Deterministic descriptions of the turbulence in the Navier-Stokes equations

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Outlook

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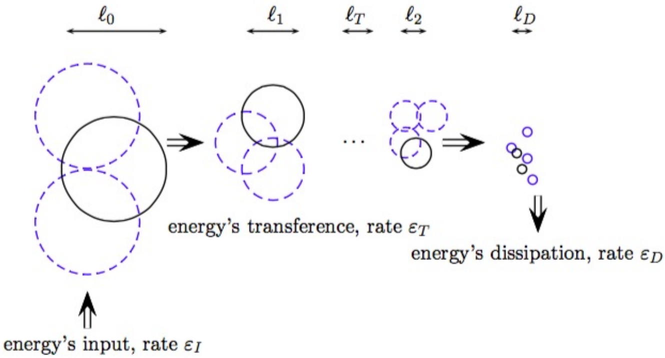
A. Kolmogorov (1903-1987)

We will focus in three laws of the K41 theory:

(1) The energy cascade model.
(2) The Kolmogorov's dissipation law.
(3) The behavior of the energy's spectrum.

(1) The energy cascade model (Richardson 1922, Kolmogorov 1941)



The Kolmogorov's dissipation law

When the fluid is in turbulent setting we have that:

$$\varepsilon_I \approx \varepsilon_T \approx \varepsilon_D := \varepsilon \approx \frac{U^3}{\ell_0}.$$

$\Rightarrow$ $U = \langle |\vec{u}|^2 \rangle^{\frac{1}{2}}$ is the fluid's averaged velocity where $\vec{u}(t,x) \in \mathbb{R}^3$ is the fluid's velocity and $\langle \cdot \rangle$ is an spatial and temporal average which we will precisely define later.

(3) The behavior of the energy's spectrum: the energy's spectrum

⇒ For $\vec{u}(t,x)$ the fluid's velocity, the energy's spectrum

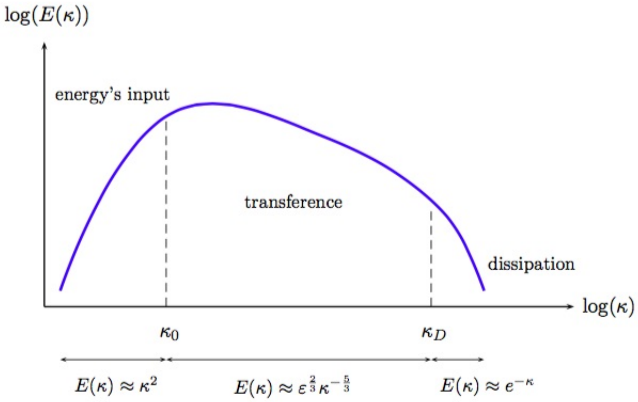
$$E(\kappa) := \int_{|\xi|=\kappa} \left| \left\langle \widehat{\vec{u}}(\cdot,\xi) \right\rangle_t \right|^2 d\sigma(\xi)$$

measures the average energy density at a certain length scale ℓ which corresponds to a frequency amplitude $\kappa = \frac{1}{\ell}$.

⇒ $\widehat{\vec{u}}$ denotes the Fourier transform of the velocity, $\langle\cdot\rangle_t$ is a temporal average and $d\sigma$ is measure of the unit sphere.

(3) The behavior of the energy's spectrum

For $\kappa_0 = \frac{1}{\ell_0}$ (for a given energy input scale $\ell_0 > 0$) and $\kappa_D = \left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{4}} = \frac{1}{\ell_D}$ (the Kolmogorov's dissipation frequency)



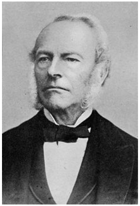
Outlook

⇒ We consider a viscous and incompressible fluid in the space $\mathbb{R}^3$ where an stationary external force $\vec{f} = \vec{f}(x)$ acts on the fluid by introducing kinetic energy independently on time and at a given energy input scale $\ell_0 > 0$.

⇒ We consider a viscous and incompressible fluid in the space $\mathbb{R}^3$ where an stationary external force $\vec{f} = \vec{f}(x)$ acts on the fluid by introducing kinetic energy independently on time and at a given energy input scale $\ell_0 > 0$.

The base equations: the incrompressible Navier-Stokes equations

H. Navier (1785-1836)



G. Stokes (1819-1903)

$$\left\{ \begin{array}{ll} \partial_t \vec{u} + \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f}, & div(\vec{u}) = 0, \quad \text{on }]0, +\infty[\times\Omega, \\ & \vec{u}(0, \cdot) = \vec{u}_0, \end{array} \right. \quad (1)$$

where $\Omega = [0, L]^3$ (periodic framework) or $\Omega = \mathbb{R}^3$ (non-periodic framework).

(1) In the periodic framework we have a convenient framework where we will introduce the basic ideas to study the Kolmogorov's dissipation law.
(2) Thereafter we will study the Kolmogorv's dissipation law in the non-periodic framework: the pass of the periodic framework to the non-periodic one is delicate.

Let be $L > 0$ and $\Omega = [0, L]^3$.

$\Rightarrow$ If $\vec{u}_0, \vec{f} \in L^2$ are $\Omega-$periodic functions such that $\int_\Omega \vec{u}_0(x)dx = \int_\Omega \vec{f}(x)dx = 0$ then there exists

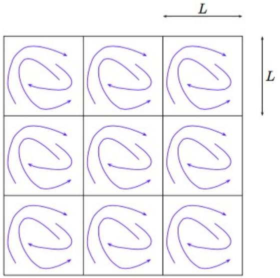
$$\vec{u} \in L^\infty(]0, +\infty[, L^2(\Omega)) \cap L^2_{loc}(]0, +\infty[, \dot{H}^1(\Omega))$$

a weak solution of the N-S equations (1) (Leray, 1943) such that:

1. $\vec{u}$ is a $\Omega-$periodic function and $\int_\Omega \vec{u}(t, x)dx = 0$ a.e. $t > 0$.
2. Moreover, for all $T > 0$

$$\|\vec{u}(T)\|^2_{L^2} + 2\nu \int_0^T \|\nabla \otimes \vec{u}(t)\|^2_{L^2} dt \leq \|\vec{u}_0\|^2_{L^2} + 2 \int_0^T \int_\Omega \vec{u}(t, x) \cdot \vec{f}(x) dx\, dt. \tag{2}$$

(A) The fluid's characteristic length is the biggest length scale where we will study the fluid's turbulent behavior. In the periodic framework this length scale in naturally given by the period $L > 0$. For simplicity we will define the input energy scale ℓ_0 by $\ell_0 = L$.



(B) The fluid's averaged velocity:

$$U = \left(\limsup_{T\to+\infty} \frac{1}{T} \int_0^T \|\vec{u}(t)\|_{L^2}^2 \frac{dt}{L^3} \right)^{\frac{1}{2}}$$

- $\vec{f}$ introduces the kinetic energy independently of time $\Rightarrow$ we consider the long-time average $\limsup_{T\to+\infty} \frac{1}{T}\int_0^T (\cdot) dt$.
- by the Poincaré's inequality (and since $\int_\Omega \vec{u}(t,x)dx = 0$) we have $\|\vec{u}(t)\|_{L^2} \leq \frac{L}{2\pi} \|\nabla \otimes \vec{u}(t)\|_{L^2}$ and then the energy inequality (2) $\Rightarrow U < +\infty$.

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(C) The energy dissipation rate:

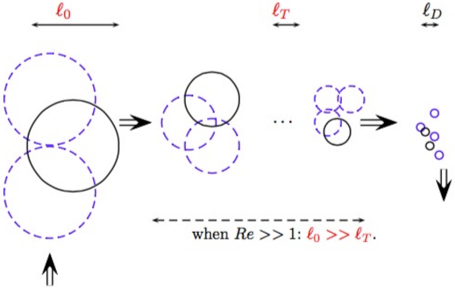
$$\varepsilon = \nu \limsup_{T \to +\infty} \frac{1}{T} \int_0^T \|\nabla \otimes \vec{u}(t)\|_{L^2}^2 \frac{dt}{L^3}.$$

The energy inequality (2) $\Rightarrow \varepsilon < +\infty$.

(D) The Reynolds numbers (Reynolds 1883):

$$Re = \frac{UL}{\nu}$$

- *Re* characterizes the ratio of the transport term: $\vec{u} \cdot \nabla \vec{u}$ to dissipation term: $\nu \Delta \vec{u}$.
- The fluid's turbulent setting is performed when $Re >> 1$.



Theorem (Doering & Foias, 2002)

Let be $L>0$ *and* $\Omega=[0,L]^3$*. Let be* $\vec{u}_0, \vec{f}\in L^2$*,* $\Omega-$*periodic functions and let be* $\vec{u}\in L^\infty_t L^2_x \cap L^2_{loc,t}\dot{H}^1_x$ *a* $\Omega-$ *periodic weak solution of the Navier-Stokes equations*

$$\begin{cases} \partial_t \vec{u} + \mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f}, \quad div(\vec{u})=0, \qquad on \quad]0,+\infty[\times\Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad \vec{u}(0,\cdot)=\vec{u}_0. \end{cases}$$

There exist two constants $c_1, c_2>0$ *independent of the physic quantities above such that*

$$\varepsilon \leq \frac{U^3}{L}\left(\frac{c_1}{Re}+c_2\right).$$

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There exist two constants $c_1, c_2 > 0$ independent of the physic quantities above such that

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Remark

(i) *If Re is large enough we get $\varepsilon \lesssim \frac{U^3}{L}$. A partial estimate of the Kolmogorov's dissipation law.*

(ii) *The other inequality $\frac{U^3}{L} \lesssim \varepsilon$ (when $Re >> 1$) is an open question.*

(iii) *In the periodic framework the fluid's characteristic length is naturally given by the period L and we have that $U < +\infty$.*

- ⇒ Now, we consider a non-periodic fluid in the whole space $\mathbb{R}^3$.
- ⇒ In this framework a convenient definition of the fluid's characteristic length L is a delicate question!

(2) The non-periodic framework

- $\Rightarrow$ Now, we consider a non-periodic fluid in the whole space $\mathbb{R}^3$.
- $\Rightarrow$ In this framework a convenient definition of the fluid's characteristic length L is a delicate question!
- $\Rightarrow$ An idea: the Constatin's model proposes to define L by using the external force $\vec{f}$ as we will see later.

(2) The non-periodic framework: the fluid's velocity

Our starting point is to define the fluid's velocity $\vec{u}$:

$\Rightarrow$ for $\vec{u}_0, \vec{f} \in L^2(\mathbb{R}^3)$ a divergence-free functions (the initial data and the external force) there exists

$$\vec{u} \in L^\infty_{loc}(]0,+\infty[, L^2(\mathbb{R}^3)) \cap L^2_{loc}(]0,+\infty[, \dot{H}^1(\mathbb{R}^3))$$

a weak solution (Leray, 1934) of

$$\left\{ \begin{array}{ll} \partial_t \vec{u} + \mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f}, \quad div(\vec{u}) = 0, & \text{on }]0,+\infty[\times\mathbb{R}^3, \\ \vec{u}(0,\cdot) = \vec{u}_0, & \end{array} \right.$$

which verifies the energy inequality: for all $T > 0$,

$$\|\vec{u}(T)\|^2_{L^2} + 2\nu \int_0^T \|\nabla\otimes\vec{u}(t)\|^2_{L^2} dt \leq \|\vec{u}_0\|^2_{L^2} + 2\int_0^T\int_{\mathbb{R}^3} \vec{u}(t,x)\cdot\vec{f}(x) dx\, dt. \tag{3}$$

(2) The non-periodic framework: the conditions on the external force

- $\Rightarrow$ According to the energy cascade model: for a given energy input scale ℓ_0 the external force $\vec{f}$ acts on the fluid only at this scale ℓ_0 and thus only at the frequencies of the order $\kappa_0 = \frac{1}{\ell_0}$.
- $\Rightarrow$ A theoretical way to model this fact is to suppose that

$$supp\left(\widehat{\vec{f}}\right) \subset \left\{\xi \in \mathbb{R}^3 : \frac{\rho_1}{\ell_0} \leq |\xi| \leq \frac{\rho_2}{\ell_0}\right\}$$

  where $0 < \rho_1 < \rho_2$ are constants.
- $\Rightarrow$ We define the averaged external force $F > 0$ by

$$F = \frac{\|\vec{f}\|_{L^2}}{\ell_0^{\frac{3}{2}}}.$$

(2) The non-periodic framework: four physic quantities (Constantin, 2003)

(A) The fluid's characteristic length :

$$L_c = \frac{F}{\|\nabla \otimes \vec{f}\|_{L^\infty}}$$

(by the Bernstein inequalities we get that $L_c \gtrsim \ell_0$).

(B) The fluid's averaged velocity:

$$U = \left(\limsup_{T\to+\infty} \frac{1}{T} \int_0^T \|\vec{u}(t)\|_{L^2}^2 \frac{dt}{{\ell_0}^3} \right)^{\frac{1}{2}}.$$

(C) The energy dissipation rate:

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(D) The Reynolds numbers:

$$Re = \frac{UL_c}{\nu}.$$

Theorem (Constantin, 2003)

Let be $\ell_0 > 0$ *and let be* $\vec{f} \in L^2(\mathbb{R}^3)$ *a divergence-free external force such that* $\widehat{\vec{f}}$ *is localized at the frequencies* $\frac{\rho_1}{\ell_0} \leq |\xi| \leq \frac{\rho_2}{\ell_0}$. *Let be* $\vec{u}_0 \in L^2(\mathbb{R}^3)$ *a divergence-free function and let be* $\vec{u} \in L^\infty_{loc,t} L^2_x \cap L^2_{loc,t} \dot{H}^1_x$ *a weak solution of*

$$\left\{ \begin{array}{l} \partial_t \vec{u} + \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f}, \quad div(\vec{u}) = 0, \quad on \quad]0, +\infty[\times \mathbb{R}^3, \\ \qquad\qquad\qquad\qquad\qquad\qquad \vec{u}(0, \cdot) = \vec{u}_0. \end{array} \right.$$

There exist a constant $c_1 > 0$, *which does not depend of the physic quantities, such that*

$$\varepsilon \leq c_1 \frac{U^3}{L_c} \left(1 + (Re)^{-\frac{1}{2}} + \frac{3}{4} (Re)^{-1} \right).$$

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Remark

As in the periodic framework we get the inequality $\varepsilon \lesssim \frac{U^3}{L_c}$ when $Re >> 1$. However, this theorem presents two lacks which we will talk about more in details.

(2) The non-periodic framework: the lacks in the Constantin's theorem

(a) The definition of the averaged velocity U:

$\Rightarrow$ for $\vec{u} \in L^{\infty}_{loc,t} L^2_x \cap L^2_{loc,t} \dot{H}^1_x$ a weak solution of the N-S equations we do not know a convenient control of $\|\vec{u}(t)\|_{L^2}$ respect to the time t: the energy inequality (3) $\Longrightarrow$ for all $t \in]0, +\infty[$,

$$\|\vec{u}(t)\|^2_{L^2} \leq \|\vec{u}_0\|^2_{L^2} + \frac{t}{2\nu} \|\vec{f}\|^2_{\dot{H}^{-1}}$$

$\Rightarrow$ we can not assure that

$$U = \left(\limsup_{T \to +\infty} \frac{1}{T} \int_0^T \|\vec{u}(t)\|^2_{L^2} \frac{dt}{{\ell_0}^3} \right)^{\frac{1}{2}} < +\infty.$$

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(b) The fluid's characteristic length $L_c = \frac{F}{\|\nabla \otimes \vec{f}_0\|_{L^\infty}}$: in order to prove the Constantin's theorem we need the inequality

$$\|\nabla \otimes \vec{f}\|_{L^2} \leq c {\ell_0}^{-\frac{3}{2}} \|\nabla \otimes \vec{f}\|_{L^\infty}$$

which is not generally verified.

$\Rightarrow$ In order to make sense the averaged velocity U we modify the N-S equations by introducing an additional term: for $\alpha > 0$ and $0 < \kappa_2 < \frac{\rho_1}{\ell_0}$ ($\widehat{\vec{f}}$ is localized at the frequencies $\frac{\rho_1}{\ell_0} \leq |\xi| \leq \frac{\rho_2}{\ell_0}$) we define

$$\widehat{\alpha P_2 \vec{u}}(t,\xi) = \alpha \mathbb{1}_{|\xi|<\kappa_2}(\xi)\widehat{\vec{u}}(t,\xi).$$

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$$\widehat{\alpha P_2 \vec{u}}(t,\xi) = \alpha \mathbb{1}_{|\xi|<\kappa_2}(\xi)\widehat{\vec{u}}(t,\xi).$$

The damped N-S equations

$$\left\{ \begin{array}{ll} \partial_t \vec{u} + \mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f} - \alpha P_2 \vec{u}, \quad div(\vec{u}) = 0, & \text{on} \quad]0,+\infty[\times\mathbb{R}^3, \\ \qquad\qquad\qquad\qquad\qquad\qquad \vec{u}(0,\cdot) = \vec{u}_0. & \end{array} \right.$$

$\Rightarrow$ For all $\alpha > 0$ there exists $\vec{u}_\alpha \in L^\infty(]0,+\infty[, L^2(\mathbb{R}^3)) \cap L^2_{loc}(]0,+\infty[, \dot{H}^1(\mathbb{R}^3))$ a weak solution.

(2) The non-periodic framework: the damped Navier-Stokes equations

$\Rightarrow$ The solution $\vec{u}_\alpha$ verifies the energy inequality: for all $t \in]0, +\infty[$,

$$\|\vec{u}_\alpha(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla \otimes \vec{u}_\alpha(s)\|_{L^2}^2 ds \quad \leq \|\vec{u}_0\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} \vec{f} \cdot \vec{u}_\alpha \, dxds$$
$$-2\alpha \int_0^t \|P_2 \vec{u}_\alpha(s)\|_{L^2}^2 ds,$$

$\Rightarrow$ by the Grönwall inequality, for $\beta > 0$, we get $\forall t \in]0, +\infty[$

$$\|\vec{u}_\alpha(t)\|_{L^2}^2 \leq \|\vec{u}_0\|_{L^2}^2 e^{-\frac{\beta}{2} t} + \frac{4}{\beta} \|\vec{f}\|_{L^2}^2 (1 - e^{-\frac{\beta}{2} t})$$

$\Rightarrow$ in this way

$$U_\alpha = \left(\limsup_{T \to +\infty} \frac{1}{T} \int_0^T \|\vec{u}_\alpha(t)\|_{L^2}^2 \frac{dt}{{\ell_0}^3} \right)^{\frac{1}{2}} < +\infty.$$

(2) The non-periodic framework: the fluid's characteristic length

For a given energy input scale $\ell_0 > 0$ and since $\vec{f} \in L^2(\mathbb{R}^3)$ is such that $\widehat{\vec{f}}$ is localized at the frequencies $\frac{\rho_1}{\ell_0} \leq |\xi| \leq \frac{\rho_2}{\ell_0} \Rightarrow F = \frac{\|\vec{f}\|_{L^2}}{\ell_0^{\frac{3}{2}}}$.

$\Rightarrow$ We introduce the parameter $\gamma := \frac{\|\vec{f}\|_{L^\infty}}{F}$ and we define

$$L = \frac{\ell_0}{\gamma}.$$

$\Rightarrow$ By the Bernstein inequalities we get that:

- $0 < \gamma \leq 1 \Rightarrow L \geq \ell_0$ and
- $c_1\, L \leq L_c \leq c_2\, L$.

Now, we fix α by $\alpha = \frac{\nu}{\ell_0^2}$ and we denote by $\vec{u} \in L^\infty_t L^2_x \cap L^2_{loc,t} \dot{H}^1_x$ the solution of

$$\left\{ \begin{array}{ll} \partial_t \vec{u} + \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f} - \frac{\nu}{\ell_0^2} P_2 \vec{u}, & div(\vec{u}) = 0, \quad]0, +\infty[\times \mathbb{R}^3, \\ & \vec{u}(0, \cdot) = \vec{u}_0. \end{array} \right. \tag{4}$$

$\Rightarrow$ We study the relation: $\varepsilon \approx \frac{U^3}{L}$ when $Re >> 1$

Theorem (2015)

Let be $\ell_0 > 0$ the energy input scale and $\vec{f} \in L^2(\mathbb{R}^3)$ the external force such that $\widehat{\vec{f}}$ is localized at the frequencies $\frac{\rho_1}{\ell_0} \leq |\xi| \leq \frac{\rho_2}{\ell_0}$. Let be $L = \frac{\ell_0}{\gamma}$ the fluid's characteristic length where $\gamma = \frac{\|\vec{f}\|_{L^\infty}}{F}$. Finally, let be $\vec{u} \in L^\infty_t L^2_x \cap L^2_{loc,t}\dot{H}^1_x$ a weak solution of the damped N-S equations (4). We define

- $U = \left(\limsup_{T\to+\infty} \frac{1}{T}\int_0^T \|\vec{u}(t)\|^2_{L^2} \frac{dt}{\ell_0^3}\right)^{\frac{1}{2}}$,
- $\varepsilon = \nu \limsup_{T\to+\infty} \frac{1}{T}\int_0^T \|\nabla \otimes \vec{u}(t)\|^2_{L^2} \frac{dt}{\ell_0^3}$ *and*
- $Re = \frac{U\,L}{\nu}$.

If $Re \geq \frac{2G_0}{\gamma^2}$ then there exist two constants $C_1(G_0), C_2(G_0) > 0$ such that

$$C_1(G_0)\varepsilon \leq \frac{U^3}{L} \leq C_2(G_0)\varepsilon,$$

where $G_0 = \frac{\|\vec{f}\|_{L^\infty}\,\ell_0^3}{\nu^2}$ is a fix and dimensionless quantity.

Remark

In the damped N-S equations

$$\partial_t \vec{u} + \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f} - \alpha P_2 \vec{u}$$

the damping term $-\alpha P_2 \, \vec{u}$ *allows us:*

(i) *to obtain a control on* $\|\vec{u}(t)\|^2_{L^2}$ *when* $t \longrightarrow +\infty$ *such that* $U < +\infty$,

(ii) *by setting* $\alpha = \frac{\nu}{\ell_0^2}$ *and* $L = \frac{\ell_0}{\gamma}$ *we have that* $\varepsilon \approx \frac{U^3}{L}$, *if Re is large enough.*

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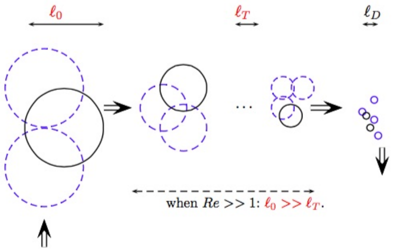
$\Rightarrow$ *However, by the term* $-\frac{\alpha}{\ell_0^2} P_2 \vec{u}$ *we can prove an additional control on the Taylor scale* $\ell_T := \left(\frac{\nu U^2}{\varepsilon}\right)^{\frac{1}{2}}$ *respect to* ℓ_0*:*

$$\ell_T \approx C_3(G_0)\ell_0$$

$\Rightarrow$ *the model given by the damped N-S equations with* $\alpha = \frac{\nu}{\ell_0^2}$ *and* $L = \frac{\ell_0}{\gamma}$ *is actually a non turbulent model.*

(2) The non-periodic framework: a non turbulent model



- $\Rightarrow$ Even in the asymptotic setting of the large Reynolds numbers *Re* we may not conclude that the deterministic model given by the damped N-S equations with with $\alpha = \frac{\nu}{\ell_0^2}$ and $L = \frac{\ell_0}{\gamma}$ is a turbulent one.
- $\Rightarrow$ This deterministic model may be seen as an artificial model of the fluid's mechanics when the Reynolds numbers large enough are not sufficient to characterize the turbulent setting.

Outlook

- ⇒ We recall that we consider a viscous and incompressible fluid in the whole space $\mathbb{R}^3$ where an *stationary* external force $\vec{f} = \vec{f}(x)$ acts on the fluid by introducing kinetic energy *independently* on time.
- ⇒ Since $\vec{f}$ does not depend on time the idea is to consider now the *stationary* N-S equations:

$$\mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f}, \quad div(\vec{u}) = 0, \quad \text{on} \quad \mathbb{R}^3.$$

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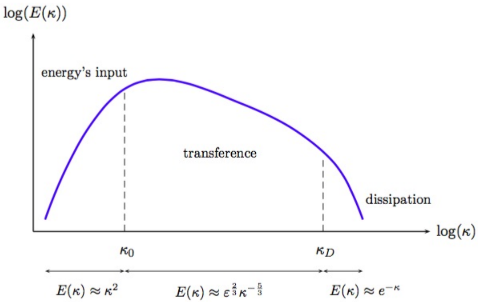
- $\Rightarrow$ We have that $\vec{u} = \vec{u}(x)$, the velocity depends only on the spatial variable.
- $\Rightarrow$ We want to study the exponential decay of $\widehat{\vec{u}}$ according the K41 theory.

Deterministic study of the energy's spectrum: motivation

⇒ The energy's spectrum $E(\kappa)$ is given by $E(\kappa) = \int_{|\xi|=\kappa} \left|\widehat{\vec{u}}(\xi)\right|^2 d\sigma(\xi)$.

According to the K41 theory we have that:



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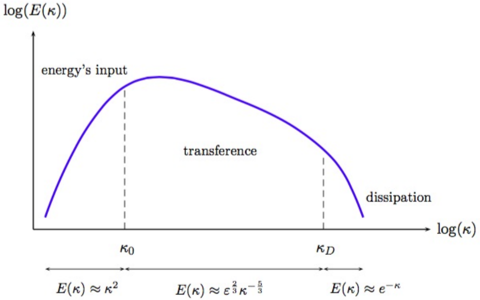

⇒ We will focus in the study of the exponential decay:

$$|\widehat{\vec{u}}(\xi)| \approx e^{-|\xi|} \Longrightarrow E(\kappa) \approx e^{-\kappa}$$

for the highs frequencies $|\xi| >> 1$.

⇒ The expected behavior $E(\kappa) \approx \kappa^2$ $(0 < \kappa < \kappa_0)$ and $E(\kappa) \approx \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}$ $(\kappa_0 < \kappa < \kappa_D)$ is completely unknown.

Deterministic study of the energy's spectrum: the exponential decay

⇒ The idea: we suppose that $\widehat{\vec{f}}$ has an exponential decay and we want to obtain a similar behavior for $\widehat{\vec{u}}$ where $\vec{u}$ is a solution of
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⇒ The idea: we suppose that $\widehat{\vec{f}}$ has an exponential decay and we want to obtain a similar behavior for $\widehat{\vec{u}}$ where $\vec{u}$ is a solution of $\mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f}, \quad div(\vec{u}) = 0.$

Theorem (2016)

Let be $\vec{f} \in \dot{H}^{-1}(\mathbb{R}^3)$ a time independent and divergence-free external force such that for $\varepsilon_0 > 0$ we have that

$$\int_{\mathbb{R}^3} e^{2\varepsilon_0|\xi|} \left|\widehat{\vec{f}}(\xi)\right|^2 \frac{d\xi}{|\xi|^2} < +\infty.$$

Then there exist $\vec{u} \in \dot{H}^1(\mathbb{R}^3)$ solution to the stationary Navier-Stokes equations in the whole space $\mathbb{R}^3$:

$$\mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f}, \quad div(\vec{u}) = 0,$$

such that $\vec{u}$ verifies the exponential frequency decay in norm L^2:

$$\int_{\mathbb{R}^3} e^{2\varepsilon_1|\xi|} \left|\widehat{\vec{u}}(\xi)\right|^2 |\xi|^2 d\xi < +\infty$$

where $\varepsilon_1 > 0$ is a constant which depends of ε_0.

For $0 \leq a < 3$ we define the space of pseudo-measures $\mathcal{PM}^a$ by

$$\mathcal{PM}^a = \left\{ g \in \mathcal{S}'(\mathbb{R}^3) : \widehat{g} \in L^1_{loc}(\mathbb{R}^3) \quad \text{and} \quad |\xi|^a \widehat{g} \in L^\infty(\mathbb{R}^3) \right\}$$

which is a Banach space provided of the norm

$$\|g\|_{\mathcal{PM}^a} = \left\| |\xi|^a \widehat{g} \right\|_{L^\infty}.$$

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Theorem (2016)

Let be $\vec{f} \in \mathcal{PM}$ an stationary and divergence-free external force. There exists a constant $\eta > 0$ such that if

$$\sup_{\xi \in \mathbb{R}^3} e^{|\xi|} \left| \widehat{\vec{f}}(\xi) \right| < \eta$$

then there exists $\vec{u} \in \mathcal{PM}^2$ solution to the stationary Navier-Stokes equations

$$\mathbb{P}(\vec{u} \cdot \nabla \vec{u}) - \nu \Delta \vec{u} = \vec{f}, \quad div(\vec{u}) = 0,$$

such that $\widehat{\vec{u}}$ verifies the following pointwise exponential frequency decay:

$$\left| \widehat{\vec{u}}(\xi) \right| \leq c \frac{e^{-|\xi|}}{|\xi|^2}.$$

Outlook

We are interested in study two properties of the solutions of the stationary Navier-Stokes equations:

$$\mathbb{P}(\vec{u}\cdot\nabla\vec{u}) - \nu\Delta\vec{u} = \vec{f}, \quad div(\vec{u}) = 0.$$

(1) Under spatial decay conditions on the external force $|\vec{f}(x)|$ we study the asymptotic behavior of $|\vec{u}(x)|$ when $|x| \to +\infty$.

(2) We study the long-time asymptotics behavior of the non-stationary Navier-Stokes equations: for $\vec{f} \in L^2$ an stationary and smooth enough external force we consider

$$\begin{cases} \partial_t \vec{v} + \mathbb{P}(\vec{v}\cdot\nabla\vec{v}) - \nu\Delta\vec{v} = \vec{f}, \quad div(\vec{v}) = 0, \quad \text{on }]0,+\infty[\times\mathbb{R}^3, \\ \qquad\qquad \vec{v}(0,\cdot) = \vec{v}_0 \in L^2, \end{cases}$$

$\Rightarrow$ we want to study the properties of

$$\vec{V}(x) = \limsup_{T \longrightarrow +\infty} \frac{1}{T} \int_0^T \vec{v}(t,x) dt.$$

$\Rightarrow$ Do we have that $\vec{V}(x) = \vec{u}(x)$?

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Thank you for attention