s.$$

Mild solutions to the PDE

Definition

u is a *mild* solution to the Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, f, g, T)$

$$(\partial_t + b \cdot \nabla + \mathcal{L}^\alpha)\, u(t,x) = f(t,x) \text{ on } [0,T) \times \mathbb{R}^d, \qquad u(T,\cdot) = g \text{ on } \mathbb{R}^d,$$

if $u \in \mathcal{C}^{0,1}$ with $\nabla u \in \mathcal{C}_b^0([0,T], \mathbb{B}_{\infty,\infty}^{\theta-1-\varepsilon})$ for any $0 < \varepsilon \ll 1$ and for $\theta = \beta + \alpha - \frac{d}{p} - \frac{\alpha}{r}$, and if u satisfies, $\forall (t,x) \in [0,T] \times \mathbb{R}^d$,

$$u(t,x) = P^\alpha_{T-t}[g](x) - \int_t^T P^\alpha_{s-t}[f - b \cdot \nabla u](s,x)\mathrm{d}s. \qquad (2)$$

$\longrightarrow$ Parabolic gain is now θ, yielding the condition on β (paraproduct rule) :

$$\beta + (\theta - 1) > 0$$

$$\Longleftrightarrow$$

$$\beta > \frac{1 - \alpha + \frac{d}{p} + \frac{\alpha}{r}}{2}$$

Martingale solutions to the SDE

Definition ([Ethier and Kurtz, 1986])

A proba measure $\mathbb{P}$ (with its canonical process (X_t)) is a solution to the martingale problem associated with $(b, \mathcal{L}^\alpha, x)$ if

- $\mathbb{P}(X_0 = x) = 1$
- $\forall f \in \mathcal{C}([0, T], \mathcal{S}(\mathbb{R}^d, \mathbb{R})), \forall g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ s.t. $\nabla g \in \mathbb{B}^{\theta-1}_{\infty,\infty}$,

$$\left(u(t, X_t) - \int_0^t f(s, X_s)\mathrm{d}s - u(0, x) \right)_{0 \le t \le T}$$

is a martingale under $\mathbb{P}$, with u a mild solution to the Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, f, g, T)$

Existence and uniqueness of solutions

Theorem ([Chaudru de Raynal and Menozzi, 2022])

Assume

$$\alpha \in \left(\frac{1 + [\frac{d}{p}]}{1 - [\frac{1}{r}]}, 2 \right] \qquad \beta \in \left(\frac{1 - \alpha + [\frac{d}{p}] + [\frac{\alpha}{r}]}{2}, 0 \right), \tag{GR}$$

Then, the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d,$$

where

- $b \in L^r([0, T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$
- $(Z_t)_{t \geq 0}$ *is an* α*-stable symmetric process,* $\alpha \in (1, 2]$

admits a unique martingale solution.

Dynamics of the SDE

Proposition ([Delarue and Diel, 2016],[Chaudru de Raynal and Menozzi, 2022])

Assume

$$\alpha \in \left(\frac{1 + [\frac{d}{p}]}{1 - [\frac{1}{r}]}, 2 \right) \qquad \beta \in \left(\frac{1 - \alpha + [\frac{2d}{p}] + [\frac{2\alpha}{r}]}{2}, 0 \right), \tag{GRD}$$

Then, the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d$$

rewrites

$$X_t = X_0 + \int_0^t \mathfrak{b}(s, X_s, \mathrm{d}s) + Z_t,$$

where $\forall 0 \leq v \leq s \leq T$,

$$\mathfrak{b}(v, x, s - v) = \int_v^s \int b(r, y) p_\alpha(r - v, y - x) \mathrm{d}y \mathrm{d}r$$

Main result

Theorem ([F.,2023])

Let $\mathbb{P}$ *be the previous solution and* $(x_t)_{t\in[s,T]}$ *the associated canonical process. Then,* $\forall t \in (s,T]$, x_t *admits a density* $p(s,t,x,\cdot)$ *s.t.* $\exists C \geq 1 : \forall (x,y) \in \mathbb{R}^d$,

$$C^{-1}\bar{p}_\alpha(t-s,y-x) \leq p(s,t,x,y) \leq C\bar{p}_\alpha(t-s,y-x),$$
$$|\nabla_x p(s,t,x,y)| \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}}\bar{p}_\alpha(t-s,y-x),$$

where $\bar{p}_\alpha(v,z) = \frac{C_\alpha}{v^{\frac{d}{\alpha}}}\frac{1}{\left(1+\frac{|z|}{v^{\frac{1}{\alpha}}}\right)^{d+\alpha}}, v > 0, z \in \mathbb{R}^d$

This also gives the following logarithmic gradient estimates :

$$|\nabla_x \log p(s,t,x,y)| = \frac{|\nabla_x p(s,t,x,y)|}{p(s,t,x,y)} \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}}.$$

Main result

Theorem ([F., 2023])

Let $\mathbb{P}$ be the previous solution and $(x_t)_{t\in[s,T]}$ the associated canonical process. Then, $\forall t \in (s,T]$, x_t admits a density $p(s,t,x,\cdot)$ s.t. $\exists C \geq 1 : \forall (x,y,y') \in \mathbb{R}^d$,

$$|p(s,t,x,y) - p(s,t,x,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}} \left(\bar{p}_\alpha(t-s,y-x) + \bar{p}_\alpha(t-s,y'-x)\right),$$

$$|\nabla_x p(s,t,x,y) - \nabla_x p(s,t,x,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}} \left(\bar{p}_\alpha(t-s,x-y) + \bar{p}_\alpha(t-s,x-y')\right),$$

for any $\rho \in (-\beta, \gamma-\beta)$, where $\gamma := \beta - \frac{1-\alpha+\frac{\alpha}{r}+\frac{d}{p}}{2} > 0$

Thermic characterization of Besov spaces

Proposition ([Triebel, 1988])

For $\vartheta \in \mathbb{R}, m \in (0,+\infty], \ell \in (0,\infty], \mathbb{B}^{\vartheta}_{\ell,m}(\mathbb{R}^d) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathcal{H}^{\vartheta}_{\ell,m},\tilde{\alpha}} < \infty \Big\}$,
where :

$$\|f\|_{\mathcal{H}^{\vartheta}_{\ell,m},\tilde{\alpha}} := \|\phi(D)f\|_{L^\ell} + \begin{cases} \left(\displaystyle\int_0^1 \frac{\mathrm{d}v}{v} v^{(n-\frac{\vartheta}{\tilde{\alpha}})m} \|\partial_v^n p_{\tilde{\alpha}}(v,\cdot) \star f\|_{L^\ell}^m \right)^{\frac{1}{m}}, & m < \infty, \\ \displaystyle\sup_{v \in (0,1]} v^{n-\frac{\vartheta}{\tilde{\alpha}}} \|\partial_v^n p_{\tilde{\alpha}}(v,\cdot) \star f\|_{L^\ell}, & m = \infty, \end{cases}$$
$$=: \|\phi(D)f\|_{L^\ell} + \mathcal{T}^{\vartheta}_{\ell,m}[f],$$

where $\phi(D)f := \mathcal{F}^{-1}[\phi] \star f$

Useful inequalities in Besov spaces

- Duality inequality :
  $\forall m, \ell, \vartheta$, with m' and ℓ' respective conjugates of m and ℓ, and $(f, g) \in \mathbb{B}^{\vartheta}_{\ell,m} \times \mathbb{B}^{-\vartheta}_{\ell',m'}$,

$$\left| \int f(y)g(y)\mathrm{d}y \right| \leq \|f\|_{\mathbb{B}^{\vartheta}_{\ell,m}} \|g\|_{\mathbb{B}^{-\vartheta}_{\ell',m'}}.$$

- Hölder inequality (product rule) :
  $\forall p, q, s$ and $\forall \rho > \max\left\{ s, d\left(\frac{1}{p} - 1\right)_+ - s \right\}$, $\forall (f, g) \in \mathbb{B}^{\rho}_{\infty,\infty} \times \mathbb{B}^{s}_{p,q}$,

$$\|f \cdot g\|_{B^s_{p,q}} \lesssim \|f\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|g\|_{\mathbb{B}^s_{p,q}}.$$

Smooth approximation of the drift

Proposition ([Chaudru de Raynal et al., 2022])

Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ with $\beta \in (-1, 0]$, $1 \leq p, q \leq \infty$. There exists a time-space sequence of smooth bounded functions $(b^m)_{m\in\mathbb{N}}$ s.t.

$$\|b - b^m\|_{L^{\tilde{r}} - \mathbb{B}^{\tilde{\beta}}_{p,q}} \underset{m\to\infty}{\longrightarrow} 0, \qquad \forall \tilde{\beta} < \beta,$$

with $\tilde{r} = r$ if $r < \infty$ and for any $\tilde{r} < \infty$ otherwise. Moreover, $\exists \kappa \geq 1$:

$$\sup_{m\in\mathbb{N}} \|b^m\|_{L^{\tilde{r}} - \mathbb{B}^{\beta}_{p,q}} \leq \kappa \|b\|_{L^{\tilde{r}} - \mathbb{B}^{\beta}_{p,q}}. \tag{3}$$

We will compute uniform estimates on the density of the *mollified* SDE

$$\mathrm{d}X^m_t = b^m(t, X^m_t)\mathrm{d}t + \mathrm{d}Z^m_t$$

Duhamel expansion

We have

$$p^m(s,t,x,y) = p_\alpha(t-s,y-x)+\int_s^t \int p^m(s,u,x,z)b^m(u,z)\nabla_z p_\alpha(t-u,y-z)\mathrm{d}z\mathrm{d}u \tag{4}$$

Denote, for fixed $(s,x) \in [0,1] \times \mathbb{R}^d$,

$$h^m_{s,x}(t,y) := \frac{p^m(s,t,x,y)}{\bar{p}_\alpha(t-s,y-x)}.$$

Besov duality

The Duhamel expansion (4) yields

$$h_{s,x}(t,y) \le C + \frac{C}{\bar{p}_\alpha(t-s,y-x)} \int_s^t \left| \int h_{s,x}(u,z) b^m(u,z) \bar{p}_\alpha(u-s,z-x) \nabla p_\alpha(t-u,y-z) \mathrm{d}z \right| \mathrm{d}u$$

- Duality :

$$h_{s,x}(t,y) \le C + \frac{1}{\bar{p}_\alpha(t-s,y-x)} \times \int_s^t \|h_{s,x}(u,\cdot) b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|\bar{p}_\alpha(u-s,\cdot - x) \nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

- Product rule : for $\rho > -\beta$ (i.e. the paraproduct rule condition),

$$h_{s,x}(t,y) \le C + \frac{1}{\bar{p}_\alpha(t-s,y-x)} \times \int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|\bar{p}_\alpha(u-s,\cdot - x) \nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

Besov norm of $h_{s,x}(u,\cdot)$

- Product rule : for $\rho > -\beta$ (i.e. the paraproduct rule condition),

$$h_{s,x}(t,y) \leq C + \frac{1}{\bar{p}_\alpha(t-s,y-x)} \times$$

$$\int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|\bar{p}_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

Recall that

$$\|h_{s,x}(u,\cdot)\|_{\mathbb{B}^\rho_{\infty,\infty}} = \|\phi(D)h_{s,x}(u,\cdot)\|_{L^\infty} + \mathcal{T}^\rho_{\infty,\infty}[h_{s,x}(u,\cdot)]$$

Thermic part of $\|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}}$

Lemma

$\forall 0 \le s \le t, \forall x \in \mathbb{R}^d, \forall \rho > -\beta,$

$$\mathcal{T}^{\rho}_{\infty,\infty}[h_{s,x}(t,\cdot)] \lesssim \frac{1}{(t-s)^{\frac{\rho}{\alpha}}} \times$$
$$\left(1 + \int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \times \right.$$
$$\left. \|\bar{p}_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}} + 1\right] \mathrm{d}u \right).$$

$\rightarrow$ Gronwall lemma on $g_{s,x}(t) := \|h_{s,x}(u,\cdot)\|_{L^\infty} + (t-s)^{\frac{\rho}{\alpha}} \mathcal{T}^{\rho}_{\infty,\infty}[h_{s,x}(u,\cdot)]$

Accounting for the singularities

For all of this to work, we need the following integral to converge :

$$\int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|\bar{p}_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}+1\right] \mathrm{d}u$$

$\rightarrow$ how does the space Besov norm $\|\cdot\|_{\mathbb{B}^{-\beta}_{p',q'}}$ translate into time singularities ?

Besov estimates for $\bar{p}_\alpha$

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$$\int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|\bar{p}_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}+1\right]\mathrm{d}u$$

$\to$ how does the space Besov norm $\|\cdot\|_{\mathbb{B}^{-\beta}_{p',q'}}$ translate into time singularities ?

Lemma

- $\forall 0\le s\le u\le t$, $\forall (x,y)\in\mathbb{R}^d$, $\forall \zeta\in(-\beta,1]$, $\forall j,k\in\{0,1\}$,

$$\|\nabla^j\bar{p}_\alpha(u-s,x-\cdot)\nabla^k p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}}$$
$$\lesssim \frac{\bar{p}_\alpha(t-s,x-y)}{(u-s)^{\frac{j}{\alpha}}(t-u)^{\frac{k}{\alpha}}}(t-s)^{\frac{\beta}{\alpha}}\left[\frac{1}{(t-u)^{\frac{d}{\alpha p}}}+\frac{1}{(u-s)^{\frac{d}{\alpha p}}}\right]$$
$$\times\left[(t-s)^{\frac{\zeta}{\alpha}}\left(\frac{1}{(t-u)^{\frac{\zeta}{\alpha}}}+\frac{1}{(u-s)^{\frac{\zeta}{\alpha}}}\right)+1\right].$$

Results extensions

- Explicit constant
- Non-trivial Hölder continuous diffusion coefficient σ that preserves the condition of non-zero spectral measure on the unit sphere

  should work

- $\alpha = 2$, standard Brownian Motion scope

  should work, requires full series expansion
  [Perkowski and van Zuijlen, 2022] for $b \in \mathbb{B}^{\beta}_{\infty,1}$

- Does a Euler scheme for such process work ? Currently working on it

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Besov estimates for $\bar{p}_\alpha$

Lemma

- $\forall 0 \le s \le u \le t$, $\forall (x,y,w) \in \mathbb{R}^d$, $\forall \zeta \in (-\beta, 1]$,

$$\left\| \bar{p}_\alpha(u-s, x-\cdot) \left[\frac{\nabla p_\alpha(t-u, w-\cdot)}{\bar{p}_\alpha(t-s, w-x)} - \frac{\nabla p_\alpha(t-u, y-\cdot)}{\bar{p}_\alpha(t-s, y-x)} \right] \right\|_{\mathbb{B}^{-\beta}_{p',q'}}$$
$$\lesssim \frac{|w-y|^\zeta}{(t-u)^{\frac{\zeta+1}{\alpha}}} (t-s)^{\frac{\beta}{\alpha}} \left[\frac{1}{(t-u)^{\frac{d}{\alpha p}}} + \frac{1}{(u-s)^{\frac{d}{\alpha p}}} \right]$$
$$\times \left[(t-s)^{\frac{\zeta}{\alpha}} \left(\frac{1}{(t-u)^{\frac{\zeta}{\alpha}}} + \frac{1}{(u-s)^{\frac{\zeta}{\alpha}}} \right) + 1 \right].$$