Optimal execution cost for liquidation through a limit order market

Etienne Chevalier[*] [†] Vathana Ly Vath [‡] [†] Alexandre Roch [§] Simone Scotti[¶]

July 6, 2015

Abstract

We study the problem of optimally liquidating a large portfolio position in a limit order book market. We allow for both limit and market orders and the optimal solution is a combination of both types of orders. Market orders deplete the order book, making future trades more expensive, whereas limit orders can be entered at more favorable prices but are not guaranteed to be filled. We model the bid-ask spread with resilience by a jump process, and the market order arrival process as a controlled Poisson process. The objective is to minimize the execution cost of the strategy. We formulate the problem as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. We conclude with a calibration of the model on recent market data and a numerical implementation.

Keywords: Liquidity Risk, Limit Order Books, Impulse Control, Viscosity Solutions, System of Variational Inequalities.

MSC2000 subject classification: 60G40, 91B70, 93E20.

[*]Université d'Evry, Laboratoire Analyse et Probabilités, France, etienne.chevalier@univ-evry.fr.
[†]This author's research was supported by the "Chaire Marchés en Mutation", Fédération Bancaire Française, and the French ANR research grant LIQUIRISK
[‡]Université d'Evry and ENSIIE, Laboratoire Analyse et Probabilités, France, lyvath@ensiie.fr.
[§]University of Quebec at Montreal (UQAM) - Faculty of Management, roch.alexandre_f@uqam.ca. This author's research was supported by the Institute of Financial Mathematics of Montreal and the Natural Sciences and Engineering Research Council of Canada.
[¶]Université Paris Diderot, LPMA, France, simone.scotti@math.univ-paris-diderot.fr.

1 Liquidity Risk in Limit Order Books

The study of market liquidity consists in quantifying the costs incurred by investors trading in markets in which supply or demand is finite, trading counterparties are not continuously available, or trading causes price impacts. Liquidity is a risk when the extent to which these properties are satisfied varies randomly through time. Liquidity and liquidity risk models vary considerably from one study to the next according to the problem at hand or the paradigm considered. For instance, Back [5] and Kyle [23] construct an equilibrium model for dealers markets with insider trading. Constantinides [15], Davis and Norman [17], and Shreve and Soner [30] study the portfolio selection problem with first order liquidity costs, namely proportional transaction costs arising from a bid-ask spread. There has also been a number of studies on large trader models ([6], [24], [28]), and dynamic supply curves ([12]), with a more recent emphasis on liquidation problems with market orders ([2], [3], [26]).

It is often assumed in financial modeling that agents are liquidity takers in the sense that they trade at the available prices, albeit with a liquidity premium that must be paid for immediacy of trading. However, in any financial market structure there must also necessarily exist market participants who are price setters (i.e. liquidity providers). For instance, in dealers markets, a market-maker (or specialist) quotes bids and offers and serves as the intermediary between public traders. However, in limit order book markets, any public trader can also play the role of liquidity provider by posting prices and quantities at which he is willing to buy or sell while waiting for a counterparty to engage in that trade. Limit orders can be entered at more favorable prices but are not guaranteed to be filled. On the other hand, a market order is filled automatically against existing limit orders, albeit at a less favorable price as it depletes the order book, making additional trades more expensive. It is therefore desirable to consider financial models with an enlarged set of admissible trading strategies by including the possibility of making both limit orders and market orders. In this paper, we consider the liquidation problem of a large portfolio position from this perspective.

Many authors have investigated the liquidation and market making problems with limit orders only, in particular [4], [7], [13], [18], [19], [20] and [25]. In these models, the arrival intensity of outside market orders that match the limit orders that are posted is typically a function of the spread between the posted price and a reference price. In a more complex model, Cartea et al. [11] develop a high-frequency limit order trading strategy in a limit order market characterized by feedback effects in market orders and the shape of the order book, and by adverse selection risk due to the presence of informed traders who make influential trades. Kühn and Muhle-Karbe [22] provide an asymptotics analysis for a small investor who sets bid and ask prices and seeks to maximize expected utility when the spread is small.

On the other hand, some authors consider a limit order market in which both limit and market orders are possible. Guilbaud and Pham [21] determine the optimal trading strategy of a market maker who makes both types of trades and seeks to maximize the expected utility over a short term horizon. Cartea and Jaimungal [9] determine the optimal liquidation schedule in a limit order market in which the liquidity cost of a market order is fixed, and the probability of passing a limit order depends on the spread between the posted price and a reference price, modeled as a Brownian motion plus drift. The investor's value function includes a quadratic penalty defined in terms of a target inventory schedule. In this work, we also consider a limit order market in which both limit and market orders are allowed, and study the problem of optimally liquidating a large portfolio position. Our contribution to the above literature is to consider spread dynamics which are impacted by both limit and market order strategies. Market orders that the investor places directly increase the observed bid-ask spread. As a result, past market orders have a direct impact on future liquidity costs. Furthermore, limit orders posted inside the bid-ask spread effectively decrease the observed spread and have an impact on the future probability distributions of its jumps.

We model the bid-ask spread with resilience (mean reversion) and a jump process, and the market order arrival process as a controlled Poisson process. See Section 2 for a description of the model. The objective is to liquidate a fixed number of shares of a risky asset by minimizing the expected liquidity premium paid. We formulate the problem in Section 3 as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. In Section 4, we numerically implement the model and calibrate it to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook*.

2 The Limit Order Book Market Model

Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space supporting a random Poisson measure M on $[0, T] \times \mathbb{R}$ with mean measure $\gamma_t \; dt \; m(dz)$ where $\gamma : [0, T] \to (0, \bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$, with $m(\mathbb{R}) < \infty$. We consider a market with a risky asset that can be traded through a limit order book. We consider a large investor whose goal is to liquidate a number $N > 0$ of shares of this risky asset. The investor sets a date T before which the position must be liquidated and attempts to minimize the price impact of the liquidation strategy. In a limit order book market, there are two types of transactions: limit orders in which quantities and prices to trade are constantly entered (and potentially later canceled) in a limit order book, and market orders which are executed against the most favorable existing limit orders. The lowest limit order price to sell is the best ask price, whereas the highest limit order price to buy is the best bid. The trades that the investor who wants to liquidate a position can make are therefore sell limit orders, which will be executed when an incoming buy market order enters the system, and sell market orders, which are automatically executed against the existing buy limit orders.

Market orders. The investor can make market orders by controlling the time and the size of his trades. This is modeled by an impulse control strategy $\beta = (\tau_i,\ \xi_i)_{i\leq n}$ where the τ_i's are stopping times representing the intervention times of the investor and the ξ_i's are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathbb{N}$ and giving the number of shares sold by a market order at time τ_i.

Limit Orders. The investor can also make limit orders. We denote by $\mathcal{A}_0$ a compact subset of $[0,\infty)$ representing the set of possible spreads *below the current best ask price* at which the investor can place a limit order to sell in the order book. We also add the admissibility condition that the limit price is above the current best bid price, otherwise the limit order would in fact be a market order. In other words, the investor can choose to place his limit price anywhere inside the bid-ask spread or at the current best ask price. Since the effect of this new limit order is that the best ask price can now be lower, we call the best ask price *excluding* the investor's limit order the *otherwise best ask price*. In practice, many limit orders could be placed, however we only consider one limit order at a time, for a fixed number of shares denoted n', to keep the problem mathematically tractable. Clearly, the higher the investor sets the price in the limit order the more profitable it is, however the less likely it is that the order will be executed. The spread *below the current best ask price* is an $\mathcal{A}_0$-valued stochastic control denoted by $\alpha = (\alpha_t)_{t\leq T}$.

Definition 2.1. (Investor's control strategy) *We define the investor's control strategy as being the full control available to the investor, thus given by a pair of controls $\delta = (\alpha, \beta)$.*

Bid-Ask Spread. Mathematically, we define the dynamics of the bid-ask spread as follows. We denote by X_t the spread between the best bid and the best ask price *excluding* the investor's limit price at time t. Since α_t represents the spread *below the otherwise best ask price* at which the investor places a limit order to sell in the order book, the true observed bid-ask spread is $X_t - \alpha_t$. Between the investor's market orders, we assume the spread X is impacted by α and follows

$$dX_t^\alpha = \mu(t, X_{t-}^\alpha, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\alpha, \alpha_t, z)\tilde{M}(dt, dz). \tag{2.1}$$

Under this construction, the limit order α sends a signal and modifies the distribution of the jumps of X, represented by G. Here $\tilde{M}$ is the compensated random measure of M, and μ is a deterministic and Lipschitz continuous function in the second argument, satisfying the following growth condition. There exists $\bar{\mu} > 0$ such that

$$\sup_{t\in[0,T],a\in\mathcal{A}_0} |\mu(t,x,a)| \leq \bar{\mu}(1+|x|), \text{ for all } x \geq 0.$$

We equally assume that μ and G are such that X remains positive. Moreover, we assume the following Lipschitz-type condition on G: there exists g such that for all x, y, a, z

$$|G(x,a,z) - G(y,a,z)| \leq |x-y| g(a,z), \tag{2.2}$$

and

$$\sup_{x\in\mathbb{R}^+,a\in\mathcal{A}_0} \int_{\mathbb{R}} |g(x,a,z)|^{p+1} m(dz) < \infty, \tag{2.3}$$

for some $p \geq 1$.

Liquidity cost

The liquidity cost due to a market order to sell is defined in terms of the structure of the limit order book. We summarize the information contained in the order book by a function $S(t,x,n)$ which gives the proceeds obtained for a sale of n shares at time t done through market orders when the spread equals x. In the order book density case, this corresponds to Equation 12 in [2]. Let A_t be a stochastic process representing the best ask price. We may then define the liquidity cost due to a market sell order of size n, denoted by $L(t,x,n)$, in terms of the best ask price as follows

$$L(t,x,n) := nA_t - S(t,x,n). \tag{2.4}$$

The slippage of a market order of size n is then defined as a fixed transaction cost, $k > 0$, plus the liquidity cost, i.e.

$$K(t,x,n) = k + L(t,x,n). \tag{2.5}$$

Example. The simplest example is a quadratic model with proportional transaction costs:

$$S(t,x,n) = (A_t - x)n - \zeta_t n^2,$$

with A_t and ζ_t two stochastic processes representing the best ask price and a measure of illiquidity. This model arises from a limit order book with constant density as shown in [29]. In the quadratic model, $L(t, x, n) = xn + \zeta_t n^2$.

We introduce the set of functions from $[0, T] \times \mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p (see (2.3)) in the second argument, uniformly in the first, and denote it by $\mathcal{P}$. For technical reasons (see Proposition (3.4)), we assume that for all $n \in \{0, .., N\}$, the function $L(\cdot, \cdot, n)$ belongs to $\mathcal{P}$.

Impact on the best bid

During a transaction, the investor's market orders are matched with the existing limit orders in the order book so that the result is a shift in the best bid price to the left by an amount denoted by $I(t, x, n)$. In [2], this quantity is called the extra spread and denoted by D_t^B. The bid-ask spread will necessarily increase by the same amount. In the quadratic model for S, the quantity I is given by $I(t, x, n) = 2\zeta_t n$ (c.f. [29]). See (4.11) and (4.12) below for a discrete model for L and I.

Dynamics of the controlled bid-ask spread. As noted in the previous section, market orders have an impact on the best bid. As a result, market orders also have an impact on the bid-ask spread X that would otherwise be observed if $\alpha = 0$, as well as the true observed bid-ask spread $X - \alpha$. The resulting dynamic for X^δ (with $\delta = (\alpha, \beta)$) taking into account both α and β is

$$\begin{cases} dX_t^\delta &= \mu(t, X_t^\delta, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\delta, \alpha_{t-}, z)\tilde{M}(dt, dz) \quad \text{if } \tau_n < t < \tau_{n+1} \\ X_{\tau_n}^\delta &= \check{X}_{\tau_n^-}^\delta + I(\tau_n, \check{X}_{\tau_n^-}^\delta, \xi_n), \end{cases} \tag{2.6}$$

where $\check{X}_{t^-}^\delta = X_{t^-}^\delta + \Delta X_t^\delta$, ΔX_t^δ is the jump of the measure M at time t. The superscripts in controlled processes will often be omitted to alleviate the notation.

Market orders arrival. The investor's limit orders are matched against other market participants' market orders. The probability of a limit order being matched by an incoming market order depends on the strategy α and is constructed as follows. We start with a time inhomogeneous Poisson process $\mathcal{N}$, independent of W and M, with intensity $\lambda(t, 0) \geq \underline{\lambda} > 0$, $t \geq 0$. The jumps of this Poisson process are denoted θ_i, $i \geq 1$. For all $x > 0$, we define intensity functions $\lambda(\cdot, x) : [0, T] \to [0, \infty)$, and assume $(\lambda(\cdot, x))_{x>0}$ is an equicontinuous family of functions, bounded below and above by constants $\underline{\lambda}, \overline{\lambda} > 0$. If the investor chooses to place a limit order at a spread α_t below the otherwise best ask price at time t, the likelihood of the execution of this order depends on the observed spread $X_t - \alpha_t$ and arrives with an intensity $\lambda(t, X_t - \alpha_t)$. At the time θ_i, the investor's limit order will go through for a random quantity equal to Y_i, less or equal to n' (the fixed size of the limit order), and independent of $\mathcal{F}_{\theta_i -}$. The fact that the jump intensity is time-dependent is particularly relevant in markets where there is well-known u-shaped trading volume pattern during the day.

Let $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^\alpha$ with $Z_0^\alpha = 1$ and

$$dZ_t^\alpha = Z_{t-}^\alpha \left(\frac{\lambda(t, X_t - \alpha_t)}{\lambda(t,0)} - 1 \right) (d\mathcal{N}_t - \lambda(t,0)dt) \, .$$

Then a control α changes the distribution of $\mathcal{N}$ under $\mathbb{P}$ to the distribution of $\mathcal{N}$ under $\mathbb{P}^\alpha$, by changing the intensity of $\mathcal{N}$ from $\lambda(t,0)$ to $\lambda(t, X_t - \alpha_t)$.

The slippage of a limit order that is matched at time θ_i is then given by $\alpha_{\theta_i} Y_i$.

Dynamics of the remaining number of shares to liquidate $N^{\delta,n,t}$. To keep track of the portfolio through time, we define a pure jump process $N^{\delta,n,t}$ representing the remaining number of shares in the portfolio (taking into consideration transactions through both limit orders and market orders) when the portfolio starts with n remaining shares at time t. The process $N^{\delta,n,t}$ thus starts at $N_t^{\delta,n,t} = n$ at time t, is piecewise constant, and jumps by $-(Y_i \wedge N_{\theta_i -}^{\delta,n,t})$ at time θ_i and by $-(\xi_i \wedge N_{\tau_i -}^{\delta,n,t})$ at time τ_i. This is understood to mean that the process jumps by $-(Y_i + \xi_j) \wedge N_{\theta_i -}^{\delta,n,t}$ if $\theta_i = \tau_j$ for some $i, j \geq 1$.

Admissible control strategies

Now, we define the set of admissible strategies. The limit orders control strategy $\alpha = (\alpha_s)_{0 \leq s \leq T}$ is assumed to be a stochastic Markov control such that $\alpha_t < X_{t-}^\delta$ for all $t \leq T$. We denote the set of Markov control by $\mathcal{A}$. Let $\mathcal{T}_{t,T}$ be the set of stopping times with values in $[t,T]$. The set of admissible strategies started at time $t \in [0,T]$ when the investor has n shares remaining in the portfolio and that the spread is equal to x is defined as

$$\begin{aligned} \mathcal{AB}(t,n,x) \quad &= \quad \{\delta = (\alpha, \beta) : \alpha \in \mathcal{A}, \beta = (\tau_i, \xi_i)_{i \leq n}, \tau_i \in \mathcal{T}_{t,T};\ \xi_i \leq n \text{ is an } \mathbb{N}\text{-valued random variable} \\ &\qquad \mathcal{F}_{\tau_i^-} - \text{measurable s.t. } \tau^{\delta,n,t} \leq T\}, \end{aligned}$$

where $\tau^{\delta,n,t} = \inf\{s \geq t : \ N_s^{\delta,n,t} = 0\}$.

The Control Problem

The investor's goal is to minimize expected slippage by balancing his actions between market orders, which are more expensive due to immediacy, and limit orders, for which the execution time is unknown and random but are executed at more favorable prices. For a strategy $\delta = (\alpha, \beta) \in \mathcal{AB}(t,n,x)$ started at time t, slippage is defined as

$$S_T^\delta = \sum_{i=1}^{n} K(\tau_i, \check{X}^\delta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i \leq \tau^\delta} + \sum_{i\geq 1} \alpha_{\theta_i} Y_i \mathbb{1}_{\theta_i \leq \tau^\delta}.$$

For $(t, x, n) \in [0, T] \times [0, +\infty) \times \mathbb{N}$, we define the optimal expected slippage function in the following way:

$$C_n(t, x) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha} S_T^\delta, \tag{2.7}$$

with $\mathbb{E}_{t,x,n,\alpha}$ the expectation under $\mathbb{P}^\alpha$, given that $N_t = n$ and $X_t = x$. For convenience, we extend this function to negative values of n by letting $C_{-i}(t, x) = 0$ for $i \in \mathbb{N}^*$. We have the following boundary condition:

$$C_n(T, x) = K(T, x, n) \text{ for all } n \in \mathbb{N}^*,$$

which follows readily from the fact that $\tau^{\delta,n,T} = T$, so that the investor must necessarily liquidate the remaining part of his portfolio with a market order at time T.

Remark 2.1. *Under the assumption that the ask price A is a martingale, which is commonly used in the literature, minimizing slippage is equivalent to maximizing the proceeds of the sales, which is expressed in terms of S_T^δ by*

$$\int_0^T A_t dN_t - S_T^\delta.$$

Indeed,

$$\mathbb{E}_{t,x,n,\alpha}\left(\int_0^T A_t dN_t - S_T^\delta\right) = \mathbb{E}_{t,x,n,\alpha}\left(N_T A_T - \int_0^T N_{t-} dA_t - N_t A_t - S_T^\delta\right) = -nA_t - \mathbb{E}_{t,x,n,\alpha} S_T^\delta,$$

in which A_t is $\mathcal{F}_t-$measurable.

2.1 Penalty Function

The maturity T is an urgency parameter. The shorter it is, the more aggressive the strategy and the higher the liquidation cost. However, in order to impose more urgency in the liquidation, it is possible to include a penalty function or a risk aversion parameter in the minimization problem. For instance, one could follow Almgren and Chriss [1] and consider

$$C_n(t, x, s) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha}\left[S_T^\delta + \eta (S_T^\delta)^2\right],$$

for some positive risk aversion constant η. In that case, however, the complexity of the problem is largely increased by the fact that the value function now depends on 3 variables (excluding n).

The other possibility is therefore to add a penalty function π in terms of the number of remaining shares at time t:

$$C_n(t,x) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha}\left[S_T^\delta + \int_t^T \pi(N_s^\delta, s)ds\right]. \tag{2.8}$$

This penalty function can be used to target a specific liquidation schedule as in Cartea and Jaimungal [9], it can be a proxy for the variance of the value of the remaining shares in the portfolio when π is of the quadratic form (see Cartea and Jaimungal [10]), or it can reflect a negative drift in the ask price or "short-term price signal" as suggested by Almgren [3]. The extension of our results to Equation 2.8 is straightforward. See Remark 3.2 below and the section on numerical results for more details.

3 Characterization of the slippage function

In this section, we prove that the function C_n is the viscosity solution of an associated quasi-variational inequality. We first introduce the infinitesimal generator of the process $(t, X_t)_{t\geq 0}$ between two market orders:

$$\mathcal{L}^a u(t,x) = \frac{\partial u}{\partial t} + \mu(t,x,a)\frac{\partial u}{\partial x} + \gamma_t \int_{\mathbb{R}} (u(t,x+G(x,a,z)) - u(t,x))m(dz),$$

and the limit orders operator:

$$\Delta_n^a u(t,x) = \lambda(t,x-a)\left[f(a) + \sum_{i=1}^{\infty} p_i C_{n-i}(t,x) - u(t,x)\right],$$

in which $p_i = \mathbb{P}(Y_1 = i)$ $(i \geq 1)$ and $f(a) = a\sum_{i=1}^{\infty} ip_i$, $a \in \mathcal{A}_0$. Finally, define the impulse function for market orders:

$$\mathcal{M}_n(t,x) = \min_{i\in\{1,\dots,n\}} \left[C_{n-i}(t,x+I(t,x,i)) + K(t,i,x)\right].$$

Notice that, for all $(t,x,n) \in [0,T]\times\mathbb{R}_+\times\mathbb{N}^*$, we deduce from (2.7) that

$$0 \leq C_n(t,x) \leq K(t,x,n) = \kappa + L(t,x,n).$$

Therefore, recalling that $\mathcal{P}$ is the set of functions from $[0,T]\times\mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, we have $C_n \in \mathcal{P}$ for all $n \in \mathbb{N}$.

Our main result of this section is the following theorem.

Theorem 3.1. *For all $n \geq 1$, C_n is the unique continuous viscosity solution in $\mathcal{P}$ of the following variational inequality:*

$$\begin{cases} \min\left(\min_{a\in\mathcal{A}_0} \mathcal{L}^a u + \Delta_n^a u;\ \mathcal{M}_n - u\right) = 0 & \textit{on } [0,T)\times[0,\infty), \\ u(T,x) = K(T,n,x) & \textit{for } x \geq 0. \end{cases} \tag{3.9}$$

Remark 3.2. *When the penalty function π is present, the variational inequality becomes*

$$\begin{cases} \min\left(\min_{a\in\mathcal{A}_0} \mathcal{L}^a u + \Delta_n^a u + \pi(n,t);\ \mathcal{M}_n - u\right) = 0 & \textit{on } [0,T)\times[0,\infty), \\ u(T,x) = K(T,n,x) & \textit{for } x \geq 0. \end{cases}$$

The proof of Theorem 3.1 is based on the following dynamic programming principles (DPP):

Proposition 3.1. *Let $n \in \mathbb{N}$. For all $(t,x) \in [0,T)\times\mathbb{R}^+$,*

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[\mathcal{M}_{\widetilde{N}_\nu}(\nu,\widetilde{X}_\nu) + \int_t^\nu \lambda(s,\widetilde{X}_s-\alpha_s)f(\alpha_s)ds].$$

(Here, $\widetilde{N}$ and $\widetilde{X}$ denote respectively the processes $N^{a,\beta}$ and $X^{a,\beta}$ obtained with $\beta \equiv 0$.) In particular, by the dynamic programming principle for optimal stopping problems we find that for all $\theta \in \mathcal{T}_t$, we have

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A},\nu\in\mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[C_{\widetilde{N}_\theta}(\theta,\widetilde{X}_\theta)\mathbb{1}_{\theta<\nu} + \mathcal{M}_{\widetilde{N}_\nu}(\nu,\widetilde{X}_\nu)\mathbb{1}_{\nu\leq\theta} + \int_t^{\nu\wedge\theta} \lambda(s,\widetilde{X}_s-\alpha_s)f(\alpha_s)ds],$$

and for all $\epsilon > 0$,

$$C_n(t,x) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}_{t,x,n,\alpha}[C_{\widetilde{N}_\varsigma}(\varsigma,\widetilde{X}_\varsigma) + \int_t^\varsigma \lambda(s,\widetilde{X}_s-\alpha_s)f(\alpha_s)ds],$$

for all $\varsigma \leq \tau^\epsilon := \inf\{u \geq t : C_{\widetilde{N}_u}(u,\widetilde{X}_u) > \mathcal{M}_{\widetilde{N}_u}(u,\widetilde{X}_u) - \epsilon\}$.

Proof: Set $H_t = \sum_{i\geq1} Y_i \mathbb{1}_{\{\theta_i\leq t\}}$. From the fact that the Y_i's are independent of $\mathcal{F}_t$ for all t and identically distributed, we know that $H_t - \int_0^t \lambda(s,X_s^\delta-\alpha_s)\mathbf{E}Y_i ds$ is a $\mathbb{P}^\alpha$-martingale. From the fact that $\mathcal{M}_t(\alpha) := \int_0^t \alpha_s(dH_s - \lambda(s,X_s^\delta-\alpha_s)\mathbf{E}Y_i ds)$ is also a $\mathbb{P}^\alpha$-martingale for all adapted predictable processes α, we can write C_n as

$$C_n(t,x) = \inf_{\delta\in\mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha}\overline{S}_T^\delta$$

with

$$\overline{S}_T^\delta = \sum_{i=1}^n K(\tau_i, \check{X}^\delta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i \le \tau^\delta} + \int_t^{\tau^\delta} \lambda(s, X_s^\delta - \alpha_s) f(\alpha_s) ds,$$

in which $f(a) = a\mathbf{E}(Y_1)$.

Using the change of measure Z^α, we can write

$$\begin{aligned} C_n(t,x) &= \frac{1}{z} \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,z} Z_T^\alpha \overline{S}_T^\delta \\ &= \frac{1}{z} \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,z} \sum_{i=1}^n Z^\alpha_{\tau_i} K(\tau_i, \check{X}^\delta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i \le \tau^\delta} + \int_t^{\tau^\delta} Z_s^\alpha \lambda(s, X_s^\delta - \alpha_s) f(\alpha_s) ds \\ &:= \frac{1}{z} \overline{C}_n(t,x,z) \end{aligned}$$

with $\mathbb{E}_{t,x,n,z}$ the expectation under $\mathbb{P}$ given that $X_t = x, N_t = n$ and $Z_t^\alpha = z$.

By Theorem 8.5 of Oksendal and Sulem [27], $\overline{C}_n(t,x,z)$ satisfies the following Dynamic Programming Principle:

$$\overline{C}_n(t,x,z) = \inf_{\alpha \in \mathcal{A}, \nu \in \mathcal{T}_t} \mathbb{E}_{t,x,n}[\overline{\mathcal{M}}_{\widetilde{N}_\nu}(\nu, \widetilde{X}_\nu, Z_\nu^\alpha) + \int_t^\nu Z_s^\alpha \lambda(s, \widetilde{X}_s - \alpha_s) f(\alpha_s) ds],$$

in which

$$\overline{\mathcal{M}}_n(t,x,z) = \min_{i \in \{1,\dots,n\}} \left[\overline{C}_{n-i}(t, x + I(t,x,i), z) + zK(t,i,x)\right].$$

Consequently,

$$\begin{aligned} C_n(t,x) &= \frac{1}{z}\overline{C}_n(t,x,z) = \frac{1}{z} \inf_{\alpha \in \mathcal{A}, \nu \in \mathcal{T}_t} \mathbb{E}_{t,x,n}[Z_\nu^\alpha \mathcal{M}_{\widetilde{N}_\nu}(\nu, \widetilde{X}_\nu) + \int_t^\nu Z_s^\alpha \lambda(s, \widetilde{X}_s - \alpha_s) f(\alpha_s) ds] \\ &= \inf_{\alpha \in \mathcal{A}, \nu \in \mathcal{T}_t} \mathbb{E}_{t,x,n,\alpha}[\mathcal{M}_{\widetilde{N}_\nu}(\nu, \widetilde{X}_\nu) + \int_t^\nu \lambda(s, \widetilde{X}_s - \alpha_s) f(\alpha_s) ds]. \end{aligned}$$

□

First we notice that $C_0 = 0$ is obviously a continuous solution of (3.2) for $n = 0$. We prove Theorem 3.1 using an induction argument on n and the following propositions.

Proposition 3.2 (Subsolution Property)**.** *Suppose C_k is a continuous function for all $k \in \{0, ..., n-1\}$. The upper semi-continuous envelope of C_n, denoted by C_n^u is then a subsolution of (3.2).*

Proposition 3.3 (Supersolution Property)**.** *Suppose that for all $k \in \{0, ..., n-1\}$ C_k is a continuous function. The lower semi-continuous envelope of C_n, denoted by C_n^l, is then a supersolution of (3.2).*

The proofs of the above propositions can be found in the appendix.

Proposition 3.4 (Comparison Principle). *Assume that for all $k \in \{0, ..., n-1\}$, C_k is a continuous function. Then if v is viscosity subsolution of (3.2) and w is a viscosity supersolution of (3.2), such that $\lim_{x\to 0} v^u \le \lim_{x\to 0} w^l$, and $v,\ w \in \mathcal{P}$ then $v^u \le w^l$.*

Proof: For ease of exposition, we omit the superscripts denoting the semi-continuous envelopes of v and w.

We assumed that v and w have at most polynomial growth of order $p > 0$. Let $\varepsilon = \min(1, 1/\underline{\lambda})$, $b > 0$ and define $\varphi(t,x) = -e^{-bt}(1 + x^{p+1}) - 1/\underline{\lambda}$, for $x \ge 0$, $t \le T$.

Let $m \ge 1$. We need to show that $\varrho := \sup_{(t,x)} v_m - w \le 0$, with $v_m := v + \frac{1}{m}\varphi$. Suppose on the contrary that $\varrho > 0$. Since,

$$\lim_{x\to\infty} v_m - w = -\infty \text{ and } \lim_{x\to 0} v_m - w \le 0, \tag{3.10}$$

it is clear that this supremum is attained at some point $(t_0, x_0) \in [0,T) \times \mathcal{O}$ in which $\mathcal{O}$ is an open subset of $\mathbb{R}_+$, i.e. $\varrho = v_m(t_0,x_0) - w(t_0,x_0)$, with $t_0 < T$ and $x_0 > 0$.

For $i \ge 1$, define $\Phi_i(t,x,y) = v_m(t,x) - w(t,y) - \frac{i}{2} \mid x - y \mid^2$. Let $\varrho_i = \sup_{[0,T]\times\mathcal{O}^2} \Phi_i(t,x,y)$, which we can assume is attained at some point $(\hat{t}_i, \hat{x}_i, \hat{y}_i) \in [0,T] \times \overline{\mathcal{O}}^2$. By taking a subsequence, we can also assume there exists a point $(\hat{t}_0, \hat{x}_0, \hat{y}_0)$ to which $(\hat{t}_i, \hat{x}_i, \hat{y}_i)$ converges as $i \to \infty$. For i large enough, we can then assume that $\hat{t}_i < T$, and $\hat{x}_i > 0$. The goal is to apply Theorem 8.3 of [16] to the functions Φ_i for each $i \ge 1$ and take a limit as $i \to \infty$. In order to do so, we first want to show that v_m is a subsolution of (3.2).

We begin by proving that the function $v_m = v + \frac{1}{m}\varphi$ is a strict subsolution of (3.2) in the sense that

$$\min(\min_{a\in\mathcal{A}_0} \mathcal{L}^a v_m + \Delta_n^a v_m, \mathcal{M}_n - v_m) \ge \frac{1}{m}\varepsilon > 0.$$

Since v is a subsolution, $\mathcal{M}_n(t,x) - v(t,x) \ge 0$ which implies that $\mathcal{M}_n(t,x) - v(t,x) - \frac{1}{m}\varphi(t,x) \ge \frac{1}{m}\varepsilon$. On the other hand, on $[0,T] \times \mathbb{R}_+$, we calculate

$$\begin{aligned} e^{bt}\mathcal{L}^a\varphi(t,x) \quad &\ge \quad b(1 + x^{p+1}) - \mu(t,x,a)(p+1)x^p - \gamma_t \int_{\mathbb{R}} ((x + G(x,a,z))^{p+1} - x^{p+1}) m(dz) \\ &\ge \quad bx^{p+1} + \sum_{i=0}^{p+1} d_i x^i + b, \end{aligned}$$

where d_i $(i \le p)$ are constants depending on model parameters. Therefore, for b large enough, we get

$\mathcal{L}^a\varphi(t,x) \geq 0$ and, consequently,

$$\min_{a\in\mathcal{A}_0}\left(\mathcal{L}^a[v+\frac{1}{m}\varphi]+\Delta_n^a[v+\frac{1}{m}\varphi]\right) \quad \geq \quad \min_{a\in\mathcal{A}_0}\left(\Delta_n^a[v+\frac{1}{m}\varphi]-\Delta_n^a v\right)$$

because v is a subsolution which implies that $\mathcal{L}^a v \geq -\Delta_n^a v$ for all a. Hence,

$$\min_{a\in\mathcal{A}_0}\left(\mathcal{L}^a[v+\frac{1}{m}\varphi]+\Delta_n^a[v+\frac{1}{m}\varphi]\right) \quad \geq \quad \min_{a\in\mathcal{A}_0}\lambda(t,x-a)\frac{1}{\underline{\lambda} m} \geq 1/m \geq \varepsilon/m.$$

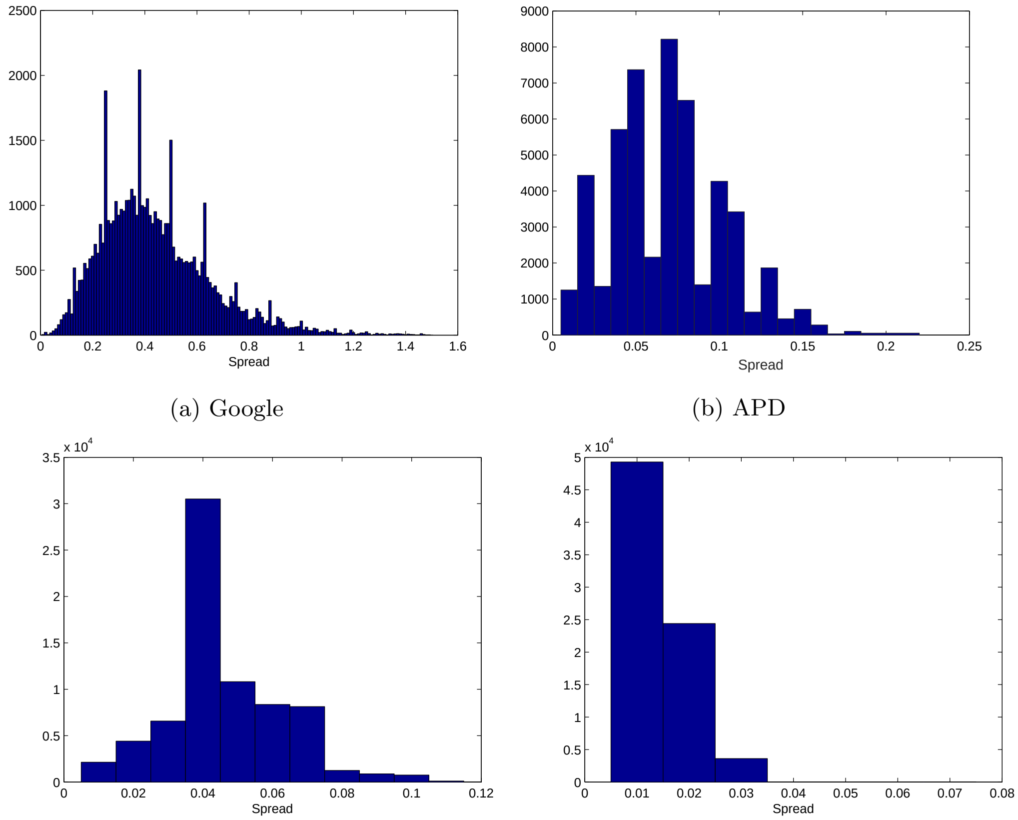
The rest of the proof is classical and consists in showing that $\lim_i \hat{x}_i = \lim_i \hat{y}_i = \hat{x}_0$ and applying Theorem 8.3 of [16] at the point $(\hat{t}_i, \hat{x}_i, \hat{y}_i)$. See for instance [14]. Condition 2.2 is needed for the convergence of the integral term in $\mathcal{L}^a[v_m](\hat{t}_i,\hat{x}_i)$ to $\mathcal{L}^a[w](\hat{t}_i,\hat{y}_i)$.

Proof of Theorem 3.1:

We know that C_0 is continuous and it is the unique viscosity solution of (3.2). By induction, suppose C_k is the unique continuous viscosity solutions of (3.2) for $k \leq n-1$. By the previous propositions, we then obtain that C_n is the unique viscosity solution of (3.2) and that the comparison result holds. In particular, C_n is continuous. □

4 Numerical Results

We calibrated the model to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook* from February 28th to March 4th, 2011. The data files obtained from NYXdata.com contains all time-stamped limit orders entered, removed, modified, filled or partially filled on the NYSE ArcaBook platform. The firms considered are Google (GOOG), Air Products & Chemicals Inc. (APD), International Business Machines Corp. (IBM), and J.P. Morgan Chase & Co. (JPM). All four stocks are very liquid and were part of the S&P500 index in 2011. Yet a major difference is that the empirical distribution of their bid-ask spreads differ considerably, as seen in Figure 1. This is due to the fact that their stock prices are of a different order of magnitude with GOOG at an average price of 606.97, APD at 91.15, IBM at 161.76 and JPM at 45.92 over the five days. In percentage, JPM and IBM have smaller spreads (0.03% of stock price) than GOOG (0.073% of stock price) and APD (0.075%). Since prices are quoted in cents, this offers a large array of values of spreads for GOOG, for which the spread varied from \$0.01 to \$2.67 during the five trading days considered. The resulting liquidation strategies are very different quantitatively and qualitatively.



(a) Google (b) APD

(c) IBM (d) JPM

Figure 1: Distribution of bid-ask spread

The discrete limit order book is constructed as follows. Let

$$x_{-j} = x + 0.01j,$$

and $n_0 > 0$ such that

$$b(x) = b(x_{-j}) = n_0, j \geq 1.$$

The quantity $b(y)$ denotes the number of shares in the order book with a price equal to the best ask price minus y at time t, and $\mathcal{B}_0 = \{x, x_{-1}, x_{-2}, x_{-3}, \dots\}$ is the support of b. For instance, $b(x)$ is the number of shares in the order book at the best bid price when x equals the bid-ask spread. Define $B(x_{-i}, x) = b(x) + \sum_{k=1}^{i} b(x_{-k})$, the number of shares offered at prices no more than x_{-i} dollars below the best ask price when the bid-ask spread equals x. The relation between L and b is then given inductively as follows:

$$L(x,n) = \begin{cases} nx, & \text{for } n \le b(x); \\ L(t,x,b(x)) + (n-b(x))x_{-1}, & \text{for } b(x) < n \le B(x_{-1},x); \\ L(t,x,B(x_{-1},x)) + (n-B(x_{-1},x))x_{-2}, & \text{for } B(x_{-1}) < n \le B(x_{-2},x); \\ L(t,x,B(x_{-2},x)) + (n-B(x_{-2},x))x_{-3}, & \text{for } B(x_{-2}) < n \le B(x_{-3},x); \\ \dots \end{cases} \tag{4.11}$$

The extra spread function is then given by

$$I(x,n) = x - \inf\{x_{-i} : n < B(x_{-i},x)\}. \tag{4.12}$$

Note that L does not depend on the current time t and can be represented as follows:

$$L(x,kn_0) = kn_0(x - 0.005) + (kn_0)^2\zeta, \tag{4.13}$$

for multiples k of n_0, with $\zeta = \frac{0.01}{2n_0}$. We follow the methodology of Blais and Protter [8], Eq. (3.2), to estimate ζ. Blais and Protter [8] compute at every point in time (e.g., every second of the day) the liquidity cost per share, as a function of the number of shares sold, as

$$\frac{L(t,x_t,n)}{n} = \hat{x}_t + n\zeta_t.$$

They then perform a linear regression with the observed liquidity cost per share as a function of n, to determine the value of ζ_t, at each point in time during the day. We define ζ in (4.13) as the average observed value of ζ_t over the five trading days, for each stock studied. Estimated values are reported in Table 1. However, in order to better compare liquidity across different stocks we also compute the liquidity cost per dollars invested squared:

$$\xi_t = \frac{\zeta_t}{S_t^2},$$

with S_t the bid price at time t. This is the liquidity cost coming from the second term in (4.13) normalized in terms of the number of dollars invested in the stock. It is a measure that can be used to compare liquidity between stocks since it is invariant after a stock split, and it gives the liquidity cost beyond the bid-ask spread for trading a fixed dollar volume. It allows to compare the liquidity of stocks based on the number of dollars invested in each stock, instead of the number of shares invested (one share of GOOG should not be compared with one share of JPM). The fitted values are given in Table

1. For example, the average second-order liquidity cost for the sale of \$1M is equal to \$1880, \$6080, \$1570 and \$1010 for GOOG, APD, IBM and JPM respectively.

Stock	$\mathbf{E}(\zeta_t)(\times 10^{-5})$	$\mathbf{E}(\xi_t)(\times 10^{-9})$	Daily price volatility
GOOG	71.43	1.88	2.49%
APD	5.06	6.08	2.21%
IBM	4.10	1.57	1.10%
JPM	21.29	1.01	1.70%

Table 1: Statistics of liquidity costs and daily volatitility

We consider a logarithmic model for λ:

$$\lambda(x) = \lambda_0 + \lambda_1 \log(x).$$

The quantity $\lambda(x)$ gives the average number of market orders in a second when the value of the spread x is observed. For each observed value of spread, we estimate the intensity of market order arrivals and regress these values against the log of the spread. The regressions are presented in Figure 2 and estimated parameters and the R^2 of each regression are presented in Table 2.

Stock	λ_0	λ_1	R^2	γ	n_0
GOOG	0.0168	-0.0277	73.65%	0.2394	7
APD	-0.0035	-0.0079	64.57%	0.1510	99
IBM	-0.0282	-0.0205	86.83%	0.2814	122
JPM	-0.0183	-0.0052	90.06%	0.1045	2348

Table 2: Fitted parameters for spread and order book dynamics. Intensities are per second.

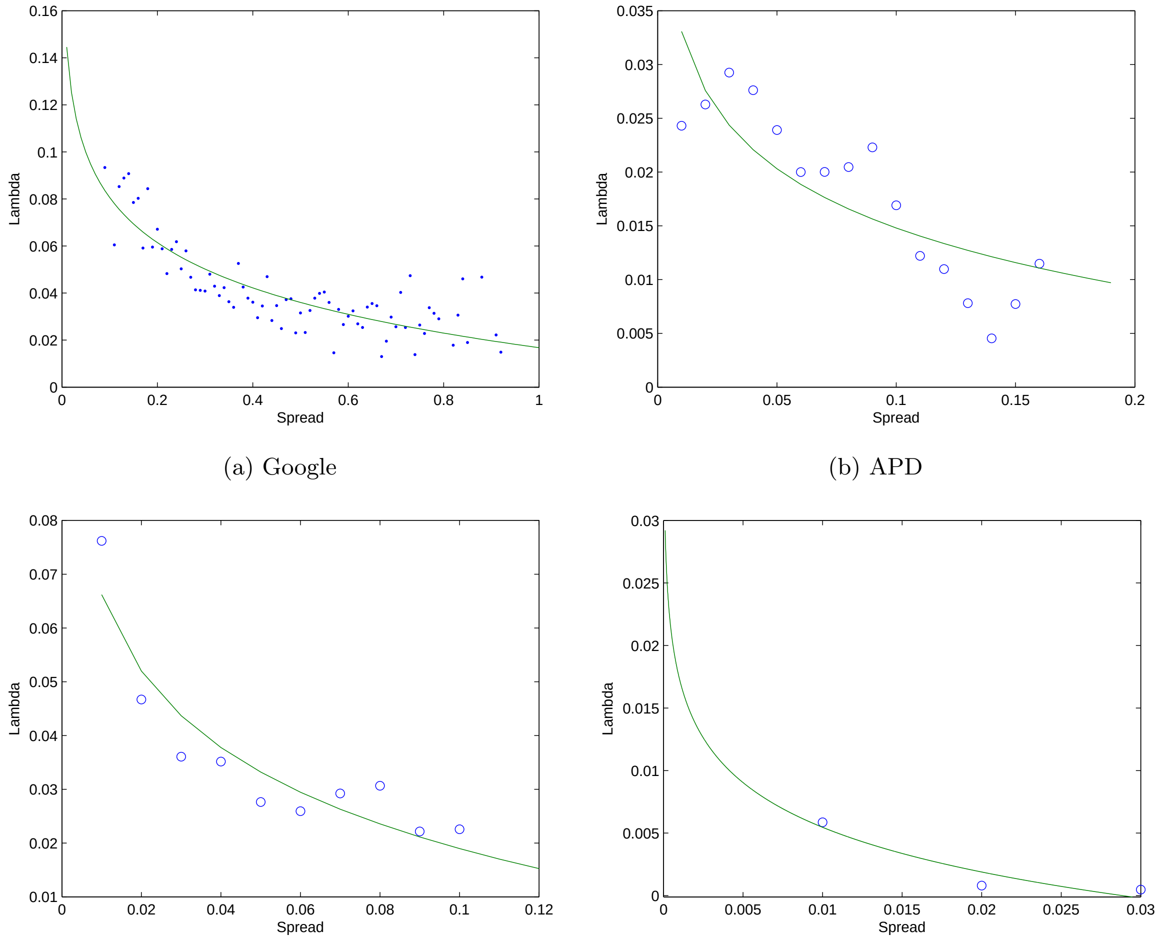
Given the availability of high frequency data, we can accurately estimate the conditional probability of increments of X_t conditioned on the prior value X_{t-}. Consequently, we estimate the function $G(\cdot, 0, \cdot)$ in (2.1) for $\alpha = 0$ from the empirical distribution of X_t given X_{t-}. The function G is then assumed to satisfy $G(x, a, z) = G(x - a, 0, z)$ for $a \leq x$. In other words, at the time of a jump,

$$P(X_t^\alpha = y | X_{t-}^\alpha = x) = P(X_t^0 = y - \alpha_{t-} | X_{t-}^0 = x - \alpha_{t-}),$$

where X^0 is defined by

$$dX_t^0 = \int_{\mathbb{R}} G(X_{t-}^0, 0, z) \tilde{M}(dt, dz).$$

The idea is that the jumps of the observed spread $X - \alpha$ do not depend on the value of α, suggesting that the market participants do not distinguish between the investor who attempts to liquidate and the rest of the market. This is coherent with our modeling of the arrival intensity of market orders λ in



(a) Google (b) APD

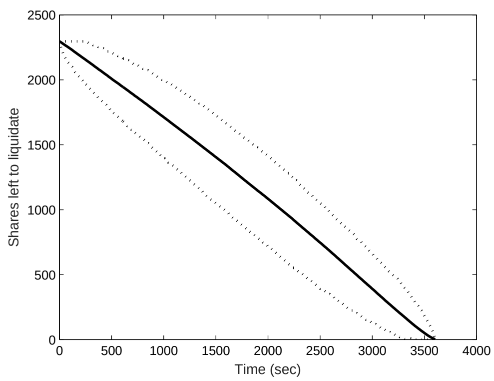
(c) IBM (d) JPM

Figure 2: Arrival intensity (λ) of market orders as function of the current value of spread (time units are seconds).

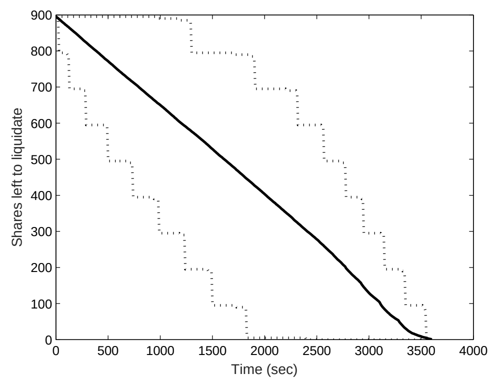
terms of the observed spread. Nevertheless, the jumps of the process X depend on the current limit price posted by the investor by a shift in the distribution of X^0.

The average number of shares posted in the limit order book at each price are reported in Table 2. Tick sizes are \$0.01, and time steps are 1 second. We take γ as constant. The fitted values are given in Table 2.

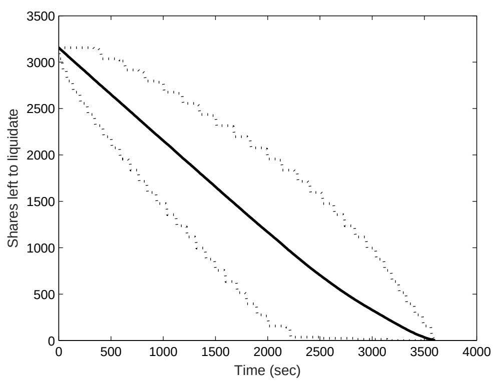
In the first numerical experiment, we compare the liquidity of stocks and the liquidation strategies using two comparison methods. First, we consider time periods of an hour and take N_0 equal to five percent of the average hourly volume traded on Arcabook. In Figure 3, we plotted the average liquidation schedule as well as the 5th and 95th percentile in the case $\pi \equiv 0$. The expected slippage and relevant statistics are given in Table 3. Second, we compute the strategy to liquidate \$1M in each stock in 10 minutes, and report the results in Table 3.



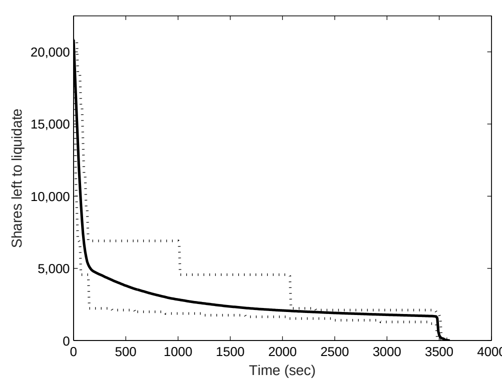
(a) Google



(b) APD



(c) IBM



(d) JPM

Figure 3: Average Liquidation Schedule. Dashed lines represent 5th and 95th percentiles.

Stock	T (minutes)	N_0	Dollar Volume	$ES_T{}^b$	$Q1^b$	$Q2^b$	$Q3^b$	Skewness
GOOG	60	2296^a	\$1.39M	357.72	336.25	354.99	375.78	0.68
APD	60	895^a	\$81,567	11.73	9.80	10.85	12.90	2.43
IBM	60	3156^a	\$0.51M	35.15	33.54	33.78	35.32	2.61
JPM	60	$20,825^a$	\$0.96M	202.91	201.07	203.41	204.58	-0.29
GOOG	10	1652	\$1M	315.91	296.32	309.68	327.74	1.57
APD	10	10,989	\$1M	462.61	390.68	449.29	517.90	1.11
IBM	10	6222	\$1M	149.32	130.00	145.96	165.68	0.56
JPM	10	21,132	\$1M	218.57	217.20	219.65	219.65	-1.16

Table 3: Numerical Results. (a) 5% of average hourly volume traded on Arcabook. (b) In dollars.

The limit order strategy (α) for JPM is almost entirely at the best ask price. This is due to the fact that each tick represents a large percentage of the ask price (approx. 2 basis points), hence it is expensive to lower the spread to increase the probability of passing limit orders given that the observed spread tends to increase when a limit order is placed inside it. On the other hand, the limit order strategy for GOOG is very active as seen in Figure 4 which shows that the strategy largely consists in targeting a number of spreads so that if X_t^α falls between two of these targeted spreads, the controlled spread $X_t^\alpha - \alpha_t$ equals the lesser of the two. The limit order strategy is more active when the maturity becomes small and the cost of holding on to a large number of shares is expensive. The limit order strategy for IBM is also active for higher values of spread. See Figure 5.

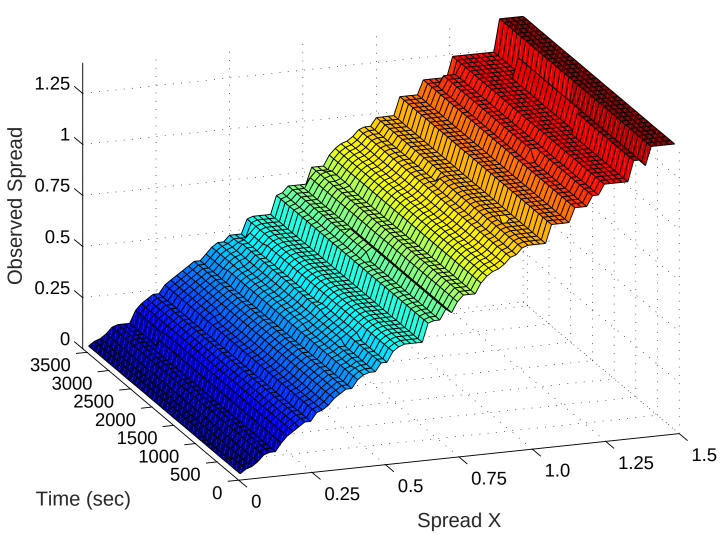


Figure 4: Limit Order Strategy for GOOG

Following Cartea and Jaimungal [10], we have included in a second numerical experiment the penalty function $\pi(n) = \sigma^2 \eta n$, where σ is the volatility of the best ask price (see Table 1). The limit order strategy does not vary qualitatively across values of η. We present the average liquidation strategy for

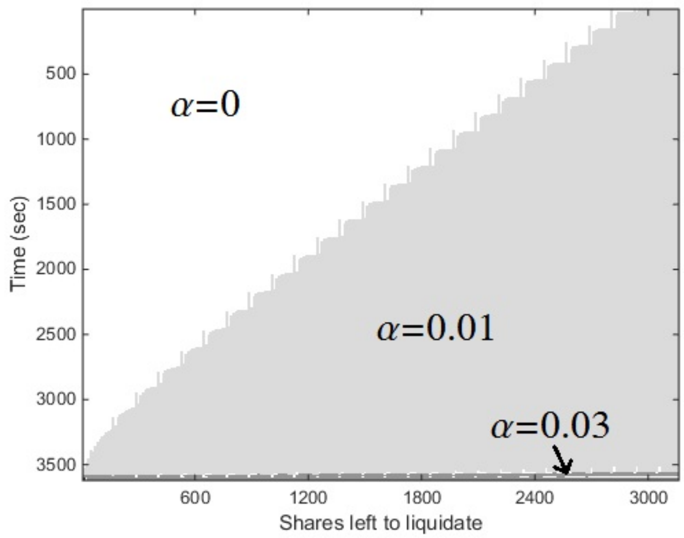


Figure 5: Limit Order Strategy for IBM, for $x = 0.13$.

IBM over 60 minutes for different values of η in Figure 6, and the relevant statistics in Table 4. The liquidation cost increases as η increases since higher values of η force the investor to liquidate more quickly which reduces the number of limit orders that will pass and increases the liquidity cost paid through market orders. Furthermore, we see that the skewness decreases as η increases. The positive skewness can be explained by the fact that when risk aversion is low, the investor gives more chance to the limit orders to pass which creates the possibility of having to liquidate larger blocks through market orders towards the end of the period if this strategy does not succeed.

η	$ES_T{}^a$	$Q1^a$	$Q2^a$	$Q3^a$	Skewness
0	35.15	33.54	33.78	35.32	2.61
0.05	38.18	35.92	36.66	39.13	1.98
0.125	42.49	39.49	41.11	44.26	1.61
0.25	48.28	45.65	47.76	50.11	1.14
0.5	56.11	53.00	55.77	58.80	0.64
1	67.41	62.49	66.77	71.56	0.58

Table 4: Statistics for IBM, liquidated over 60 minutes with N_0 equal to 5% of the average hourly volume traded on Arcabook. (a) In dollars.

5 Conclusion

We studied the problem of optimally liquidating a large portfolio position in a limit order book market through both market and limit orders. We modeled the bid-ask spread by a jump process for which the value is directly affected by the market orders, and for which the distribution of the jumps is

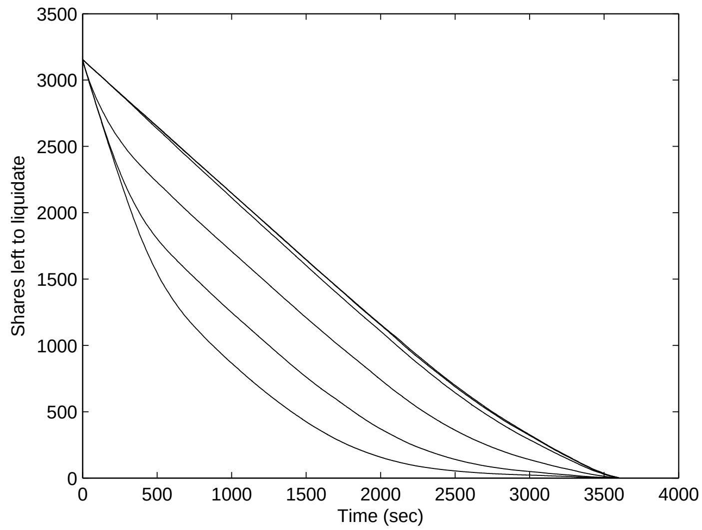


Figure 6: Average Liquidation Schedule for IBM over 60 minutes for $\eta = 0, 0.05, 0.125, 0.25, 0.5, 1$ (in this order, from right to left on the graph).

impacted by the limit order strategy. The arrival of market orders is on the other hand modeled by a controlled Poisson process for which the arrival intensity is a function of the current limit order. The objective is to minimize the execution cost of the strategy. We showed that the value function is the unique viscosity solution of the associated system of variational inequalities from which we can obtain the optimal strategy of this mixed stochastic continuous control and impulse problem. The model was numerically implemented on a quadratic liquidity cost function and a logarithmic market order arrival function. The optimal strategy was computed for four stocks with very different historical bid-ask spread distributions. The optimal limit order strategy varies widely between stocks, whereas the average market order strategy is typically convex with respect to time. The optimal limit order strategy is very active for stocks for which one tick represents a small percentage of the stock price. The extension to a quadratic penalty function makes the average liquidation schedule more aggressive in the beginning, but does not change the qualitative properties of the strategy.

A Appendix

Proof of subsolution property: Let $(t_0, x_0) \in [0,\ T) \times (0, +\infty)$ and $\phi \in \mathcal{C}^{1,2}\left([0,\ T) \times (0, +\infty)\right)$ such that

$$\phi(t_0, x_0) = C_n^u(t_0, x_0) \text{ and } \phi \geq C_n^u \text{ on } [0,\ T) \times (0, +\infty).$$

We have to prove that

$$\min\Big(\min_a \mathcal{L}^a\phi(t_0,x_0)+\Delta_n^a\phi(t_0,x_0);\ [\mathcal{M}_n-\phi](t_0,x_0)\Big)\geq 0.$$

However, we obviously have $\mathcal{M}_n \geq C_n$ and $\mathcal{M}_n$ is continuous so $\mathcal{M}_n(t_0,x_0)\geq C_n^u(t_0,x_0)=\phi(t_0,x_0)$. Hence, we just have to show that $\mathcal{L}^a\phi(t_0,x_0)+\Delta_n^a\phi(t_0,x_0)\geq 0$ for all $a\in\mathcal{A}_0$. We introduce a sequence $(t_m,x_m)_{m\geq 0}$ such that

$$\lim_{m\to+\infty}(t_m,x_m)=(t_0,x_0) \text{ and } \lim_{m\to+\infty}C_n(t_m,x_m)=C_n^u(t_0,x_0).$$

Let $\varepsilon>0$. From the continuity of ϕ, $\mathcal{L}^a\phi$ and C_i for $0\leq i<n$, we deduce that there exists $\eta>0$ such that for all (t,x) such that $\mid t-t_0\mid<\eta$ and $\mid x-x_0\mid<\eta$, we have

$$\mid\phi(t,x)-\phi(t_0,x_0)\mid+\mid\mathcal{L}^a\phi(t,x)-\mathcal{L}^a\phi(t_0,x_0)\mid+\sum_{i=1}^{n-1}\mid C_i(t,x)-C_i(t_0,x_0)\mid\leq\varepsilon.$$

As ϕ is continuous, we have that $\gamma_m:=C_n(t_m,x_m)-\phi(t_m,x_m)$ converges to 0 when m goes to infinity. Set $h_m=\sqrt{\gamma_m}$. Take m large enough so that $t_m+h_m<T\wedge(t_0+\eta)$ and $\mathcal{B}(x_m,\frac{\eta}{2})\subset\mathcal{B}(x_0,\eta)$.

We consider a strategy with no market orders before ν_m, the infimum between t_m+h_m and the first exit time of the associated process X from $\mathcal{B}(x_m,\frac{\eta}{2})\subset\mathcal{B}(x_0,\eta)$, i.e.

$$\nu_m=\inf\{t\geq t_m:\mid X_t-x_m\mid\geq\frac{\eta}{2}\}\wedge(t_m+h_m),$$

with $X_{t_m}=x_m$. We denote by $\theta(\alpha)$ the first jump time after t_m of $\mathcal{N}^\alpha$ and set $\hat\nu_m=\nu_m\wedge\theta(\alpha)$. From the DPP, we know that we have

$$\begin{aligned}
&\gamma_m+\phi(t_m,x_m)=C_n(t_m,x_m)\\
\leq\ & \inf_{\alpha\in\mathcal{A}}\mathbb{E}_{t_m,x_m,n}[C_{N^\alpha_{\hat\nu_m}}(\hat\nu_m,X_{\hat\nu_m})+\int_{t_m}^{\hat\nu_m}\lambda(s,X_s-\alpha_s)f(\alpha_s)ds]\\
\leq\ & \mathbb{E}_{t_m,x_m,n}[C_n(\hat\nu_m,X_{\hat\nu_m})\mathbb{1}_{\{\theta(a)>\hat\nu_m\}}+C_{N^\alpha_{\theta(a)}}(\theta(a),X_{\theta(a)})\mathbb{1}_{\{\theta(a)=\hat\nu_m\}}+\int_{t_m}^{\hat\nu_m}\lambda(s,X_s-a)f(a)ds]\\
\leq\ & \mathbb{E}_{t_m,x_m,n}[\phi(\hat\nu_m,X_{\hat\nu_m})\mathbb{1}_{\{\theta(a)>\hat\nu_m\}}+C_{N^\alpha_{\theta(a)}}(\theta(a),X_{\theta(a)})\mathbb{1}_{\{\theta(a)=\hat\nu_m\}}+\int_{t_m}^{\hat\nu_m}\lambda(s,X_s-a)f(a)ds]\\
=\ & \mathbb{E}_{t_m,x_m,n}[\phi(\hat\nu_m,X_{\hat\nu_m})+\Big(C_{N^\alpha_{\theta(a)}}(\theta(a),X_{\theta(a)})-\phi(\theta(a),X_{\theta(a)})\Big)\mathbb{1}_{\{\theta(a)=\hat\nu_m\}}+\int_{t_m}^{\hat\nu_m}\lambda(s,X_s-a)f(a)ds]
\end{aligned}$$

for all $a\in\mathcal{A}_0$. On the other hand, we can apply Itô's formula to the process $(\phi(t,X_t))_{t\geq 0}$ between t_m

and $\hat{\nu}_m$. We get

$$\phi(t_m, x_m) = \mathbb{E}_{t_m,x_m,n}[\phi(\hat{\nu}_m, X_{\hat{\nu}_m}) - \int_{t_m}^{\hat{\nu}_m} \mathcal{L}^a\phi(t, X_t)\, dt].$$

Combining the last equations and inequalities, we obtain:

$$\begin{aligned}
& \mathbb{E}_{t_m,x_m,n}[\int_{t_m}^{\hat{\nu}_m} \mathcal{L}^a\phi(t, X_t)\, dt + \int_{t_m}^{\hat{\nu}_m} \lambda(s, X_s - a) f(a) ds] \\
\geq\; & \mathbb{E}_{t_m,x_m,n}[\Big(\phi(\theta(a), X_{\theta(a)}) - C_{N^a_{\theta(a)}}(\theta(a), X_{\theta(a)})\Big)\, \mathbb{1}_{\{\theta(a)=\hat{\nu}_m\}}] + \gamma_m \\
\geq\; & \gamma_m + (\phi(t_0, x_0) - \varepsilon)\, \mathbb{P}\left(\theta(a) = \hat{\nu}_m\right) \\
& - \sum_{k=1}^{\infty} \mathbb{E}_{t_m,x_m,n}\left[C_{n-k}(\theta(a), X_{\theta(a)}) \mathbb{1}_{\{\theta(a)=\hat{\nu}_m\}}\right] p_k \\
\geq\; & \gamma_m + \left(\phi(t_0, x_0) - n\varepsilon - \sum_{k=1}^{\infty} C_{n-k}(t_0, x_0) p_k\right) \mathbb{P}\left(\theta(a) \leq \nu_m\right)
\end{aligned}$$

for any $a \in \mathcal{A}_0$. Dividing the last inequality by $\mathbb{E}_{t_m,x_m,n}(\hat{\nu}_m - t_m)$, letting m going to infinity and then ε to 0, we obtain the following inequality:

$$\mathcal{L}^a\phi(t_0, x_0) + \lambda(t_0, X_{t_0} - a) f(a) \geq \lambda(t_0, X_{t_0} - a) \left(\phi(t_0, x_0) - \sum_{k=1}^{\infty} C_{n-k}(t_0, x_0) p_k\right).$$

□

Proof of supersolution property: Let $(t_0, x_0) \in [0,\ T) \times (0, +\infty)$ and $\phi \in \mathcal{C}^{1,2}\left([0,\ T) \times (0, +\infty)\right)$ such that $\phi(t_0, x_0) = C_n^l(t_0, x_0)$ and $\phi \leq C_n^l$ on $[0,\ T) \times (0, +\infty)$. We have to prove that

$$\min\Big(\min_a \mathcal{L}^a\phi(t_0, x_0) + \Delta_n^a\phi(t_0, x_0);\ [\mathcal{M}_n - \phi](t_0, x_0)\Big) \leq 0.$$

If $[\mathcal{M}_n - \phi](t_0, x_0) \leq 0$, then it is automatically satisfied. Therefore, let us assume that $[\mathcal{M}_n - \phi](t_0, x_0) > 0$. Let $\epsilon = \frac{1}{2}[\mathcal{M}_n - \phi](t_0, x_0)$.

We introduce, as before, a sequence $(t_m, x_m)_{m\geq 0}$ such that

$$\lim_{m\to+\infty} (t_m, x_m) = (t_0, x_0) \text{ and } \lim_{m\to+\infty} C_n(t_m, x_m) = C_n^l(t_0, x_0) = \phi(t_0, x_0)$$

and take m large enough to satisfy

$$\mathcal{B}^m := (t_m, (t_m + \eta/2) \wedge T) \times (x_m - \eta/2, x_m + \eta/2) \subset (t_0, (t_0 + \eta) \wedge T) \times (x_0 - \eta, x_0 + \eta).$$

Define the stopping time $\nu_{\mathcal{B}^m}$ as the first exit time of $\mathcal{B}^m$.

Let $\varepsilon > 0$. From the continuity of ϕ, $\mathcal{L}^a\phi$ and C_i for $0 \leq i < n$, we deduce that for all $a \in \mathcal{A}_0$, there exists $\eta > 0$ such that for all (t, x) such that $\mid t - t_0 \mid < \eta$ and $\mid x - x_0 \mid < \eta$, we have

$$\mid \phi(t,x) - \phi(t_0, x_0) \mid + \mid \mathcal{L}^a \phi(t,x) - \mathcal{L}^a \phi(t_0,x_0) \mid + \sum_{i=1}^{n-1} \mid C_i(t,x) - C_i(t_0,x_0) \mid \leq \varepsilon.$$

As ϕ is continuous, we have that $\gamma_m := C_n(t_m, x_m) - \phi(t_m, x_m)$ converges to 0 when m goes to infinity. Set $h_m = \sqrt{\gamma_m}$. Take m large enough so that $t_m + h_m < T \wedge (t_0 + \eta)$.

It follows from the DPP that

$$C_n(t_m, x_m) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}_{t_m, x_m, n}[C_{N^\alpha_\varsigma}(\varsigma, X_\varsigma) + \int_{t_m}^{\varsigma} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds],$$

for all $\varsigma \leq \tau^{\epsilon,m,\alpha} := \inf\{u \geq t_m : C_{N^\alpha_u}(u, X_u) > \mathcal{M}_{N^\alpha_u}(u, X_u) - \epsilon\}$. Note that $\tau^{\epsilon,m,\alpha} > t_m$ a.s. for m large enough. We define the following stopping time $\tilde{\nu}^m = \tau^{\epsilon,m,\alpha} \wedge \nu_{\mathcal{B}^m} \wedge (t_m + h_m)$. From the above DPP, it follows that

$$\begin{aligned}
C_n(t_m, x_m) &= \inf_{\alpha \in \mathcal{A}} \mathbb{E}_{t_m,x_m,n}[C_{N^\alpha_{\theta(\alpha)\wedge\tilde{\nu}^m}}(\theta(\alpha) \wedge \tilde{\nu}^m, X_{\theta(\alpha)\wedge\tilde{\nu}^m}) + \int_{t_m}^{\theta(\alpha)\wedge\tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds] \\
&= \inf_{\alpha \in \mathcal{A}} \mathbb{E}\Big[C_n(\tilde{\nu}^m, X_{\tilde{\nu}^m}) \mathbb{1}_{\{\tilde{\nu}^m < \theta(\alpha)\}} + C_{N^\alpha_{\theta(\alpha)}}(\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} \\
&\qquad + \int_{t_m}^{\theta(\alpha)\wedge\tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds\Big]
\end{aligned}$$

in which $\theta(\alpha)$ is the first jump time after t_m of the process H. However, we have $C_n \geq C_n^l \geq \phi$, so

$$\begin{aligned}
& C_n(t_m, x_m) \\
\geq & \inf_{\alpha \in \mathcal{A}} \mathbb{E}[\phi(\tilde{\nu}^m, X_{\tilde{\nu}^m}) \mathbb{1}_{\{\tilde{\nu}^m < \theta(\alpha)\}} + C_{N^\alpha_{\theta(\alpha)}}(\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} + \int_{t_m}^{\theta(\alpha)\wedge\tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds] \\
= & \inf_{\alpha \in \mathcal{A}} \mathbb{E}[\phi(\tilde{\nu}^m \wedge \theta(\alpha), X_{\tilde{\nu}^m \wedge \theta(\alpha)}) + [C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} + \int_{t_m}^{\theta(\alpha)\wedge\tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds].
\end{aligned}$$

Hence, we can apply Itô's formula to the process $(\phi(t, X_t))_{t_m \leq t \leq T}$ between t_m and $\tilde{\nu}^m \wedge \theta(\alpha)$ to obtain:

$$\gamma_m \geq \inf_{\alpha \in \mathcal{A}} \mathbb{E}[\int_{t_m}^{\tilde{\nu}^m \wedge \theta(\alpha)} \mathcal{L}^{\alpha_s} \phi(s, X_s)\, ds + [C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} + \int_{t_m}^{\theta(\alpha)\wedge\tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds].$$

We also have

$$\begin{aligned}
\mathbb{E}[[C_{N^\alpha_{\theta(\alpha)}} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}}] &= \sum_{k=1}^{+\infty} \mathbb{E}\left[[C_{n-k} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}}\right] p_k \\
&\geq \mathbb{E}\left[\int_{t_m}^{\tilde{\nu}^m \wedge \theta(\alpha)} \lambda(s, X_s - \alpha_s)\, ds\right] \left(\sum_{k=1}^{+\infty} [C_{n-k} - \phi](t_m, X_{t_m}) p_k - 2\varepsilon\right).
\end{aligned}$$

For any $\alpha \in \mathcal{A}$, the result is that the quantity

$$\frac{\mathbb{E}[\int_{t_m}^{\tilde{\nu}^m \wedge \theta(\alpha)} \mathcal{L}_s^{\alpha} \phi(s, X_s)\, ds + [C_{N^{\alpha}_{\theta(\alpha)}} - \phi](\theta(\alpha), X_{\theta(\alpha)}) \mathbb{1}_{\{\theta(\alpha) \leq \tilde{\nu}^m\}} + \int_{t_m}^{\theta(\alpha) \wedge \tilde{\nu}^m} \lambda(s, X_s - \alpha_s) f(\alpha_s) ds]}{\mathbb{E}_{t_m, x_m, n}(\tilde{\nu}^m \wedge \theta(\alpha) - t_m)}$$

converges to

$$\mathcal{L}^{\alpha_{t_0}} \phi(t_0, x_0) + \lambda(t_0, X_{t_0} - \alpha_{t_0}) \left(f(\alpha_{t_0}) + \sum_{k=1}^{\infty} C_{n-k}(t_0, x_0) p_k - \phi(t_0, x_0) \right)$$

when m goes to infinity and then ε to 0. We finally obtain

$$\min_a \mathcal{L}^a \phi(t_0, x_0) + \Delta_n^a \phi(t_0, x_0) \leq 0$$

by taking the minimum over $\mathcal{A}_0$. $\square$

References

[1] Almgren, R., and N. Chriss (1999) ”Value under liquidation”, *Risk*, December 1999.

[2] Alfonsi, A., A. Fruth and A. Schied (2010): “Optimal execution strategies in limit order books with general shape functions”, *Quantitative Finance*, **10**, 143-157.

[3] Almgren, R. (2012): “Optimal trading with stochastic liquidity and volatility”, *SIAM Journal on Financial Mathematics*, **3**, 163-181.

[4] Avellaneda, M. and S. Stoikov (2008): “High-frequency trading in a limit order book”, *Quantitative Finance*, **8**, 217-224.

[5] Back, K. (1992): “Insider trading in continuous time”, *Review of Financial Studies*, **5**, 387-409.

[6] Bank, P. and D. Baum (2004): “Hedging and portfolio optimization in illiquid financial markets with a large trader”, *Mathematical Finance*, **14**, 1-18.

[7] Bayraktar, E. and M. Ludkovski (2012): “Liquidation in Limit Order Books with Controlled Intensity”, *Mathematical Finance*, Forthcoming.

[8] Blais, M. and P. Protter (2010): “An Analysis of the Supply Curve for Liquidity Risk through Book Data”, *International Journal of Theoretical and Applied Finance*, **13**, 821-838.

[9] Cartea, A. and S. Jaimungal (2013): “Optimal Execution with Limit and Market Orders”, *Quantitative Finance*, Forthcoming.

[10] Cartea, A and S. Jaimungal (2013): “Risk Metrics and Fine Tuning of High Frequency Trading Strategies”, *Mathematical Finance*, Forthcoming

[11] Cartea, A., S. Jaimungal and J. Ricci (2014). “Buy low, sell high: A high frequency trading perspective.” *SIAM Journal on Financial Mathematics,* **5**, 415-444.

[12] Cetin, U., R. Jarrow and P. Protter (2004): “Liquidity risk and arbitrage pricing theory”, *Finance and Stochastics*, **8**, 311-341.

[13] Chevalier E., M. Gaigi, V. Ly Vath and M. Mnif (2013): “Optimal market making strategies under inventory constraints”, preprint.

[14] Chevalier E., V. Ly Vath, A. Roch and S. Scotti (2015): “Optimal exit strategies for investment projects”, *Journal of Mathematical Analysis and Applications*, **425**, 666-694.

[15] Constantinides, G. (1986): "Capital market equilibrium with transaction costs", *J. Political Economy*, **94**, 842-862.

[16] Crandall M., Ishii H. and P.L. Lions (1992): "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[17] Davis, M. H. A. and A. R. Norman (1990): "Portfolio Selection with Transaction Costs", *Mathematics of Operations Research*, **15**, 676-713.

[18] Guéant, O., and C. Lehalle (2013): "General Intensity Shapes In Optimal Liquidation", *Mathematical Finance*, Forthcoming.

[19] Guéant, O., C. Lehalle, and J. Tapia (2011): "Optimal portfolio liquidation with limit orders", *SIAM J. Finan. Math.*, **3**, 740-764.

[20] Guéant O., C. Lehalle, and J. Tapia (2012): Dealing with the inventory risk: a solution to the market making problem", *Mathematics and Financial Economics*, Forthcoming.

[21] Guilbaud, F. and H. Pham (2013): "Optimal high frequency trading with limit and market orders", *Quantitative Finance*, **13**, 79-94.

[22] Kühn, C., and Muhle-Karbe, J. (2014). Optimal liquidity provision in limit order markets. Swiss Finance Institute Research Paper, No. 13-71.

[23] Kyle, A. (1985): "Continuous actions and insider trading", *Econometrica*, **53**, 1315-1335.

[24] Ly Vath, V., M. Mnif and H. Pham (2007): "A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact", *Finance and Stochastics*, **11**, 51-90.

[25] Nyström, K., S. M. Ould Aly, and C. Zhang (2014): "Market Making And Portfolio Liquidation Under Uncertainty", *International Journal of Theoretical and Applied Finance*, **17**, 1450034 (33 pages).

[26] Obizhaeva, A. and J. Wang (2013): "Optimal Trading Strategy and Supply/Demand Dynamics", *Journal of Financial Markets*, **16**, 1-32.

[27] Oksendal, B. and A. Sulem. (2007): Applied Stochastic Control of Jump Diffusions, Universitext, Springer Verlag, Berlin.

[28] Roch, A. (2011): "Liquidity risk, price impacts and the replication problem", *Finance and Stochastics*, **15**, 399-419.

[29] Roch, A. and M. Soner (2013): "Resilient price impact of trading and the cost of illiquidity", *International Journal of Theoretical and Applied Finance* **16**, 1-27.

[30] Shreve, S. E., and H. M. Soner (1994): "Optimal investment and consumption with transaction costs" *The Annals of Applied Probability* **4**, 609-692.