Heat kernel estimates for stable-driven SDEs with distributional drift

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Main SDE

We consider the *formal* SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{E}$$

where

- $b \in L^r([0, T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$ is a Besov drift
- Z_t is a symmetric α-stable process, $\alpha \in (1, 2)$, with spectral measure equivalent to the Lebesgue measure on $\mathbb{S}^{d-1}$.

We will denote by $\mathcal{L}^\alpha$ the generator of Z, p_α its density and P^α_t the associated semi-group :

$$\forall \phi, P^\alpha_t[\phi](x) := \int_{\mathbb{R}^d} p_\alpha(t, y - x)\phi(y)\mathrm{d}y$$

Expectation, heuristics

Take a time homogeneous b.

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + Z_t$$

- We expect that we can define $\int_0^t b(X_s)\mathrm{d}s$ if this term is more regular than the noise Z (which has regularity $\frac{1}{\alpha}$)
- Assume $b \in \mathcal{C}^\beta$ for some $\beta > 0$. Then (informally), $b(X_\cdot) \in \mathcal{C}^{\frac{\beta}{\alpha}}$, so we need

$$1 + \frac{\beta}{\alpha} > \frac{1}{\alpha} \iff \alpha + \beta > 1$$

- If $b \in L^p \subset \mathbb{B}^{-\frac{d}{p}}_{\infty,\infty} = \mathcal{C}^{-\frac{d}{p}}$, the same reasoning gives

$$\frac{d}{p} < \alpha - 1$$

which is consistent with the KR condition $\frac{d}{p} + \frac{\alpha}{q} < \alpha - 1$.

Expectation, heuristics

For $b \in L^r([0,T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d)) \approx \mathcal{C}^{\beta}$ with $\beta < 0$, this is not sufficient, because we need to work on the underlying PDE

$$(\partial_t + b \cdot \nabla + \mathcal{L}^{\alpha}) u(t,x) = f(t,x) \text{ on } [0,T] \times \mathbb{R}^d$$

where $\mathcal{L}^{\alpha}$ is the generator of the α-stable noise and f some source term.

- $\alpha + \beta > 1 \iff$ we can define ∇u
- By paraproduct rules, if we want to define the generalized functions product $b \cdot \nabla u$, we need $\beta + (\beta + \alpha - 1) > 0$

$$\text{i.e. } \beta > \frac{1-\alpha}{2} \tag{1}$$

- [Delarue and Diel, 2016] : weak 1D existence and uniqueness under $\beta > -\frac{2}{3}$ with structure on b, with $\alpha = 2$
- [Flandoli et al., 2017] : virtual solutions under (1), $\alpha = 2$
- [Athreya et al., 2020] : strong 1D existence and uniqueness under (1)
- [Kremp and Perkowski, 2022] : weak $\mathbb{R}^d$ existence and uniqueness under $\beta > \frac{2-2\alpha}{3}$ with structure on b

Mild solutions to the PDE

Definition

u is a *mild* solution to the Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, f, g, T)$

$$(\partial_t + b \cdot \nabla + \mathcal{L}^\alpha)\, u(t,x) = f(t,x) \text{ on } [0,T) \times \mathbb{R}^d, \qquad u(T,\cdot) = g \text{ on } \mathbb{R}^d,$$

if $u \in \mathcal{C}^{0,1}$ with $\nabla u \in \mathcal{C}^0_b([0,T], \mathbb{B}^{\theta-1-\varepsilon}_{\infty,\infty})$ for any $0 < \varepsilon \ll 1$ and for $\theta = \alpha + \beta - \frac{d}{p} - \frac{\alpha}{r}$, and if u satisfies, $\forall (t,x) \in [0,T] \times \mathbb{R}^d$,

$$u(t,x) = P^\alpha_{T-t}[g](x) - \int_t^T P^\alpha_{s-t}[f - b \cdot \nabla u](s,x)\mathrm{d}s. \tag{2}$$

$\longrightarrow$ Parabolic gain is now θ, yielding the condition on β (paraproduct rule) :

$$\beta + (\theta - 1) > 0$$

$$\iff$$

$$\beta > \frac{1 - \alpha + \frac{d}{p} + \frac{\alpha}{r}}{2}$$

Martingale solutions to the SDE

Definition ([Ethier and Kurtz, 1986])

A proba measure $\mathbb{P}$ (with its canonical process (X_t)) is a solution to the martingale problem associated with $(b, \mathcal{L}^\alpha, x)$ if

- $\mathbb{P}(X_0 = x) = 1$
- $\forall f \in \mathcal{C}([0, T], \mathcal{S}(\mathbb{R}^d, \mathbb{R})), \forall g \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ s.t. $\nabla g \in \mathbb{B}^{\theta-1}_{\infty,\infty}$,

$$\left(u(t, X_t) - \int_0^t f(s, X_s)\mathrm{d}s - u(0, x) \right)_{0 \le t \le T}$$

is a martingale under $\mathbb{P}$, with u a mild solution to the Cauchy problem $\mathcal{C}(b, \mathcal{L}^\alpha, f, g, T)$

Existence and uniqueness of solutions

Theorem ([Chaudru de Raynal and Menozzi, 2022])

Assume

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2 \right] \qquad \beta \in \left(\frac{1-\alpha+\frac{d}{p}+\frac{\alpha}{r}}{2}, 0 \right), \tag{GR}$$

Then, the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d,$$

where

- $b \in L^r([0, T], \mathbb{B}^{\beta}_{p,q}(\mathbb{R}^d, \mathbb{R}^d))$
- $(Z_t)_{t \geq 0}$ *is an* α*-stable symmetric process,* $\alpha \in (1, 2]$

admits a unique martingale solution.

Dynamics of the SDE

Proposition
([Delarue and Diel, 2016],[Chaudru de Raynal and Menozzi, 2022])

Assume

$$\alpha \in \left(\frac{1+\frac{d}{p}}{1-\frac{1}{r}}, 2\right) \qquad \beta \in \left(\frac{1-\alpha+\frac{2d}{p}+\frac{2\alpha}{r}}{2}, 0\right), \tag{GRD}$$

Then, the formal SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t, \qquad X_0 = x \in \mathbb{R}^d$$

rewrites

$$X_t = X_0 + \int_0^t \mathfrak{b}(s, X_s, \mathrm{d}s) + Z_t,$$

where $\forall 0 \leq v \leq s \leq T$,

$$\mathfrak{b}(v, x, s-v) = \int_v^s \int b(r, y) p_\alpha(r-v, y-x)\mathrm{d}y\mathrm{d}r$$

Main result

Theorem ([F.,2023])

Let $\mathbb{P}$ be the previous solution and $(x_t)_{t\in[s,T]}$ the associated canonical process. Then, $\forall t \in (s,T]$, x_t admits a density $p(s,t,x,\cdot)$ s.t. $\exists C \geq 1 : \forall (x,y) \in \mathbb{R}^d$,

$$C^{-1}p_\alpha(t-s,y-x) \leq p(s,t,x,y) \leq Cp_\alpha(t-s,y-x),$$

$$|\nabla_x p(s,t,x,y)| \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}} p_\alpha(t-s,y-x),$$

This also gives the following logarithmic gradient estimates :

$$|\nabla_x \log p(s,t,x,y)| = \frac{|\nabla_x p(s,t,x,y)|}{p(s,t,x,y)} \leq \frac{C}{(t-s)^{\frac{1}{\alpha}}}.$$

Main result

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Let $\mathbb{P}$ be the previous solution and $(x_t)_{t\in[s,T]}$ the associated canonical process. Then, $\forall t \in (s,T]$, x_t admits a density $p(s,t,x,\cdot)$ s.t. $\exists C \geq 1 : \forall (x,y,y') \in \mathbb{R}^d$,

$$|p(s,t,x,y) - p(s,t,x,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho}{\alpha}}}\left(p_\alpha(t-s,y-x) + p_\alpha(t-s,y'-x)\right),$$

$$|\nabla_x p(s,t,x,y) - \nabla_x p(s,t,x,y')| \leq \frac{C|y-y'|^\rho}{(t-s)^{\frac{\rho+1}{\alpha}}}\left(p_\alpha(t-s,x-y) + p_\alpha(t-s,x-y')\right),$$

for any $\rho \in (-\beta, \gamma-\beta)$, where $\gamma := \beta - \frac{1-\alpha+\frac{\alpha}{r}+\frac{d}{p}}{2} > 0$

Euler schemes for singular drifts

- [Bencheikh and Jourdain, 2020] $b \in L^\infty$ error in TV : $h^{\frac{1}{2}}$.
- [Le and Ling, 2021] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$ condition, strong error rate $h^{\frac{1}{2}} \ln(h)$.
- [Dareiotis et al., 2022] $b \in L^\infty \cap \dot{W}^\beta_m$, strong error rate $h^{\frac{1+\beta}{2}-}$
- [Jourdain and Menozzi, 2021] $b \in L^q - L^p$ under $\frac{d}{p} + \frac{2}{q} < 1$ condition, weak error rate $h^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}}$.

Euler scheme definition

Joint work in progress with E. Issoglio and S. Menozzi.

For $b \in L^r - \mathbb{B}^{\beta}_{p,q}$, we define the Euler scheme as :

$$X^h_{t_{k+1}} = X^h_{t_k} + (Z_{t_{k+1}} - Z_{t_k}) + \int_{t_k}^{t_{k+1}} \int p_\alpha(r - t_k, y - X^h_{t_k}) b(U_k, y) \mathrm{d}y \mathrm{d}r,$$

where $U_k \sim \mathcal{U}([t_k, t_{k+1}])$.

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where $U_k \sim \mathcal{U}([t_k, t_{k+1}])$.

Recall that the formal SDE $\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \mathrm{d}Z_t$ rewrites

$$X_t = X_s + \int_s^t \int p_\alpha(r - s, y - X_s) b(r, y) \mathrm{d}y \mathrm{d}r + Z_t$$

Euler scheme density estimates and convergence rate

Proposition (Density estimates for the Euler scheme)

X_t^h admits a density $\Gamma^h(t_k, x, t, y)$, for which
$\exists C > 0 : \forall h, \forall k \in \{0, ..., n-1\}, t \in (t_k, T], x, y, y' \in \mathbb{R}^d$,

$$|\Gamma^h(t_k, x, t, y') - \Gamma^h(t_k, x, t, y)| \leq C \frac{|y - y'|^\gamma \wedge (t - t_k)^{\frac{\gamma}{\alpha}}}{(t - t_k)^{\frac{\gamma}{\alpha}}} \left(p_\alpha(t - t_k, y - x) + p_\alpha(t - t_k, y' - x)\right). \quad (3)$$

Also, for all $0 \leq k < \ell < n,\ t \in [t_\ell, t_{\ell+1}], x, y \in \mathbb{R}^d$,

$$|\Gamma^h(t_k, x, t, y) - \Gamma^h(t_k, x, t_\ell, y)| \leq C \frac{(t - t_\ell)^{\frac{\gamma}{\alpha}}}{(t_\ell - t_k)^{\frac{\gamma}{\alpha}}} p_\alpha(t - t_k, y - x). \quad (4)$$

$\Rightarrow$ expected weak error rate :

$$|\Gamma^h(0, x, t, y) - p(0, x, t, y)| \leq C h^{\frac{\gamma}{\alpha}} p_\alpha(t, y - x)$$

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Smooth approximation of the drift

Proposition ([Chaudru de Raynal et al., 2022])

Let $b \in L^r - \mathbb{B}^{\beta}_{p,q}$ with $\beta \in (-1, 0]$, $1 \leq p, q \leq \infty$. There exists a time-space sequence of smooth bounded functions $(b^m)_{m\in\mathbb{N}}$ s.t.

$$\|b - b^m\|_{L^{\tilde{r}} - \mathbb{B}^{\tilde{\beta}}_{p,q}} \underset{m\to\infty}{\longrightarrow} 0, \qquad \forall \tilde{\beta} < \beta,$$

with $\tilde{r} = r$ if $r < \infty$ and for any $\tilde{r} < \infty$ otherwise. Moreover, $\exists \kappa \geq 1$:

$$\sup_{m\in\mathbb{N}} \|b^m\|_{L^{\tilde{r}} - \mathbb{B}^{\beta}_{p,q}} \leq \kappa \|b\|_{L^{\tilde{r}} - \mathbb{B}^{\beta}_{p,q}}. \tag{5}$$

We will compute uniform estimates on the density of the *mollified* SDE

$$\mathrm{d}X^m_t = b^m(t, X^m_t)\mathrm{d}t + \mathrm{d}Z^m_t$$

Duhamel expansion

We have

$$p^m(s,t,x,y) = p_\alpha(t-s,y-x) + \int_s^t \int p^m(s,u,x,z) b^m(u,z) \nabla_z p_\alpha(t-u,y-z) \mathrm{d}z\mathrm{d}u \tag{6}$$

Denote, for fixed $(s,x) \in [0,1] \times \mathbb{R}^d$,

$$h^m_{s,x}(t,y) := \frac{p^m(s,t,x,y)}{p_\alpha(t-s,y-x)}.$$

Besov duality

The Duhamel expansion (6) yields

$$h_{s,x}(t,y) \le C + \frac{C}{p_\alpha(t-s,y-x)} \int_s^t \left| \int h_{s,x}(u,z) b^m(u,z) p_\alpha(u-s,z-x) \nabla p_\alpha(t-u,y-z) \mathrm{d}z \right| \mathrm{d}u$$

- Duality :

$$h_{s,x}(t,y) \le C + \frac{1}{p_\alpha(t-s,y-x)} \times \int_s^t \|h_{s,x}(u,\cdot) b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|p_\alpha(u-s,\cdot-x) \nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

- Product rule : for $\rho > -\beta$ (i.e. the paraproduct rule condition),

$$h_{s,x}(t,y) \le C + \frac{1}{p_\alpha(t-s,y-x)} \times \int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|p_\alpha(u-s,\cdot-x) \nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

Besov norm of $h_{s,x}(u,\cdot)$

- Product rule : for $\rho > -\beta$ (i.e. the paraproduct rule condition),

$$h_{s,x}(t,y) \leq C + \frac{1}{p_\alpha(t-s,y-x)} \times$$

$$\int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|p_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \mathrm{d}u$$

Recall that

$$\|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}} = \|\phi(D)h_{s,x}(u,\cdot)\|_{L^\infty} + \mathcal{T}^{\rho}_{\infty,\infty}[h_{s,x}(u,\cdot)]$$

Thermic part of $\|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}}$

Lemma

$\forall 0 \le s \le t, \forall x \in \mathbb{R}^d, \forall \rho > -\beta,$

$$\mathcal{T}^{\rho}_{\infty,\infty}[h_{s,x}(t,\cdot)] \lesssim \frac{1}{(t-s)^{\frac{\rho}{\alpha}}} \times \left(1 + \int_s^t \|h_{s,x}(u,\cdot)\|_{\mathbb{B}^{\rho}_{\infty,\infty}} \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \times \|p_\alpha(u-s,\cdot - x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}} + 1\right] \mathrm{d}u\right).$$

$\rightarrow$ Gronwall lemma on $g_{s,x}(t) := \|h_{s,x}(u,\cdot)\|_{L^\infty} + (t-s)^{\frac{\rho}{\alpha}} \mathcal{T}^{\rho}_{\infty,\infty}[h_{s,x}(u,\cdot)]$

Accounting for the singularities

For all of this to work, we need the following integral to converge :

$$\int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^{\beta}_{p,q}} \|p_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}+1\right]\mathrm{d}u$$

$\rightarrow$ how does the space Besov norm $\|\cdot\|_{\mathbb{B}^{-\beta}_{p',q'}}$ translate into time singularities ?

Besov estimates for p_α

For all of this to work, we need the following integral to converge :

$$\int_s^t \|b^m(u,\cdot)\|_{\mathbb{B}^\beta_{p,q}} \|p_\alpha(u-s,\cdot-x)\nabla p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \left[\frac{(t-s)^{\frac{\rho}{\alpha}}}{(t-u)^{\frac{\rho}{\alpha}}}+1\right] \mathrm{d}u$$

$\to$ how does the space Besov norm $\|\cdot\|_{\mathbb{B}^{-\beta}_{p',q'}}$ translate into time singularities ?

Lemma

- $\forall 0 \le s \le u \le t$, $\forall (x,y) \in \mathbb{R}^d$, $\forall \zeta \in (-\beta, 1]$, $\forall j,k \in \{0,1\}$,

$$\begin{aligned}
&\|\nabla^j p_\alpha(u-s,x-\cdot)\nabla^k p_\alpha(t-u,y-\cdot)\|_{\mathbb{B}^{-\beta}_{p',q'}} \\
&\lesssim \frac{p_\alpha(t-s,x-y)}{(u-s)^{\frac{j}{\alpha}}(t-u)^{\frac{k}{\alpha}}}(t-s)^{\frac{\beta}{\alpha}}\left[\frac{1}{(t-u)^{\frac{d}{\alpha p}}}+\frac{1}{(u-s)^{\frac{d}{\alpha p}}}\right] \\
&\quad \times \left[(t-s)^{\frac{\zeta}{\alpha}}\left(\frac{1}{(t-u)^{\frac{\zeta}{\alpha}}}+\frac{1}{(u-s)^{\frac{\zeta}{\alpha}}}\right)+1\right].
\end{aligned}$$

Besov estimates for p_α

Lemma

- $\forall 0 \le s \le u \le t$, $\forall (x,y,w) \in \mathbb{R}^d$, $\forall \zeta \in (-\beta, 1]$,

$$\left\| \bar{p}_\alpha(u-s, x-\cdot)\left[\frac{\nabla p_\alpha(t-u, w-\cdot)}{\bar{p}_\alpha(t-s, w-x)} - \frac{\nabla p_\alpha(t-u, y-\cdot)}{\bar{p}_\alpha(t-s, y-x)}\right]\right\|_{\mathbb{B}^{-\beta}_{p',q'}}$$

$$\lesssim \frac{|w-y|^\zeta}{(t-u)^{\frac{\zeta+1}{\alpha}}}(t-s)^{\frac{\beta}{\alpha}}\left[\frac{1}{(t-u)^{\frac{d}{\alpha p}}} + \frac{1}{(u-s)^{\frac{d}{\alpha p}}}\right]$$

$$\times \left[(t-s)^{\frac{\zeta}{\alpha}}\left(\frac{1}{(t-u)^{\frac{\zeta}{\alpha}}} + \frac{1}{(u-s)^{\frac{\zeta}{\alpha}}}\right) + 1\right].$$