Université d'Évry Val d'Essonne

Année 2015

Habilitation à Diriger la Recherche
Spécialité : Mathématiques Appliquées

présentée par
LY VATH Vathana

Quelques Contributions en Finance Mathématique
Risque de Liquidité et Finance d'Entreprise

Soutenue le 23 Novembre 2015, devant le jury composé de :

Pr. ABERGEL Frédéric, Ecole Centrale de Paris
Pr. CREPEY Stéphane, Université d'Evry Val d'Essonne
Pr. JEANBLANC Monique, Université d'Evry Val d'Essonne
Pr. PHAM Huyên, Université Paris 7
Dr. SULEM Agnès, INRIA
Pr. TOUZI Nizar, Ecole Polytechnique
Pr. VILLENEUVE Stéphane, Université de Toulouse 1
Pr. ZERVOS Mihail, London School of Economics

au vu des rapports de :

Dr. SULEM Agnès, INRIA
Pr. VILLENEUVE Stéphane, Université de Toulouse 1
Pr. ZERVOS Mihail, London School of Economics

A Alyssa, Thibault, Maxime et Annie

Remerciements

Mes premiers remerciements vont naturellement aux membres du Laboratoire de Mathématiques et Modélisation d'Evry (LaMME), en particulier ceux de l'équipe Probabilités et Mathématiques Financières, au sein de laquelle j'ai effectué ma recherche depuis la fin de ma thèse. Je tiens à remercier en particulier Monique Jeanblanc pour sa disponibilité, ses soutiens et conseils tout au long de ces années. Je tiens également à remercier Stéphane Crépey, Arnaud Gloter, Pierre-Gilles Lemarié et Laurent Denis.

Je suis très reconnaissant envers Agnès Sulem, Stéphane Villeneuve et Mihalis Zervos pour leur intérêt dans mes travaux d'HDR et leur disponibilité en acceptant de la rapporter.

Je remercie également Frédéric Abergel, Stéphane Crépey, Huyên Pham et Nizar Touzi d'avoir accepté de participer au jury.

Je suis également reconnaissant envers tous mes co-auteurs, en particulier, Etienne Chevalier, Simone Scotti, Alexandre Roch, Mhamed Gaigi, Mohamed Mnif et Thomas Lim.

Mes remerciements vont également vers mes collègues enseignants-chercheurs et personnel administratif de l'ENSIIE, en particulier Menad Sidhamed et Nicolas Brunel.

J'exprime également ma reconnaissance envers Valérie Picot pour toutes ses aides administratives pendant ces années passées au laboratoire.

Mes pensées vont enfin aux collègues et amis du laboratoire, de l'ENSIIE ou d'ailleurs, pour la bonne ambiance et surtout pour leurs amitiés. Je tiens à remercier, en particulier, Vincent, Alexandre, Jean-François, Etienne, Erhan, Nicolas, Thomas, Sergio, Pierre, Eric, Arnaud, Michèle, Yvette, Marie-Luce, Nadia, Sylvain, Guillaume, Stéphane, Gael, Christophe...

TABLE DES MATIÈRES

Résumé

Dans ce rapport, je présente l'ensemble de mes travaux effectués entre 2007 et 2015. Depuis ma soutenance de thèse, mes travaux portent essentiellement sur deux problématiques séparées :

- Problèmes d'optimisation et de modélisation en risques de liquidité
- Problèmes de contrôle stochastique appliqué à la finance d'entreprise

Ces problématiques présentent deux principaux challenges, le premier dans leur modélisation et formulation mathématique et le deuxième dans leur résolution mathématique et numérique. Ces problèmes sont souvent formulés comme des problèmes de contrôle stochastique sous contraintes non-standard. Cela nécessite des analyses assez fines au niveau mathématique, aussi bien dans la partie théorique qui consiste à caractériser les fonctions valeurs via des approches de viscosité en prouvant l'existence et l'unicité que dans la description des différentes regions caractérisant les politiques optimales des problèmes. De plus, pour entièrement résoudre ces problèmes, il est souvent nécessaire d'avoir recours aux méthodes de résolutions numériques (probabilistes ou EDP).

Partie 1 : Problèmes d'optimisation et de modélisation de risques de liquidité

Dans cette partie, j'ai principalement travaillé sur des problèmes liés aux risques de liquidité, de modélisation de bid-ask spread, de carnet d'ordres et de market making. La modélisation et la compréhension de ces aspects nous permettraient de répondre à de nombreux problèmes importants dans la gestion des risques financiers et la gestion d'actifs. Cette partie contient trois chapitres. Le premier chapitre correspond aux travaux menés depuis ma thèse sur un problème de gestion de portefeuille sous contrainte de liquidité [A3]. La partie numérique correspond à l'article [A10], à paraître dans AMO et co-écrit avec des collègues de l'ENIT et Mhamed Gaigi, notre doctorant en co-tutelle ENIT-Evry. Le deuxième chapitre de cette partie est une collaboration avec Etienne Chevalier, Alexandre Roch (UQAM, Montréal) et Simone Scotti et contient l'article [A9] traitant un problème de liquidation optimale dans un marché LOB (Limit Order Book). Enfin, le troisième chapitre est un travail sur un problème de market making sous contrainte d'inventaire [S12], co-écrit avec Etienne Chevalier, Mhamed Gaigi, et Mohamed Mnif.

Partie 2 : Problèmes de contrôle stochastique appliqué à la finance d'entreprise

Dans cette partie, nous nous intéressons aux problèmes d'optimisation stochastique, assimilables aux options réelles. Par analogie avec l'option du financier, on parle d'option réelle pour caractériser la position d'un industriel qui bénéficie d'une certaine flexibilité dans

la gestion de l'entreprise, par exemple, un projet d'investissement. Il est, en effet, possible de limiter ou d'accroître le niveau d'investissement compte tenu de l'évolution des perspectives économiques et de rentabilité, tout comme un financier peut exercer ou non son option sur un sous-jacent. Cette flexibilité détient une valeur qui est tout simplement la valeur de l'option réelle. Dans cette partie, nous nous intéressons aux problèmes de décisions d'investissement et de politiques optimales de distribution de dividende. Mathématiquement, ces problèmes sont formulés comme des problèmes de contrôles stochastiques, principalement des problèmes de contrôle singulier et de changement de régime avec des contraintes dans un cadre multidimensionnel.

Cette partie contient un chapitre sur des problèmes d'"optimal multiple-switching" [A5], co-écrit avec Huyên Pham et Xunyu Zhou, et d'"optimal exit strategies" [A8], co-écrit avec Etienne Chevalier, Alexandre Roch et Simone Scotti. Enfin, le dernier Chapitre contient un ensemble de problèmes sur les politiques optimales de dividende et d'investissement. Le premier problème traite un problème couplé de contrôle singulier et de changement de régime pour une politique de dividende avec investissement réversible [A4]. Le deuxième problème [A7], co-écrit avec Etienne Chevalier et Simone Scotti, publié dans Siam Journal in Financial Mathematics, traite un problème de dividende et d'investissement sous contrainte de dette. Enfin, le dernier traite le cas où l'on prend en compte le problème lié aux contraintes de liquidité [S11]. Cet [S11] est co-écrit avec E. Chevalier et M. Gaigi.

Articles publiés ou acceptés pour publication

[A1] Competitive market equilibrium under asymmetric information, Decisions in Economics and Finance, 2007, 30, pp. 79-94.

[A2] Explicit solution to an optimal switching problem in the two regime case, avec H. Pham, Siam Journal on Control and Optim., 2007, 46, pp. 395-426.

[A3] A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact,avec M. Mnif and H. Pham, Finance and Stochastics, 2007, 11, pp. 51-90.

[A4] A mixed singular/switching control problem for a dividend policy with reversible technology investment, avec H. Pham et S. Villeneuve, Annals of Applied Probability, 2008, 18, pp. 1164-1200.

[A5] Optimal switching over multiple regimes, avec H. Pham et X.Y. Zhou, Siam Journal on Control and Optim., 2009, 48, pp. 2217-2253.

[A6] Bid-Ask Spread modelling, a perturbation approach, avec T. Lim, JM. Sahut, and S. Scotti, 2011, Seminar on Stochastic Analysis, Random Fields and Applications VII (Ascona 2011).

[A7] An Optimal Dividend and Investment Control Problem under Debt Constraints, avec E. Chevalier, et S. Scotti, 2013, SIAM J. Finan. Math., 4(1), 297 - 326.

[A8] Exit Optimal exit strategies for investment projects, avec E. Chevalier, A. Roch and S. Scotti, 2015, Journal of Mathematical Analysis and Applications, Vol.425(2), pp.666-694.

[A9] Optimal execution cost for liquidation through a limit order market, avec E. Chevalier, A. Roch and S. Scotti, 2014, to appear in International Journal of Theoretical and Applied Finance.
[A10] Numerical approximation for a portfolio optimization problem under liquidity risk and costs, avec M. Gaigi, M. Mnif, et S. Toumi, 2014, to appear in Applied Mathematics and Optimization.

Travaux soumis

[S11] Liquidity risk and optimal dividend/investment strategies, avec E. Chevalier et M. Gaigi, 2015.
[S12] Optimal market dealing under constraints, with E. Chevalier, M. Gaigi, et M. Mnif, 2013.

Working papers et travaux en cours

[W13] Wages and Employment in Economies with Multi-Worker Firms, Uncertainty and Labor Turnover Costs, avec S. Scotti, A. Vindigni, 2015, working paper.
[W14] An optimal capital structure control problem under uncertainty, avec E. Bayraktar et E. Chevalier, en cours.
[W15] Optimal execution in a one-sided order book with stochastic resilience, avec E. Chevalier et S. Pulido, en cours.
[W16] Optimal dividend and capital injection policy with external audit, avec E. Chevalier et A. Roch, en cours.

Part I

Liquidity risk modelling and portfolio selection and liquidation

Classical financial models in mathematical finance assume perfect elasticity of traded assets: traders act as price takers, so that they buy and sell with arbitrary size without changing the price. Relaxing this assumption is very important in the study of many financial problems such as option hedging, optimal allocation and liquidation problems. It is particularly important when dealing in markets where the transactions frequency or the number of operators is low. The market liquidity crunch we have witnessed during the financial crisis in 2008 is a case in point.

The study of market liquidity mainly consists in quantifying the costs incurred by investors trading in markets in which supply and demand is finite, trading counterparties are not continuously available, or trading causes prices impacts. Liquidity is a risk when the extent to which these properties are satisfied varies randomly through time. Liquidity and liquidity risk models varies considerably from one study to the next according to the problem at hand or the paradigm considered. One way to have a better understanding of the liquidity risk is to examine it through the types of financial markets, market participants, and transaction orders, which are relevant in the study of liquidity risk.

It is clear from the very structure of the financial markets that, in addition to the presence of price-takers, there must necessarily exist a second type of market participants who are price-setters or liquidity providers. These different types of market participants are related to the types of transaction orders they post. Price-takers or liquidity takers are market participants who post market orders, whereas price-setters or liquidity providers post limit orders. As far as trading transactions are concerned, we may clearly distinguish two different and important types of markets: order-driven markets and quote-driven markets. In order-driven markets, also called "Limit Order Book" markets, traders can post both market orders or limit orders. The aggregated limit orders constitute the so-called limit order book with the best bid and best ask prices forming the bid-ask prices and spread. These limit orders contain the prices and quantities at which traders are willing to buy or sell while waiting for a counterparty to engage in that trade. In quote-driven markets or dealers' markets, registered market makers quote bids and offers and serve as intermediary between public traders. More precisely, registered market makers act as counterparties when an investor wishes to buy or sell securities.

Most studies on liquidity risk have the same objectives: modelling the market structure in order to better quantify the cost and impact of the lack of liquidity in many important financial problems such as portfolio selection and assets liquidation. In the mathematical finance literature, there are several approaches in modelling liquidity risk. We may refer, for instance, to the literature on insider trading, transactions costs, market manipulations. However, a quite natural approach is to classify studies on liquidity risk in terms of different types of market participants, i.e. price-takers or price-setters, being investigated. The below way of classifying liquidity risk problems is by no way exhaustive as we have no pretention to be able describe all the liquidity problems studied whether they are from financial and economic or mathematical finance literature.

The first approach which corresponds mostly to the first wave of research and interest in liquidity risk is to relax the infinite liquidity assumption by introducing for instance some forms of cost or/and price impact for trading transactions submitted by investor. In [77] and [8], the impact of trading strategies on prices is explained by the presence of an insider. In the market manipulation literature, prices are assumed to depend directly on the trading strategies. For instance, the paper [39] considers a diffusion model for the price dynamics with coefficients depending on the large investor's strategy, while [53], [94], [93], [9] or [32] develop a continuous-time model where prices depend on strategies via a reaction function. Transaction costs, which corresponds to a way to model bid-ask spread are also considered in some studies, see for instance [74]. It is clear that in the above studies, investors are assumed to be price-takers as they uniquely submit market orders. In other words, in these above papers, the set of admissible strategies contains uniquely market buy and sell orders. This approach may be considered as the first real mathematical attempt to incorporate liquidity risk and costs in financial problems. Within this context, in [A3], liquidity risk is expressed by the presence of transaction costs and market manipulation. Our model is inspired from [103] and [64], and may be described roughly as follows. Trading on illiquid assets is not allowed continuously due to some fixed costs but only at any discrete times. These liquidity constraints on strategies are in accordance with practitioner literature and consistent with the academic literature on fixed transaction costs, see e.g. [86]. We study an optimal portfolio choice problem over a finite horizon : the investor maximizes his expected utility from terminal liquidation wealth and under a natural economic solvency constraint. The main goal in this paper is to obtain a rigorous characterization result on the value function through the associated HJB quasi-variational inequality. In order to completely deal with this impulse control problem, we study in [A10] its numerical resolution.

A second and more recent approach is therefore to consider financial models with an enlarged set of admissible trading strategies by including the possibility of making both limit and market orders. This second approach is related to a recent emphasis on liquidation and market making problems in a limit order book markets. Many authors have investigated these problems with limit orders only, in particular [7], [14], [56], [57], [58] and [87]. In these models, the arrival intensity of outside market orders that match the limit orders that are posted is typically a function of the spread between the posted price and a reference price. In a more complex model, Cartea et al. [31] develop a high-frequency limit order trading strategy in a limit order market characterized by feedback effects in market orders and the shape of the order book, and by adverse selection risk due to the presence of informed traders who make influential trades. Kühn and Muhle-Karbe [75] provide an asymptotics analysis for a small investor who sets bid and ask prices and seeks to maximize expected utility when the spread is small. On the other hand, some authors consider a limit order market in which both limit and market orders are possible, see for instance Guilbaud and Pham [59] and Cartea and Jaimungal [29]. As part of my research, I was also interested in this approach. In particular, in [A9], we consider a limit order market in which both limit and market orders are allowed, and study the problem of optimally liquidating a large portfolio position. Our contribution to the above literature is to consider spread dynamics which are impacted by both limit and market order strategies. Market orders that the investor places directly increase the observed bid-ask spread.

To complete our study, a third approach is to study liquidity risk at the microstructure level. Market microstructure modelling have attracted many interests, in particular from a statistical point of view. The main focus of this growing literature is to provide good high frequency volatility measures. We may refer for instance to Bouchaud et al. [21], Cont [36], Hansen and Lunde [63] and Rosenbaum [99] for some studies on the existence of tick value and order book. The main focus of this growing literature is to provide good high frequency volatility measures, see for instance Almgren and Chriss [5] and Barndorff-Nielsen et al. [12] for some other studies in high frequency setting. In the the study of liquidity risk at the microstructure level, another approach is to emphasize on trading strategies by directly describing the dynamics of the prices without referring to an efficient price or to model the order book, see Avellaneda and Stoikov [7], Bouchaud et al. [20], Cont et al. [37]. Within this context, in [S12], we investigate the problem of dealers or market makers operating in dealers markets or quote-driven markets. As opposed to “Limit Order Book” markets, which was considered in previous approaches, in quote-driven markets, only registered market makers are to place bid and ask prices. The role of market makers is very important in the trading of illiquid assets as she acts a facilitator of trades between different investors. We may refer to many studies on market making problems, but most of them are more related to the finance and economic literature, see for instance [66] and [85]. More recent works from the mathematical finance community may equally be worth mentioning, for instance [7] and [57]. Under this approach, in [S12], we consider a market dealer acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset in a quote-driven market. In this optimal market dealing problem, important features and constraints characterizing market making problems are no longer ignored. A related work on the subject is [A6] where we attempt to explain and model the financial and economic rationale behind the existence of the bid-ask spread.

Chapter 1

A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact: Theoretical and Numerical Aspects

This chapter is based on two papers with the first paper written with M. MNIF and H. Pham and the second one written with M. Gaigi, M. Mnif and S. Toumi.

[A3]. A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact,avec M. Mnif and H. Pham, Finance and Stochastics, 2007, 11, pp. 51-90.

[A10]. Numerical approximation for a portfolio optimization problem under liquidity risk and costs, avec M. Gaigi, M. Mnif, et S. Toumi, 2014, to appear in Applied Mathematics and Optimization.

1.1 Introduction

In this paper, we propose a model of liquidity risk and price impact that adopts both market manipulation and transaction costs. Our model is inspired from the recent papers [103] and [64], and may be described roughly as follows. Trading on illiquid assets is not allowed continuously due to some fixed costs but only at any discrete times. These liquidity constraints on strategies are in accordance with practitioner literature and consistent with the academic literature on fixed transaction costs, see e.g. [86]. There is an investor, who is large in the sense that his strategies affect asset prices. In this context, we study an optimal portfolio choice problem over a finite horizon : the investor maximizes his expected utility from terminal liquidation wealth and under a natural economic solvency constraint. In some sense, our problem may be viewed as a continuous-time version of the recent discrete-time one proposed in [33]. We mention also the paper [5], which studies an optimal trade execution problem in a discrete time setting with permanent and temporary market impact.

Our optimization problem is formulated as a parabolic impulse control problem with three variables (besides time variable) related to the cash holdings, number of stock shares and price. This problem is known to be associated by the dynamic programming principle to a Hamilton-Jacobi-Bellman (HJB) quasi-variational inequality, see [15]. We refer to [71], [74], [25] or [92] for some papers involving applications of impulse controls in finance, mostly over an infinite horizon and in dimension 1, except [74] and [92] in dimension 2. There is in addition, in our context, an important aspect related to the economic solvency condition requiring that liquidation wealth is nonnegative, which is translated into a state constraint involving a non-smooth boundary domain.

The features of our stochastic control problem make appear several technical difficulties related to the nonlinearity of the impulse transaction function and the solvency constraint. In particular, the liquidation net wealth may grow after transaction, which makes nontrivial the finiteness of the value function. Hence, the Merton bound does not provide as e.g. in transaction cost models, a natural upper bound on the value function. Instead, we provide a suitable "linearization" of the liquidation value that provides a sharp upper bound of the value function. The solvency region (or state domain) is not convex and its boundary even not smooth, in contrast with transaction cost model (see [41]), so that continuity of the value function is not direct. Moreover, the boundary of the solvency region is not absorbing as in transaction cost models and singular control problems, and the value function may be discontinuous on some parts of the boundary. Singularity of our impulse control problem appears also at the liquidation date, which translates into discontinuity of the value function at the terminal date.

In our general set-up, it is then natural to consider the HJB equation with the concept of (discontinuous) viscosity solutions, which provides by now a well established method for dealing with stochastic control problems, see e.g. the book [52]. More precisely, we need to consider constrained viscosity solutions to handle the state constraints. Our first main result is to prove that the value function is a constrained viscosity solution to its associated HJB quasi-variational inequality. Our second main result is a new comparison principle for the state constraint HJB quasi-variational inequality, which ensures a PDE characterization for the value function of our problem. Finally, we provide a numerical method based on quantization calculus and give some numerical results for the optimal transaction strategy.

1.2 The Model

This section presents briefly the model studied in [A3]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ supporting an one-dimensional Brownian motion W on a finite horizon $[0,T]$, $T < \infty$. We consider a continuous time financial market model consisting of a money market account yielding a constant interest rate $r \geq 0$ and a risky asset (or stock) of price process $P = (P_t)$ modelled by a geometric Browninan motion. We denote by X_t the amount of money (or cash holdings) and by Y_t the number of shares in the stock held by the investor at time t. The associated state process is denoted by $Z_t := (X_t, Y_t, P_t)$. We assume that the investor can only trade discretely on $[0, T)$ which is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$, where the non-decreasing stopping times $\tau_1 \leq \ldots \tau_n \leq \ldots < T$ represent the intervention times of the investor and ζ_n, $n \geq 1$, are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{R}$ and represent the number of stock purchased if $\zeta_n \geq 0$ or sold if $\zeta_n < 0$ at these times. The sequence (τ_n, ζ_n) may be a priori finite or infinite. Each intervention occurs with a transaction cost k and a exponential price impact. More precisely, when the investor makes an investment of ζ shares, her cash outflow is $\zeta p e^{-\lambda\zeta} - k$, where λ is a positive constant. In this model, the price impact is considered to be permanent.

Investment problem. The optimal investment problem is about maximizing the expected utility from terminal liquidation wealth over a finite horizon. The value function is defined as follows :

$$v(t,z) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[e^{-r(T-t)} U_L(Z_T)\right], \qquad (t,z) \in [0,T] \times \bar{\mathcal{S}}, \tag{1.2.1}$$

where U_L is defined on $\bar{\mathcal{S}}$ by $U_L(z) := U(L(z))$, with U being the utility function, L the liquidation value $L(x,y,p) = x + (ype^{-\lambda y} - k)_+ \mathbb{1}_{y\geq 0} + (ype^{-\lambda y} - k)\mathbb{1}_{y<0}$, and $\mathcal{S}$ the solvency region defined as $\mathcal{S} := \left\{z = (x,y,p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \ : L(z) > 0\right\}$ with $\partial\mathcal{S}$ and $\bar{\mathcal{S}}$, being respectively its boundary and its closure. The set $\mathcal{A}(t,z)$ is the set of admissible strategies which keeps the state processes inside the solvency region.
The HJB equations. The HJB quasi-variational inequality satisfied by the value function (1.2.1) is as follows:

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v \ , \ v - \mathcal{H}v\right] \quad = \quad 0, \qquad \text{on} \quad [0,T) \times \mathcal{S} \tag{1.2.2}$$

where $\mathcal{L}\varphi = rx\dfrac{\partial\varphi}{\partial x} + bp\dfrac{\partial\varphi}{\partial p} + \dfrac{1}{2}\sigma^2 p^2 \dfrac{\partial^2\varphi}{\partial p^2} - r\varphi$ is the infinitesimal generator associated to the non-controlled state process when there is no trading.

The operator $\mathcal{H}$ is the impulse operator defined by

$$\mathcal{H}\varphi(t,z) \quad = \quad \sup_{\zeta \in \mathcal{C}(z)} \varphi(t, \Gamma(z,\zeta)), \quad (t,z) \in [0,T] \times \bar{\mathcal{S}}$$

and the function Γ is the impulse transaction function defined from $\bar{\mathcal{S}} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$ as $\Gamma(z,\zeta) = (x - \zeta p e^{\lambda\zeta} - k, y + \zeta, p e^{\lambda\zeta})$, $\ z = (x,y,p) \in \bar{\mathcal{S}}$, $\zeta \in \mathbb{R}$. The set $\mathcal{C}(z)$ is the set of admissible transactions $\mathcal{C}(z) = \left\{\zeta \in \mathbb{R} \ : \Gamma(z,\zeta) \ \in \ \bar{\mathcal{S}}\right\} \ = \ \{\zeta \in \mathbb{R} \ : L(\Gamma(z,\zeta)) \ \geq \ 0\}$.

1.3 Viscosity characterization

The value function (1.2.1) and the associated HJB quasi-variational inequality (1.2.2) are related by means of constrained viscosity solutions. We may obtain the main results in $[A3]$ as follows:

Theorem 1.3.1. *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to* (1.2.2) *satisfying the boundary and terminal condition :*

$$\begin{aligned} \lim_{\substack{(t',z') \to (t,z) \\ z' \in \mathcal{S}}} v(t',z') &= 0, \quad \forall (t,z) \in [0,T) \times D_0 \qquad (1.3.3) \\ \lim_{\substack{(t,z') \to (T,z) \\ t<T,\, z' \in \mathcal{S}}} v(t,z') &= \max[U_L(z), \mathcal{H}U_L(z)], \quad \forall z \in \bar{\mathcal{S}} \qquad (1.3.4) \end{aligned}$$

and the growth condition :

$$|v(t,z)| \leq K\left(1 + \left(x + \frac{p}{\lambda}\right)\right)^{\gamma}, \quad \forall (t,z) \in [0,T) \times \mathcal{S} \qquad (1.3.5)$$

for some positive constant $K < \infty$.

Remark 1.3.1. *In* [A10]*, the authors have also shown that the value function lies in the set of functions satisfying the growth condition :*

$$\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}) = \left\{ v : [0,T] \times \bar{\mathcal{S}} \to \mathbb{R};\ \sup_{[0,T]\times\bar{\mathcal{S}}} \frac{|\, v(t,z)\, |}{1 + (x + \frac{p}{\lambda})^\gamma} < \infty \right\} \qquad (1.3.6)$$

For simplifying notation and when there is no ambiguity, this set will be noted $\mathcal{G}_\gamma$.

1.4 Numerical scheme and convergence results

1.4.1 Approximation scheme

For a time step $h > 0$ on the interval $[0,T]$, let us consider the following approximation scheme:

$$S^h(t,z,v^h(t,z),v^h) = 0 \qquad (t,z) \in [0,T] \times \overline{\mathcal{S}} \qquad (1.4.7)$$

where $S^h : [0,T] \times \bar{\mathcal{S}} \times \mathbb{R} \times \mathcal{G}_\gamma \to \mathbb{R}$ is defined by

$$S^h(t,z,g,\varphi) := \begin{cases} min\Big[g - \mathbb{E}[\varphi(t+h, Z^{0,t,z}_{t+h})], g - \mathcal{H}\varphi(t,z)\Big], & t \in [0, T-h] \\ min\Big[g - \mathbb{E}[\varphi(T, Z^{0,t,z}_{T})], g - \mathcal{H}\varphi(t,z)\Big], & t \in (T-h, T) \\ min\Big[g - U_L(z), g - \mathcal{H}U_L(z)\Big], & t = T \end{cases} \qquad (1.4.8)$$

We consider a time step $h = T/m$, $m \in \mathbb{N}\backslash\{0\}$ and denote by $\mathbb{T}_m = \{t_i = ih, i = 0,...,m\}$ the regular grid over the interval $[0,T]$. Thus, the time discretization of step h for the QVI (1.2.2) leads to the explicit backward scheme:

$$\begin{aligned} v^{h,n+1}(t_m,z) &= \max\Big[U_L(z), \sup_{\zeta \in \mathcal{C}(z)} U_L(\Gamma(z,\zeta))\Big] \qquad (1.4.9) \\ v^{h,n+1}(t_i,z) &= \max\Big[\mathbb{E}[v^{h,n+1}(t_{i+1}, Z^{0,t_i,z}_{t_{i+1}})], \sup_{\zeta \in \mathcal{C}(z)} v^{h,n}(t_i, \Gamma(z,\zeta))\Big] \qquad (1.4.10) \end{aligned}$$

for $i = 0, ..., m-1$, $z = (x,y,p) \in \bar{\mathcal{S}}$ and starting from $v^{h,0}(t) = \mathbb{E}[U_L(Z^{0,t,z}_{t_m})]$, where $\mathbb{Z}_l = \{z = (x,y,p) \in \mathbb{X}_l \times \mathbb{Y}_l \times \mathbb{P}_l; z \in \bar{S}_{loc}\}$ is the grid which discretizes $\mathcal{S}$ and bounded by the quantity R, and $\mathcal{C}_{M,R}(z) = \{\zeta_i = \zeta_{min} + \frac{i}{M}(\zeta_{max} - \zeta_{min}); 0 \leq i \leq M/\Gamma(z,\zeta_i) \in \bar{S}_{loc}\}$ la grille à M is the grid which discretizes the set of admissible controls, and

$$\mathcal{E}^{N,R}[v^{h,n}(t, Z^{0,s,z}_t)] := \sum_{i_1=1}^{N_1} .. \sum_{i_{d(N)}=1}^{N_{d(N)}} \mathbb{P}_{i_1..i_{d(N)}} v^{h,n}(t, Z^{0,s,z}_{N,R}(t)) \quad \forall\, s \leq t$$

où

$$Z^{0,s,z}_{N,R}(t) := \Big(x, y, \Pi_{[0,p_{max}]}(p \exp\left\{(b - \frac{\sigma^2}{2})(t-s) + \sigma W^N_{i_1..i_{d(N)}}(t-s)\right\})\Big).$$

1.4.2 Convergence results

We first show that the value function could be obtained as the limit of an iterative procedure where each step is an optimal stopping problem and the reward function is related to the impulse operator.

Theorem 1.4.2. *We define $\varphi_n(t,z)$ iteratively as a sequence of optimal stopping problems:*

$$\begin{aligned} \varphi_{n+1}(t,z) &= \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} \mathcal{H}\varphi_n(\tau, Z^{0,t,z}_\tau)\right] \\ \varphi_0(t,z) &= v_0(t,z) \end{aligned}$$

where $\mathcal{S}_{t,T}$ is the set of stopping times in $[t,T]$. Then

$$\varphi_n(t,z) \;=\; v_n(t,z)$$

, where the value functions v_n is obtained when the investor is allowed to trade at most n times.

Following Barles and Souganidis (1991), we may obtain the convergence results of our scheme, once we prove that it satisfies monotonicity, stability and consistency properties.

Theorem 1.4.3. *For all $(t,z) \in [0,T) \times \mathcal{S}$ we have the following uniform convergence*

$$\lim_{\substack{(t',z') \to (t,z) \\ (h,M,N,R) \to (0,+\infty)}} v^{h,M,N,R}(t',z') = v(t,z),$$

where $v^{h,R,N,M}$ is the solution of the discrete HJB inequality without considering an iterative scheme and v is the solution of (1.2.2).

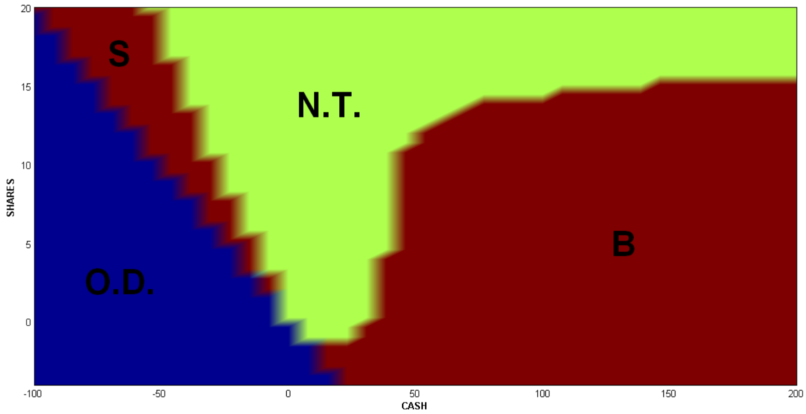

Figure 1.1: *The optimal policy in* (x, y)

1.5 Numerical results

In this section, we present numerical results, such as the value function and the optimal transaction strategy.

Figure 1.1: The shape of the optimal transaction strategy (different regions).

Tables 1.1**,** 1.2**, and** 1.3**: Convergence analysis in function of** N**,** M **and** R

In Table 1.1 and 1.2, we show some values of the value function for different values of N and M at a time $t = 0.05$, and for fixed nodes of the grid $z_1 = (x_1, y_1, p_1) = (42.10, 7.36, 23.68)$ and $z_2 = (x_2, y_2, p_2) = (121.052, 13.68, 36.84)$.

N	96	200
$v(t, z_1)$	15.8841	15.8807
$v(t, z_2)$	26.3895	26.3867

Table 1.1 : *Values of the value function for different values of* N *and* z.

M	200	250
$v(t, z_1)$	15.8825	15.8849
$v(t, z_2)$	26.3990	26.4011

Table 1.2 : *Values of the value function for different values of* M *and* z.

Chapter 2

Optimal execution cost for liquidation through a limit order market

This Chapter is based on
[$A9$]. Optimal execution cost for liquidation through a limit order market, with E. Chevalier, A. Roch and S. Scotti, 2014, to appear in International Journal of Theoretical and Applied Finance.

2.1 Liquidity Risk in Limit Order Books

In limit order book markets, any public trader can play the role of liquidity provider by posting prices and quantities at which he is willing to buy or sell while waiting for a counterparty to engage in that trade. Limit orders can be entered at more favorable prices than market orders but are not guaranteed to be filled. On the other hand, a market order is filled automatically against existing limit orders, albeit at a less favorable price as it depletes the order book, making additional trades more expensive. It is therefore desirable to consider financial models with an enlarged set of admissible trading strategies by including the possibility of making both limit orders and market orders. In this paper, we consider the liquidation problem of a large portfolio position from this perspective.

Many authors have investigated the liquidation and market making problems with limit orders only, in particular [7], [14], [56], [57], [58] and [87]. In these models, the arrival intensity of outside market orders that match the limit orders that are posted is typically a function of the spread between the posted price and a reference price. In a more complex model, Cartea et al. [31] develop a high-frequency limit order trading strategy in a limit order market characterized by feedback effects in market orders and the shape of the order book, and by adverse selection risk due to the presence of informed traders who make influential trades. Kühn and Muhle-Karbe [75] provide an asymptotics analysis for a small investor who sets bid and ask prices and seeks to maximize expected utility when the spread is small.

On the other hand, some authors consider a limit order market in which both limit and market orders are possible. Guilbaud and Pham [59] determine the optimal trading strategy of a market maker who makes both types of trades and seeks to maximize the expected utility over a short term horizon. Cartea and Jaimungal [29] determine the optimal liquidation schedule in a limit order market in which the liquidity cost of a market order is fixed, and the probability of passing a limit order depends on the spread between the posted price and a reference price, modeled as a Brownian motion plus drift. The investor's value function includes a quadratic penalty defined in terms of a target inventory schedule. In this work, we also consider a limit order market in which both limit and market orders are allowed, and study the problem of optimally liquidating a large portfolio position.

Our contribution to the above literature is to consider spread dynamics which are impacted by both limit and market order strategies. Market orders that the investor places directly increase the observed bid-ask spread. As a result, past market orders have a direct impact on future liquidity costs. Furthermore, limit orders posted inside the bid-ask spread effectively decrease the observed spread and have an impact on the future probability distributions of its jumps.

We model the bid-ask spread with resilience (mean reversion) and a jump process, and the market order arrival process as a controlled Poisson process. The objective is to liquidate a fixed number of shares of a risky asset by minimizing the expected liquidity premium paid. We formulate the problem as a mixed stochastic continuous control and impulse problem for which the value function is shown to be the unique viscosity solution of the associated system of variational inequalities. Finally, we numerically implement the model and calibrate it to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook*.

2.2 The Limit Order Book Market Model

Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered probability space supporting a random Poisson measure M on $[0, T] \times \mathbb{R}$ with mean measure $\gamma_t \, dt \, m(dz)$ where $\gamma : [0, T] \to (0, \bar{\gamma}]$ and m is a probability measure on $\mathbb{R}$, with $m(\mathbb{R}) < \infty$. We consider a market with a risky asset that can be traded through a limit order book. We consider a large investor whose goal is to liquidate a number $N > 0$ of shares of this risky asset. The investor sets a date T before which the position must be liquidated and attempts to minimize the price impact of the liquidation strategy.

Market orders. The investor can make market orders by controlling the time and the size of his trades. This is modeled by an impulse control strategy $\beta = (\tau_i, \ \xi_i)_{i \leq n}$ where the τ_i's are stopping times representing the intervention times of the investor and the ξ_i's are $\mathcal{F}_{\tau_i}$-measurable random variables valued in $\mathbb{N}$ and giving the number of shares sold by a market order at time τ_i.

Limit Orders. The investor can also make limit orders. We denote by $\mathcal{A}_0$ a compact subset of $[0,\infty)$. The investor can choose to place his limit price anywhere inside the bid-ask spread or at the current best ask price. Since the effect of this new limit order is that the best ask price can now be lower, we call the best ask price *excluding* the investor's limit order the *otherwise best ask price*. The spread *below the current best ask price* is an $\mathcal{A}_0$-valued stochastic control denoted by $\alpha = (\alpha_t)_{t\leq T}$.

Definition 2.2.1. (Investor's control strategy) *We define the investor's control strategy as being the full control available to the investor, thus given by a pair of controls* $\delta = (\alpha, \beta)$.

Bid-Ask Spread. We denote by X_t the spread between the best bid and the best ask price *excluding* the investor's limit price at time t. Between the investor's market orders, we assume the spread X is impacted by α and follows

$$dX_t^\alpha = \mu(t, X_{t-}^\alpha, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\alpha, \alpha_t, z)\tilde{M}(dt, dz). \tag{2.2.1}$$

Under this construction, the limit order α sends a signal and modifies the distribution of the jumps of X, represented by G. Here $\tilde{M}$ is the compensated random measure of M, and μ is a deterministic and Lipschitz continuous function in the second argument.

Liquidity cost

We summarize the information contained in the order book by a function $S(t,x,n)$ which gives the proceeds obtained for a sale of n shares at time t done through market orders when the spread equals x. In the order book density case, this corresponds to Equation 12 in [3]. Let A_t be a stochastic process representing the best ask price. We may then define the liquidity cost due to a market sell order of size n, denoted by $L(t,x,n)$, in terms of the best ask price as $L(t,x,n) := nA_t - S(t,x,n)$. The slippage of a market order of size n is then defined as a fixed transaction cost, $k > 0$, plus the liquidity cost, i.e. $K(t,x,n) = k + L(t,x,n)$.

The simplest example is a quadratic model with proportional transaction costs:

$$S(t,x,n) = (A_t - x)n - \zeta_t n^2,$$

with A_t and ζ_t two stochastic processes representing the best ask price and a measure of illiquidity. This model arises from a limit order book with constant density as shown in [98]. In the quadratic model, $L(t,x,n) = xn + \zeta_t n^2$.

Impact on the best bid

During a transaction, the investor's market orders are matched with the existing limit orders in the order book so that the result is a shift in the best bid price to the left by an amount denoted by $I(t,x,n)$. In [3], this quantity is called the extra spread and denoted by D_t^B.

Dynamics of the controlled bid-ask spread. The resulting dynamic for X^δ (with $\delta = (\alpha, \beta)$) taking into account both α and β is

$$\begin{cases} dX_t^\delta &= \mu(t, X_t^\delta, \alpha_t)dt + \int_{\mathbb{R}} G(X_{t-}^\delta, \alpha_{t-}, z)\tilde{M}(dt, dz) \quad \text{if } \tau_n < t < \tau_{n+1} \\ X_{\tau_n}^\delta &= \check{X}_{\tau_n^-}^\delta + I(\tau_n, \check{X}_{\tau_n^-}^\delta, \xi_n), \end{cases} \tag{2.2.2}$$

where $\check{X}_{t-}^\delta = X_{t-}^\delta + \Delta X_t^\delta$, ΔX_t^δ is the jump of the measure M at time t. The superscripts in controlled processes will often be omitted to alleviate the notation.

Market orders arrival. We start with a time inhomogeneous Poisson process $\mathcal{N}$, independent of W and M, with intensity $\lambda(t,0) \geq \underline{\lambda} > 0$, $t \geq 0$. The jumps of this Poisson process are denoted θ_i, $i \geq 1$. For all $x > 0$, we define intensity functions $\lambda(\cdot, x) : [0,T] \to [0,\infty)$, and assume $(\lambda(\cdot,x))_{x>0}$ is an equicontinuous family of functions, bounded below and above by constants $\underline{\lambda}, \overline{\lambda} > 0$. If the investor chooses to place a limit order at a spread α_t below the otherwise best ask price at time t, the likelihood of the execution of this order depends on the observed spread $X_t - \alpha_t$ and arrives with an intensity $\lambda(t, X_t - \alpha_t)$. At the time θ_i, the investor's limit order will go through for a random quantity equal to Y_i, less or equal to n' (the fixed size of the limit order), and independent of $\mathcal{F}_{\theta_i-}$. The fact that the jump intensity is time-dependent is particularly relevant in markets where there is well-known u-shaped trading volume pattern during the day.

Let $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t^\alpha$ with $Z_0^\alpha = 1$ and

$$dZ_t^\alpha = Z_{t-}^\alpha \left(\frac{\lambda(t, X_t - \alpha_t)}{\lambda(t,0)} - 1 \right) (d\mathcal{N}_t - \lambda(t,0)dt).$$

Then a control α changes the distribution of $\mathcal{N}$ under $\mathbb{P}$ to the distribution of $\mathcal{N}$ under $\mathbb{P}^\alpha$, by changing the intensity of $\mathcal{N}$ from $\lambda(t,0)$ to $\lambda(t, X_t - \alpha_t)$.

The slippage of a limit order that is matched at time θ_i is then given by $\alpha_{\theta_i} Y_i$.

$N^{\delta,n,t}$: dynamics of the remaining number of shares to liquidate.

Admissible control strategies

Now, we define the set of admissible strategies. The limit orders control strategy $\alpha = (\alpha_s)_{0 \leq s \leq T}$ is assumed to be a stochastic Markov control such that $\alpha_t < X_{t-}^\delta$ for all $t \leq T$. We denote the set of Markov control by $\mathcal{A}$. Let $\mathcal{T}_{t,T}$ be the set of stopping times with values in $[t,T]$. The set of admissible strategies started at time $t \in [0,T]$ when the investor has n shares remaining in the portfolio and that the spread is equal to x is defined as

$$\begin{aligned} \mathcal{AB}(t,n,x) \quad = \quad & \{\delta = (\alpha,\beta) : \alpha \in \mathcal{A}, \beta = (\tau_i, \xi_i)_{i \leq n}, \tau_i \in \mathcal{T}_{t,T};\ \xi_i \leq n \text{ is an } \mathbb{N}\text{-valued random variable} \\ & \mathcal{F}_{\tau_i^-} - \text{measurable s.t. } \tau^{\delta,n,t} \leq T\}, \end{aligned}$$

where $\tau^{\delta,n,t} = \inf\{s \geq t: \ N_s^{\delta,n,t} = 0\}$.

The Control Problem

The investor's goal is to minimize expected slippage by balancing his actions between market orders, which are more expensive due to immediacy, and limit orders, for which the execution time is unknown and random but are executed at more favorable prices. For a strategy $\delta = (\alpha, \beta) \in \mathcal{AB}(t, n, x)$ started at time t, slippage is defined as

$$S_T^\delta = \sum_{i=1}^{n} K(\tau_i, \check{X}^\delta_{\tau_i^-}, \xi_i)\mathbb{1}_{\tau_i \leq \tau^\delta} + \sum_{i\geq 1} \alpha_{\theta_i} Y_i \mathbb{1}_{\theta_i \leq \tau^\delta}.$$

For $(t, x, n) \in [0, T] \times [0, +\infty) \times \mathbb{N}$, we define the optimal expected slippage function in the following way:

$$C_n(t, x) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha} S_T^\delta, \tag{2.2.3}$$

with the following boundary condition: $C_n(T, x) = K(T, x, n)$ for all $n \in \mathbb{N}^*$, which follows readily from the fact that $\tau^{\delta,n,T} = T$, so that the investor must necessarily liquidate the remaining part of his portfolio with a market order at time T.

2.2.1 Penalty Function

The maturity T is an urgency parameter. The shorter it is, the more aggressive the strategy and the higher the liquidation cost. However, in order to impose more urgency in the liquidation, it is possible to include a penalty function or a risk aversion parameter in the minimization problem. We may add a penalty function π in terms of the number of remaining shares at time t:

$$C_n(t, x) = \inf_{\delta \in \mathcal{AB}(t,x,n)} \mathbb{E}_{t,x,n,\alpha} \left[S_T^\delta + \int_t^T \pi(N_s^\delta, s) ds \right]. \tag{2.2.4}$$

This penalty function can be used to target a specific liquidation schedule as in Cartea and Jaimungal [29], it can be a proxy for the variance of the value of the remaining shares in the portfolio when π is of the quadratic form (see Cartea and Jaimungal [30]).

2.3 Characterization of the slippage function

In this section, we prove that the function C_n is the viscosity solution of an associated quasi-variational inequality. We first introduce the infinitesimal generator of the process $(t, X_t)_{t\geq 0}$ between two market orders:

$$\mathcal{L}^a u(t, x) = \frac{\partial u}{\partial t} + \mu(t, x, a)\frac{\partial u}{\partial x} + \gamma_t \int_{\mathbb{R}} (u(t, x + G(x, a, z)) - u(t, x))m(dz),$$

and the limit orders operator:

$$\Delta_n^a u(t,x) = \lambda(t, x-a)\left[f(a) + \sum_{i=1}^{\infty} p_i C_{n-i}(t,x) - u(t,x)\right],$$

in which $p_i = \mathbb{P}(Y_1 = i)$ $(i \geq 1)$ and $f(a) = a\sum_{i=1}^{\infty} ip_i$, $a \in \mathcal{A}_0$. Finally, define the impulse function for market orders:

$$\mathcal{M}_n(t,x) = \min_{i\in\{1,\dots,n\}} \left[C_{n-i}(t, x + I(t,x,i)) + K(t,i,x)\right].$$

Notice that, for all $(t,x,n) \in [0,T]\times\mathbb{R}_+\times\mathbb{N}^*$, we deduce from (2.2.3) that

$$0 \leq C_n(t,x) \leq K(t,x,n) = \kappa + L(t,x,n).$$

Therefore, recalling that $\mathcal{P}$ is the set of functions from $[0,T]\times\mathbb{R}_+$ to $\mathbb{R}$ with at most polynomial growth of degree p in the second argument, we have $C_n \in \mathcal{P}$ for all $n \in \mathbb{N}$.

Our main result of this section is the following theorem.

Theorem 2.3.1. *For all $n \geq 1$, C_n is the unique continuous viscosity solution in $\mathcal{P}$ of the following variational inequality:*

$$\begin{cases} \min\left(\min_{a\in\mathcal{A}_0} \mathcal{L}^a u + \Delta_n^a u;\ \mathcal{M}_n - u\right) = 0 & \textit{on } [0,T)\times[0,\infty), \\ u(T,x) = K(T,n,x) & \textit{for } x \geq 0. \end{cases} \tag{2.3.5}$$

2.4 Numerical Results

We calibrated the model to market data corresponding to four different firms traded on the NYSE exchange through the *ArcaBook* from February 28th to March 4th, 2011. The data files obtained from NYXdata.com contains all time-stamped limit orders entered, removed, modified, filled or partially filled on the NYSE ArcaBook platform. The firms considered are Google (GOOG), Air Products & Chemicals Inc. (APD), International Business Machines Corp. (IBM), and J.P. Morgan Chase & Co. (JPM). All four stocks are very liquid and were part of the S&P500 index in 2011. Yet a major difference is that the empirical distribution of their bid-ask spreads differ considerably. This is due to the fact that their stock prices are of a different order of magnitude with GOOG at an average price of 606.97, APD at 91.15, IBM at 161.76 and JPM at 45.92 over the five days. In percentage, JPM and IBM have smaller spreads (0.03% of stock price) than GOOG (0.073% of stock price) and APD (0.075%). Since prices are quoted in cents, this offers a large array of values of spreads for GOOG, for which the spread varied from \$0.01 to \$2.67 during the five trading days considered. The resulting liquidation strategies are very different quantitatively and qualitatively.

Chapter 3

Optimal market dealing under constraints

This Chapter is based on
[A12]. Optimal market dealing under constraints, with E. Chevalier, M. Gaigi, et M. Mnif, soumis.

3.1 Introduction

In this paper, we consider an equity quote-driven market with a single risky equity assets. In the trading of most equity assets in either Nasdaq or LSE, there are several registered market makers in competition. In order to focus on the modelling of the market making strategies, we consider there is only one "representative" registered market maker. We assume that the market maker has a contractual obligation to permanently quote bid and ask prices and therefore has to satisfy any sell and buy market order from investors. The market maker may benefit from the bid-ask spread but faces a number of constraints, in particular the liquidity and inventory constraints. The structural constraints imposed upon market makers in dealer markets are proved to be a major challenge. In the study of market making problems, we may refer to Avellaneda and Stoikov [7], Ho and Stoll [66], and Mildenstein and Schleef [85]. In [7] and [66], the authors consider a market making problem as described above but within a financial market in which the risky assets has a reference price or a fair price S_t which is assumed to follow an arithmetic brownian process. The market maker quotes her ask and bid prices as respectively $S_t + \delta_t^a$ and $S_t - \delta_t^b$, where (δ_t^a, δ_t^b) represent the strategy control of the market maker. The price processes are therefore mainly driven by the reference price process.

In our study, we do not assume the existence of a reference price. The prices are therefore uniquely driven by the equilibrium between buy and sell market orders. In terms of mathematical modelling and resolution, a difficult challenge to overcome is to take into account the inventory constraints that the market maker faces. First, we consider, as [85], that the market maker has the obligation to respect the risk constraint imposed upon her by her company's risk department. We may refer to [58] which investigates the impact of the inventory constraints on the market making problem studied in [7]. The stock inventory of the market maker is assumed to have upper and lower bounds. However, unlike in [58], once the inventory reaches the lower (upper) bound, we do not allow the market maker to stop submitting limit ask (bid) order since allowing such move violates the agreement that the market maker's firm has agreed with the financial stock exchange to continuously quote bid and ask prices.

A second important difference with the problems studied in [7], [66] and [85], comes from the assumption that the market maker may liquidate her stock inventory on terminal date at the reference price or a constant price independent of the inventory. In our paper, we assume that when the market maker has to liquidate her stock inventory, she incurs a liquidity cost and the price per share received (paid) are lower (higher) than the mid-price in the case of a long (short) position. Our assumption on the form of liquidation function is mainly inspired by [64] and [103].

Furthermore, to take into account the microstructure of the financial markets, we no longer consider continuous price processes. Bid and ask prices quoted by the market maker are assumed to be discrete prices, i.e. in multiple of a tick value.

The contributions of our study, as compared to previous studies [7], [66], [85], concern both the modelling aspects and the dynamic structure of the control strategies. Important features and constraints characterizing market making problems are no longer ignored.

We provide rigorous mathematical characterization and analysis to our control problem by proving that our value functions are the unique viscosity solutions to the associated HJB system. It is always a technical challenge when applying viscosity techniques to non-standard control problems under constraints. In the proof of our comparison theorem, a major problem is to circumvent the difficulty arising from the discontinuity of our HJB operator on some parts of the solvency region boundary.

3.2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ where T is a finite horizon. We consider a single dealer market, in which there is a risky assets. In this market, the market maker has the obligation to permanently quote bid and ask prices and to act as a counterparty to investors' market orders. We equally assume that investors, considered as price-takers, may only submit either buy or sell market orders.

3.2.1 Model settings

Trading orders. We denote by $(\theta_i^a)_{i \geq 1}$ (resp. $(\theta_i^b)_{i \geq 1}$) the sequence of non-decreasing $\mathbb{F}$-stopping times corresponding to the arrivals of buy (resp. sell) market orders. We denote by $(\xi_i)_{i\geq 1}$ the sequence of these trading times. When a buy (resp. sell) market order arrives at time θ_i^a (resp. θ_j^b), the market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by P^a (resp. P^b).

Market making strategies. We define a strategy control as being a $\mathbb{F}$-predictable process $\alpha = (\alpha_t)_{(0\leq t\leq T)} = (\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b)_{0\leq t\leq T}$ where the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ take values in $\{\chi_{min}, .., \chi_{max}\}$, with $-\chi_{min} \in \mathbb{N}$ and $\chi_{max} \in \mathbb{N}^*$.
We assume that when a sell market order arrives at time θ_j^b, the market maker may either keep the bid and ask prices constant or decrease one or both of them by at most χ_{max} ticks or increase one or both of them by at most χ_{min} ticks. Notice the market maker may decide to change the bid/ask prices but transaction prices are assumed to be based on the one quoted before the prices changes. In here, a tick value is denoted by a strictly positive constant δ. On the opposite side, when a buy market order arrives at time θ_k^a, the market maker may either keep the bid and ask prices constant or increase one or both of them by at most χ_{max} ticks or decrease one or both of them by at most χ_{min} ticks.

Bid-Ask spread modelling.
We denote by $P^a = (P_t^a)_{0\leq t\leq T}$ (resp. $P^b = (P_t^b)_{0\leq t\leq T}$) the price quoted by the market maker to buyers (resp. sellers). Notice that $P^a \geq P^b$.
The dynamics of $P^{a,b}$ evolves according to the following equations

$$\begin{aligned} dP_t^{a,b} &= 0, \ \xi_i < t < \xi_{i+1} \\ P_{\theta_{j+1}^b}^{a,b} &= P_{\theta_{j+1}^{b-}}^{a,b} - \delta\epsilon_{\theta_{j+1}^b}^{a,b} \\ P_{\theta_{k+1}^a}^{a,b} &= P_{\theta_{k+1}^{a-}}^{a,b} + \delta\eta_{\theta_{k+1}^a}^{a,b}. \end{aligned}$$

where i is the number of transactions before time t, j the number of buy transactions before time t for the market maker, k the number of sell transactions before time t, and δ represents one tick.
We denote by P the mid-price and S the bid-ask spread of the stocks. The dynamics of the process (P, S) is given by

$$\begin{aligned} dP_t &= 0, \ \xi_i < t < \xi_{i+1} && (3.2.1) \\ P_{\theta_{j+1}^b} &= P_{\theta_{j+1}^{b-}} - \frac{\delta}{2}(\epsilon_{\theta_{j+1}^b}^a + \epsilon_{\theta_{j+1}^b}^b) && (3.2.2) \\ P_{\theta_{k+1}^a} &= P_{\theta_{k+1}^{a-}} + \frac{\delta}{2}(\eta_{\theta_{k+1}^a}^a + \eta_{\theta_{k+1}^a}^b), && (3.2.3) \end{aligned}$$

$$\begin{aligned} dS_t &= 0, \ \xi_i < t < \xi_{i+1} && (3.2.4) \\ S_{\theta_{j+1}^b} &= S_{\theta_{j+1}^{b-}} - \delta(\epsilon_{\theta_{j+1}^b}^a - \epsilon_{\theta_{j+1}^b}^b) && (3.2.5) \\ S_{\theta_{k+1}^a} &= S_{\theta_{k+1}^{a-}} + \delta(\eta_{\theta_{k+1}^a}^a - \eta_{\theta_{k+1}^a}^b). && (3.2.6) \end{aligned}$$

Regime switching. We first consider the tick time clock associated to a Poisson process $(R_t)_{0\le t\le T}$ with deterministic intensity λ defined on $[0,T]$, and representing the random times where the intensity of the orders arrival jumps.
We define a discrete-time stationary Markov chain $(\hat{I}_k)_{k\in\mathbb{N}}$, valued in the finite state space $\{1,...,m\}$, with probability transition matrix $(p_{ij})_{1\le i,j\le m}$, i.e. $\mathbb{P}[\hat{I}_{k+1}=j|\hat{I}_k=i]=p_{ij}$ s.t. $p_{ii}=0$, independent of R. We define the process

$$I_t = \hat{I}_{R_t},\ t\ge 0 \tag{3.2.7}$$

$(I_t)_t$ is a continuous time Markov chain with intensity matrix $\Gamma=(\gamma_{ij})_{1\le i,j\le m}$, where $\gamma_{ij}=\lambda p_{ij}$ for $i\neq j$, and $\gamma_{ii}=-\sum_{j\neq i}\gamma_{ij}$.
We model the arrivals of buy and sell market orders by two Cox processes N^a and N^b. The intensity rate of N^a_t and N^b_t is given respectively by $\lambda^a(t,I_t,P_t,S_t)$ and $\lambda^b(t,I_t,P_t,S_t)$ where λ^a and λ^b are continuous functions valued in $\mathbb{R}$ and defined on $[0,T]\times\{1,...,m\}\times\frac{\delta}{2}\mathbb{N}\times\delta\mathbb{N}$.

We assume that:

$$\bar{\lambda} := \sup_{[0,T]\times\{1,...,m\}\times\frac{\delta}{2}\mathbb{N}\times\delta\mathbb{N}}\Big(\max(\lambda^a,\lambda^b,\lambda)\Big) < +\infty.$$

We now define θ^a_k (resp. θ^b_k) as the k^{th} jump time of N^a (resp. N^b), which corresponds to the k^{th} buy (sell) market order.
We introduce the following stopping times $\rho_j(t)=\inf\{u\ge t, I_u=j\}$ and $\rho(t)=\inf\{u\ge t, R_u>R_t\}$ for $0\le t\le T$ and the notation $Z^{t,i,z,\alpha}$ is the state process associated to the control α such that $(I_t,\ Z^{t,i,z,\alpha}_t)=(i,z)$.

3.2.2 The control problem

Stock holdings. The number of shares held by the market maker at time $t\in[0,T]$ is denoted by Y_t, and Y satisfies the following equations

$$\begin{aligned} dY_t &= 0,\ \xi_i<t<\xi_{i+1} && (3.2.8)\\ Y_{\theta^b_{j+1}} &= Y_{\theta^{b-}_{j+1}}+1 && (3.2.9)\\ Y_{\theta^a_{k+1}} &= Y_{\theta^{a-}_{k+1}}-1, && (3.2.10)\end{aligned}$$

As in [58] and [85], we consider that the market maker has the obligation to respect the risk constraint imposed upon her by her company. We impose the following inventory constraint

$$y_{min}\le Y_t\le y_{max} \text{ a.s. } 0\le t\le T. \tag{3.2.11}$$

Cash holdings. We denote by $r>0$ the instantaneous interest rate. The bank account follows the below equation between two trading times

$$dX_t = rX_tdt,\ \xi_i<t<\xi_{i+1}. \tag{3.2.12}$$

When a discrete trading occurs at time θ^b_{j+1} (resp. θ^a_{k+1}), the cash amount becomes

$$\begin{aligned} X_{\theta^b_{j+1}} &= X_{\theta^{b-}_{j+1}} - P^b_{\theta^{b-}_{j+1}} && (3.2.13) \\ X_{\theta^a_{k+1}} &= X_{\theta^{a-}_{k+1}} + P^a_{\theta^{a-}_{k+1}}, && (3.2.14) \end{aligned}$$

State process. We define the state process as follows:

$$Z = (X,\ Y,\ P := \frac{P^a + P^b}{2},\ S := P^a - P^b). \tag{3.2.15}$$

Cost of liquidation of the portfolio. If the current mid-price at time $t < T$ is p and the market maker decides to liquidate her portfolio, then we assume that the price she actually gets is

$$Q(t, y, p, s) = (p - \mathrm{sign}(y)\frac{s}{2})f(t, y), \tag{3.2.16}$$

where f is an impact function defined from $[0, T] \times \mathbb{R}$ into $\mathbb{R}_+$.

Liquidation value and Solvency constraints. A key issue for the market maker is to maximize the value of the net wealth at time T. In our framework, we impose a constraint on the spread i.e.

$$0 < S_t \leq K\delta,\ 0 \leq t \leq T.$$

We also impose that the bid price remains positive, therefore the market maker has to use controls such that

$$P_t - S_t/2 > 0.$$

When the market maker has to liquidate her portfolio at time t, her wealth will be $L(t, X_t, Y_t, P_t, S_t)$ where L is the liquidation function defined as follows

$$L(t, x, y, p, s) = x + yQ(t, y, p, s),$$

with Q as defined in 3.2.16.
We may now introduce the following state space

$$\mathbb{S} = (x_{min}, +\infty) \times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, ..., K\}.$$

and then the solvency region

$$\mathcal{S} = \{(t, x, y, p, s) \in [0, T] \times \mathbb{S} :\ p - \frac{s}{2} \geq \delta\}.$$

We denote its boundary and its closure by

$$\partial_x\mathcal{S} \;=\; \Big\{(t, x, y, p, s) \in [0, T] \times \bar{\mathbb{S}} :\ x = x_{min}\Big\} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S} \cup \partial_x\mathcal{S}.$$

Admissible trading strategies. Given $(t,z) := (t,x,y,p,s) \in \mathcal{S}$, we say that the strategy $\alpha = (\epsilon^a_u, \epsilon^b_u, \eta^a_u, \eta^b_u)_{t\leq u\leq T}$ is admissible, if the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{\chi_{min}, ..., \chi_{max}\}$ and for all $u \in [t,T]$, $(u, Z^{t,i,z,\alpha}_u) \in \mathcal{S}$. We denote by $\mathcal{A}(t,z)$ the set of all these admissible policies.

Value functions. We set g a non-negative penalty function defined on $\{y_{min}, ..., y_{\max}\}$. This penalty may be compared to the holding costs function introduced in [85]. For numerical purposes, we will consider as in [85] a quadratic penalty cost function.

We also consider an exponential utility function U i.e. there exists $\gamma > 0$ such that $U(x) = 1 - e^{-\gamma x}$ for $x \in \mathbb{R}$. We set $U_L = U \text{o} L$.
As such, we consider the following value functions $(v_i)_{i\in\{1,\dots,m\}}$ which are defined on $\mathcal{S}$ by

$$v_i(t,z) := \sup_{\alpha\in\mathcal{A}(t,z)} J^\alpha_i(t,z) \tag{3.2.17}$$

where we have set

$$\begin{aligned} J^\alpha_i(t,z) &:= \mathbb{E}^{t,i,z}\left[U_L(T\wedge\tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}) - \int_t^{T\wedge\tau^{t,i,z,\alpha}} g(Y^{t,i,y,\alpha}_s)ds\right], \\ \tau^{t,i,z,\alpha} &:= \inf\{u \geq t : X^{t,i,x,\alpha}_u \leq x_{min} \text{ or } Y^{t,i,y,\alpha}_u \in \{y_{min}-1, y_{max}+1\}\}. \end{aligned}$$

3.3 Analytical properties and viscosity characterization

We use a dynamic programming approach to derive the system of partial differential equations satisfied by the value functions. Once we obtain, we obtain the upper and lower bounds of the value function and its uniform continuity, we may state the following

Theorem 3.3.2. ***Dynamic programming principle (DPP)***
Let $(i,t,z) := (i,t,x,y,p,s) \in \{1,...,m\} \times \mathcal{S}$. Let ν be a stopping time in $\mathcal{T}_{t,T}$, we have

$$v_i(t,z) = \sup_{\alpha\in\mathcal{A}(t,z)} \hat{J}^{\alpha,\nu}_i(t,z), \tag{3.3.18}$$

where, for $\alpha \in \mathcal{A}(t,z)$, we have set

$$\begin{aligned} \hat{J}^{\alpha,\nu}_i(t,z) &= \mathbb{E}\Big[- g(y)\left(\nu\wedge\hat\theta\wedge\hat\tau^\alpha - t\right) + v_{I_{\nu\wedge\hat\theta}}\left(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\right) 1\!\!1_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}} \\ &\quad + U_L\left(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha - t)}, y, p, s\right) 1\!\!1_{\{\hat\tau^\alpha\leq\nu\wedge\hat\theta\}}\Big], \end{aligned} \tag{3.3.19}$$

with $\hat\tau^\alpha = \tau^{t,i,z,\alpha}\wedge T$, $\rho = \inf\{u \geq t : R_u > R_{u^-}\}$, $\theta^w = \inf\{u \geq t : N^{w,i,t,z}_u > N^{w,i,t,z}_{u^-}\}$, for $w \in \{a,b\}$ and $\hat\theta = \rho\wedge\theta^a\wedge\theta^b$.

We now turn to the characterization of the value functions. We first define the following set:

$$\begin{aligned} A(t,z) := \Big\{\alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{-\chi_{min}, ..., \chi_{max}\}^4 \text{ s.t. } p - \frac{s}{2} - \delta\varepsilon^b \geq \delta, \\ \delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta, \text{ and } \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta\Big\}. \end{aligned}$$

For all $(i,t,x,y,p,s) := (i,t,z) \in \{1,...,m\} \times \mathcal{S}$ and $\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in A(t,z)$, we introduce the two following operators:

$$\mathcal{A}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{min},p,s) & \text{if} \quad y = y_{min} \\ v_i(t, x+p+\frac{s}{2}, y-1, p+\frac{\delta}{2}(\eta^a+\eta^b), s+\delta(\eta^a-\eta^b)) & \text{otherwise} . \end{cases}$$

$$\mathcal{B}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{max},p,s) & \text{if} \quad y = y_{max} \\ U_L(t,z) & \text{if} \quad x < x_{min}+p-\frac{s}{2} \\ U_L(t,z) & \text{if} \quad x = x_{min}+p-\frac{s}{2} < 0 \\ v_i(t, x-p+\frac{s}{2}, y+1, p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b), s-\delta(\varepsilon^a-\varepsilon^b)) & \text{otherwise} . \end{cases}$$

On the open set $\{1,...,m\} \times \mathcal{S}$, we have:

$$-\frac{\partial v_i}{\partial t} - \mathcal{H}_i(t,z,v_i,\frac{\partial v_i}{\partial x}) = 0, \ (t,z) \in \mathcal{S}, \tag{3.3.20}$$

where $\mathcal{H}_i$ is the Hamiltonian associated with state i:

$$\begin{aligned} \mathcal{H}_i(t,z,v_i,\frac{\partial v_i}{\partial x}) \quad &= \quad rx\frac{\partial v_i}{\partial x} + \sum_{j\neq i} \gamma_{ij}\left(v_j(t,x,y,p,s) - v_i(t,x,y,p,s)\right) - g(y) \\ &+ \sup_{\alpha\in A(t,z)} \Big[\lambda_i^a(t,p,s)\left(\mathcal{A}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s)\right) \\ &+ \lambda_i^b(t,p,s)\left(\mathcal{B}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s)\right)\Big] = 0. \end{aligned}$$

The boundary and terminal conditions are given by :

$$v_i(t,x_{min},y,p,s) \quad = \quad U_L(t,x_{min},y,p,s) \tag{3.3.21}$$
$$v_i(T,x,y,p,s) \quad = \quad U_L(T,x,y,p,s). \tag{3.3.22}$$

We now provide a rigorous characterization for the value function by means of viscosity solutions to the HJB equation (3.3.20) together with the appropriate boundary terminal conditions. The uniqueness property is particularly crucial to numerically solve the associated HJB. Since the value functions v_i is continuous, we shall work with the notion of continuous viscosity solutions.

The following theorem relates the value function v_i to the HJB (3.3.20) for all $1 \leq i \leq m$.

Theorem 3.3.3. *The family of value functions $(v_i)_{1\leq i\leq m}$ is the unique family of functions such that*

i) Continuity condition: For all $(i,y,p,s) \in \{1,..,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}\times\delta\{1,..,K\}$, $(t,x) \to v_i(t,x,y,p,s)$ is continuous on $\{(t,x) \in [0,T) \times [x_{min},+\infty) : (t,x,y,p,s) \in \mathcal{S}\}$.

ii) Growth condition: There exist C_1, C_2 and C_3 positive constants such that

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t,x,y,p,s) \leq 1, \qquad \text{on } \{1,..,m\} \times \mathcal{S}. \tag{3.3.23}$$

iii) *Boundary and terminal conditions:*

$$v_i(t, x_{min}, y, p, s) = U_L(t, x_{min}, y, p, s) \text{ and } v_i(T, x, y, p, s) = U_L(T, x, y, p, s). \quad (3.3.24)$$

iv) *Viscosity solution:* *$(v_i)_{1 \le i \le m}$ is a viscosity solution of the system of variational inequalities (3.3.20) on $\{1, ..., m\} \times \mathcal{S}$.*

3.4 Numerical Results

In this paragraph, we present the results of the numerical method we used to approximate the solution of the system of equations (3.3.20).

3.4.1 Numerical scheme

To solve the HJB equation (3.3.20) arising from the stochastic control problem (3.3.18), one can use either probabilistic or deterministic numerical method. We choose to use a deterministic method based on a finite difference scheme, which is well known to have the monotonicity, consistency and stability properties. These properties ensure the convergence of this scheme, see [11].

Shape of the value function

We represent in Figure 3.1 the shape of the value function associated to the regime 1 for fixed (t, x, y) such that y is positive.

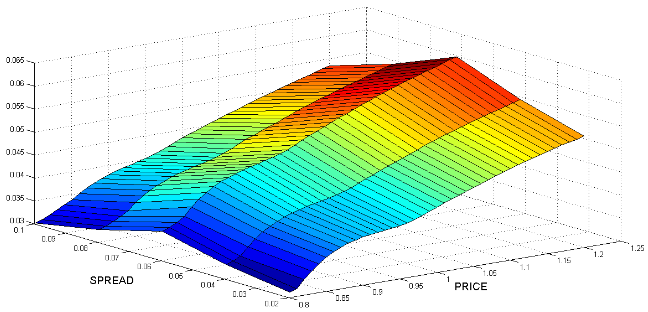

Figure 3.1: *Value function for $y \geq 0$*

Optimal market making strategies

Figure 3.2 describes the optimal control strategies for the market marker when a sell market order arrives and when the market maker's inventory is around zero. We may make the following observations:

– when the spread is very low, the market maker has to decrease the bid price more than the ask price, see region where the spread value is below 0.07.
– when the spread is high and close to the maximum spread allowed, the market maker should decrease the ask price. She should decrease the spread in order to encourage trades.

Notice that the market maker may make a profit of 3 ticks in the favorable case, i.e., the next market order is a buy order.

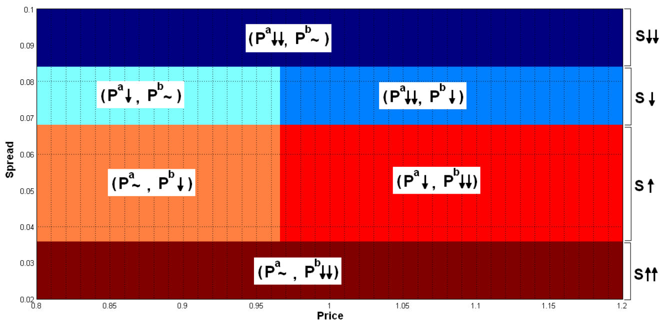

Figure 3.2: *Optimal strategy when a sell market order arrives*

Some simulated paths

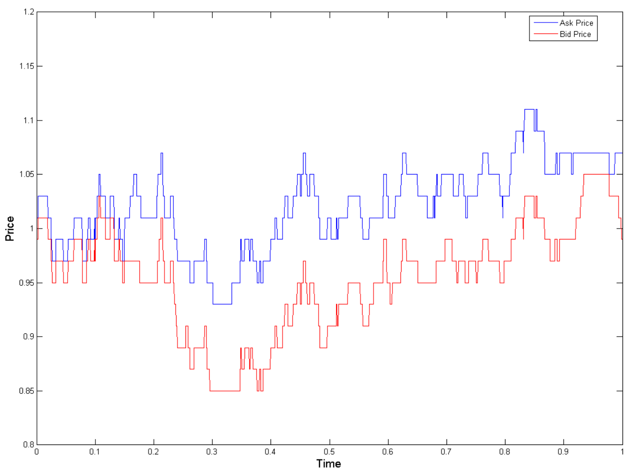

Figure 3.3: *Bid and Ask Price Paths*

Part II

STOCHASTIC CONTROL AND CORPORATE FINANCE

The theory of optimal stochastic control problem, developed in the seventies, has over the recent years once again drawn a significance of interest, especially from the applied mathematics community with the main focus on its applications in a variety of fields including economics and finance. For instance, the use of powerful tools developed in stochastic control theory has provided new approaches and sometime the first mathematical approaches in solving problems arising from corporate finance. It is mainly about finding the best optimal decision strategy for managers whose firms operate under uncertain environment whether it is financial or operational. A number of corporate finance problems have been studied, or at least revisited, with this optimal stochastic control approach. There is a vast literature on firm's investment decisions in stochastic environments, see for instance [23] and [47], [22], [45], [81], [84], [89] and [108].

In this thesis, we mainly focus on firm managerial decisions on dividend distribution policy and investment decisions. These studied problems are mathematically formulated as singular control problems, switching control problems, optimal stopping time or a combined of these control problems. We focus on the modelling side as well as the subsequent rigorous mathematical resolution. Indeed, on the modelling parts, we look at corporate finance problems by no longer assuming unrealistic financial assumptions such as the absence of corporate debt or of any other frictions. By doing so, we transform simple control problems in corporate finance into optimal stochastic control problems under different types of constraints, making therefore their resolution much more challenging tasks. This second Part of the thesis contains Chapters 4 and 5.

In Chapter 4, we are interested in two distinct problems, an optimal switching over multiple regimes problem and an optimal exit strategies for investment projects.

In the study of optimal switching control problems, a variety of problems are investigated, including problems on management of power station [27], [62], resource extraction [22], firm investment [49], marketing strategy [82], and optimal trading strategies [40], [111]. Other related works on optimal control switching problems include [13]. As part of my Ph.D. Thesis, in [A2], using viscosity techniques, we explicitly solve an optimal two-regime switching problem on infinite horizon for one-dimensional diffusions. In the above studies, only problems involving the two-regime case are investigated. There are still few studies on the multi-regime switching problems. The main additional feature in the multiple regime problems consists not only in determining the switching region as opposed to the continuation region, but also in identifying the optimal regime to where to switch. This additional feature sharply increases the complexity of the multi-regime switching problems, see [A5].

The second study in Chapter 4 deals with the problem of optimal exit strategies for an investment projet which is unprofitable. The objective is to find the optimal decision to whether wait for a buyer or liquidate the assets at immediate liquidity and termination costs. In relation to our studies, Dixit and Pindyck [47] consider various firm's decisions problems with entry, exit, suspension and/or abandonment scenarios in the case of an asset given by a geometric Brownian motion. The firm's strategy can then be described in terms of stopping times given by the time when the value of the assets hit certain threshold levels characterized as free boundaries of a variational problem. Duckworth and Zervos [49], and Lumley and Zervos [83] solve an optimal investment decision problem with switching costs in which the firm controls the production rate and must decide at which time it exits and re-enters production. In [A8], the firm, we consider, must decide between liquidating the assets of an underperforming project and waiting for the project to become once again profitable, in a setting where the liquidation costs and the value of the assets are given by general diffusion processes. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting.

In Chapter 5, we investigate a number of problems related to optimal dividend distribution policy and investment decisions, which will lead us to a variety of combined singular and switching control problems. One of the first corporate finance problems using singular stochastic control theory was the study of the optimal dividend strategy, see for instance [35], [6] and [71]. These papers focus on the study of a singular stochastic control problem arising from the research on optimal dividend policy for a firm whose cash reserve follows a stochastic process. The cash reserve may either grow when the firm makes profits or decrease when it is loss-making. The firm goes into bankruptcy when its cash reserve reaches zero. In these studies, some strong assumptions are made. The firm holds no debt and it is not possible to make any investment for future growth. Furthermore, it is clearly assumed that the firm does have the possibility to dispose of parts of its assets for some cash to avoid bankruptcy when the cash reserve approaches zero. Tackling this new issues is precisely the subject of our studies in this chapter.

In [A4], we consider a combined stochastic control problem which studies the interactions between dividend policy and investment under uncertainty. This paper consider the problem studied by Decamps and Villeneuve who investigated the problem where investment is irreversible. By relaxing the irreversible feature of the growth opportunity, the mathematical formulation of our problem becomes a combined singular/switching control problem.

In [A7], in a different setting, we consider the problem of determining an optimal control on the dividend and investment policy of a firm under debt constraints. We allowed the company to make investment by increasing its outstanding indebtedness, impacting therefore its capital structure and risk profile. The presence of a high-level of debt is a challenging constraint to any firm as it is no other than the threshold below which the firm value should never go to avoid bankruptcy. The formulation of this financial problem has led to a combined singular and multi-switching control problem under constraints. Studying such a combined control problem turns out to be a real challenge to us, especially when our objective is to provide quasi-explicit solutions to our problems.

In [S11], we no longer simplify the optimal dividend and investment problem by assuming that firm's assets are infinitely liquid. For the same reason as highlighted in financial market problems, it is necessary to take into account the liquidity constraints. More precisely, investment (for instance acquiring producing assets) and disinvestment (selling assets) should be possible but not necessarily at their fair value. The firm may have to face some liquidity costs when buying or selling assets. While taking into account liquidity constraints and costs has become the norm in recent financial markets problems, it is still not the case in the corporate finance, to the best of our knowledge, in particular in the studies of optimal dividend and investment strategies. By incorporating uncertainty into illiquid assets value, we no longer have to deal with a uni-dimensional control problem as in [A4] and [A7] but a bi-dimensional singular and multi-regime switching control problem. In such a setting, it is clear that it will be no longer possible to easily get explicit or quasi-explicit optimal strategies.

Chapter 4

Optimal switching control problem and exit strategies

This Chapter is based on

[A5]. "Optimal switching over multiple regimes", with H. Pham and X.Y. Zhou, Siam Journal on Control and Optim., 2009, 48, pp. 2217-2253.

[A8]. "Exit Optimal exit strategies for investment projects", with E. Chevalier, A. Roch and S. Scotti, 2015, Journal of Mathematical Analysis and Applications, Vol.425(2), pp.666-694.

Summary. In this chapter, we look at two different problems arising in corporate finance. This first problem deals with optimal switching problem for a general one-dimensional diffusion with multiple (more than two) regimes. This is motivated in the real options literature by the investment problem of a firm managing several production modes while facing uncertainties. A viscosity solutions approach is employed to carry out a fine analysis on the associated system of variational inequalities, leading to sharp qualitative characterizations of the switching regions. These characterizations, in turn, reduce the switching problem into one of finding a finite number of threshold values in state that would trigger switchings. In the second paper, we study the problem of an optimal exit strategy for an investment project which is unprofitable and for which the liquidation costs evolve stochastically. The firm has the option to keep the project going while waiting for a buyer, or liquidating the assets at immediate liquidity and termination costs. The liquidity and termination costs are governed by a mean-reverting stochastic process whereas the rate of arrival of buyers is governed by a regime-shifting Markov process. We formulate this problem as a multidimensional optimal stopping time problem with random maturity. We characterize the objective function and derive explicit solutions and numerical examples in the case of power and logarithmic utility functions when the liquidity premium factor follows a mean-reverting CIR process.

4.1 Optimal switching over multiple regimes

4.1.1 Introduction

Optimal multiple switching is the problem of determining an optimal sequence of stopping times for a stochastic process with several regimes (or modes).This is a classical and important problem, extensively studied. Actually, optimal switching provides a suitable model to capture the value of managerial flexibility in making decisions under uncertainty, and has been used in the pioneering works by Brennan and Schwarz [23] for resource extraction, and Dixit [46] for production facility problems.

The optimal two-regime switching problem has been the most largely studied in the literature, and is often referred to as the starting-and-stopping problem, see Brekke and Øksendal [22], Duckworth and Zervos [49], Zervos [110], and Hamadène and Jeanblanc [62], Bayraktar and Egami [13], and Guo and Tomecek [60] and Ly Vath and Pham [A2].

The applications of the starting-and-stopping problem to real options, for example the management of a power plant, are limited to the case of two modes, e.g. operating and closed. In practice, however, the efficient management of a power plant requires more than two production modes to include intermediate operating modes corresponding to different subsets of turbine running. There is little work addressing a complete treatment and mathematical resolution of the optimal multiple switching problem, especially in terms of determining the switching regions. The difficulty with a multi-regime problem in the determination of the switching regions is evident: In sharp contrast with the two-regime problem, a multiple switching problem needs to decide not only when to switch, but also *where* to switch. Djehiche, Hamadène and Popier [48], and Hu and Tang [67] have studied optimal multiple switching problems for general adapted processes by means of reflected BSDEs, and they are mainly concerned with the existence and uniqueness of solution to these reflected BSDEs. However, the important issue as to which regime to optimally switch has been left completely open.

In this paper, we consider the optimal multiple switching problem on infinite horizon for a general one-dimensional diffusion. The multiple regimes are differentiated via their profit functions, which are of very general form. The numbering of the regimes is ordered by increasing level of profitability. The transition from one regime to another one is realized sequentially at random times (which are part of the decisions), and incurs a fixed cost. Our objective is to provide an explicit characterization of the switching regions showing when and where it is optimal to change the regime. We adopt a direct solution method via the viscosity solutions technique. By carrying out a detailed analysis of the system of variational inequalities associated with the optimal switching problem, we give a precise and sharp qualitative description of the switching regions. Specifically, we give conditions under which one should switch to a regime with higher profit, and to a regime with lower profit, and we identify these destination regimes. This extends the results of [A2] for the two-regime case, to the multiple regime case. The switching regions take various structures, depending on model parameters via explicit conditions, which have meaningful economic interpretations. It appears that in some situations, it is optimal to switch to a regime for a range of state values, and to switch to another regime for a different set of state values. Such a feature is new with respect to the two-regime case, where the choice of a destination regime is not an issue. We showcase our general results by the three-regime case, where we present a complete picture of the situations as to when and where it is optimal to switch, and we reduce the problem into one of finding a finite number of threshold values of the switching regions. We also design an algorithm to compute these critical values based on the computations of expectation functionals of hitting times for one-dimensional diffusions.

4.1.2 Model and problem formulation

We present our general model and emphasize the key assumptions. The state process X is a one-dimensional diffusion on $(0,\infty)$ whose dynamics are given by :

$$dX_t \quad = \quad b(X_t)dt + \sigma(X_t)dW_t, \tag{4.1.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the *usual* conditions, and b, σ are measurable functions on $(0,\infty)$. We assume that the SDE (4.1.1) has a unique strong solution, denoted by X^x, given an initial condition $X_0 = x \in (0,\infty)$.

Throughout the paper, we denote by $\mathcal{L}$ the infinitesimal generator of the diffusion X, i.e.

$$\mathcal{L}\varphi(x) \quad = \quad b(x)\varphi'(x) + \frac{1}{2}\sigma^2(x)\varphi''(x).$$

The operational regimes are characterized by their running reward functions $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i \in \mathbb{I}_d = \{1,\ldots,d\}$. We assume that for each $i \in \mathbb{I}_d$, the function f_i is nonnegative, without loss of generality (w.l.o.g.) $f_i(0) = 0$, continuous, and satisfies the linear growth condition:

$$f_i(x) \quad \leq \quad C(1+|x|), \quad \forall x \in \mathbb{R}_+, \tag{4.1.2}$$

for some positive constant C. The numbering $i = 1,\ldots,d$, on the regimes is ordered by increasing level of profitability, which roughly means that the sequence of functions f_i is increasing in i.

Switching from regime i to j incurs an instantaneous cost, denoted by g_{ij}, with the convention $g_{ii} = 0$ and the following triangular condition:

$$g_{ik} \quad < \quad g_{ij} + g_{jk}, \quad j \neq i,k, \tag{4.1.3}$$

4.1.3 The optimal switching problem

A decision (strategy) for the operator is an impulse control α consisting of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \iota_1, \ldots, \iota_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where $\tau_n \in \mathbb{N}^*$, are $\mathbb{F}$-stopping times in $[0, \infty]$, denoted by $\tau_n \in \mathcal{T}$, $\tau_n < \tau_{n+1}$ and $\tau_n \to \infty$ a.s., representing the decision on "when to switch", and ι_n are $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$, representing the new value of the regime at time τ_n until time τ_{n+1} or the decision on "where to switch". We denote by $\mathcal{A}$ the set of all such impulse controls. Given an initial regime value $i \in \mathbb{I}_d$, and a control $\alpha = (\tau_n, \iota_n)_{n \geq 1} \in \mathcal{A}$, we denote

$$I_t^i \;=\; \sum_{n \geq 0} \iota_n 1_{[\tau_n, \tau_{n+1})}(t), \quad t \geq 0, \quad I_{0^-}^i \;=\; i.$$

The objective is to maximize this expected total profit over $\mathcal{A}$. Accordingly, we define the value functions

$$v_i(x) \;=\; \sup_{\alpha \in \mathcal{A}} E\left[\int_0^\infty e^{-rt} f_{I_t^i}(X_t^x)dt - \sum_{n=1}^\infty e^{-r\tau_n} g_{\iota_{n-1}, \iota_n}\right], \quad x > 0, \; i \in \mathbb{I}_d. \quad (4.1.4)$$

4.1.4 Dynamic programming PDE characterization

We obtain PDE characterizations of the value functions by using the dynamic programming approach. To do so, we first obtain the linear growth property and the boundary condition on the value functions.

Theorem 4.1.1. *The value functions v_i, $i \in \mathbb{I}_d$, are the unique viscosity solutions to the system of variational inequalities :*

$$\min\left\{rv_i - \mathcal{L}_i v_i - f_i \,,\; v_i - \max_{j \neq i}(v_j - g_{ij})\right\} \;=\; 0, \quad x \in (0, \infty), \quad i \in \mathbb{I}_d, \quad (4.1.5)$$

in the following sense:
(1) Viscosity property . For each $i \in \mathbb{I}_d$, v_i is a viscosity solution to

$$\min\left\{rv_i - \mathcal{L}_i v_i - f_i \,,\; v_i - \max_{j \neq i}(v_j - g_{ij})\right\} \;=\; 0, \quad x \in (0, \infty). \quad (4.1.6)$$

(2) Uniqueness property . If w_i, $i \in \mathbb{I}_d$, are viscosity solutions with linear growth conditions on $(0, \infty)$ and boundary conditions $w_i(0^+) = \max_{j \in \mathbb{I}_d}[-g_{ij}]$ to the system of variational inequalities (4.1.5) *, then $v_i = w_i$ on $(0, \infty)$.*

We also quote the useful smooth fit property on the value functions, proved in [95].

Theorem 4.1.2. *For all $i \in \mathbb{I}_d$, the value function v_i is continuously differentiable on $(0, \infty)$.*

For any regime $i \in \mathbb{I}_d$, we introduce the switching region :

$$\mathcal{S}_i = \left\{ x \in (0,\infty) \ : v_i(x) = \max_{j \neq i}(v_j - g_{ij})(x) \right\}. \tag{4.1.7}$$

$\mathcal{S}_i$ is a closed subset of $(0,\infty)$ and corresponds to the region where it is optimal for the controller to change regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in $(0,\infty)$ is the so-called continuation region :

$$\mathcal{C}_i = \left\{ x \in (0,\infty) \ : v_i(x) > \max_{j \neq i}(v_j - g_{ij})(x) \right\},$$

where it is optimal to stay in regime i. In this open domain, the value function v_i is smooth (C^2 on $\mathcal{C}_i$) and satisfies in a classical sense :

$$rv_i(x) - \mathcal{L}_i v_i(x) - f_i(x) = 0, \quad x \in \mathcal{C}_i.$$

4.1.5 Qualitative properties of the switching regions

In this section, we focus on the qualitative aspects of deriving the solution to the switching problem. Basically, we raise the following questions : When and where does one switch?

From the definition (4.1.7) of the switching regions, we have the elementary decomposition property :

$$\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}, \quad i \in \mathbb{I}_d,$$

where

$$\mathcal{S}_{ij} = \{x \in (0,\infty) \ : v_i(x) = (v_j - g_{ij})(x)\}$$

is the switching region from regime i to regime j.

We now formalize the difference between the operational regimes. We consider the following ordering conditions on the regimes through their reward functions :

$$(\mathbf{Hf}) \qquad f_1 \prec f_2 \prec \ldots \prec f_d$$

$\Longleftrightarrow$

for all $i < j \in \mathbb{I}_d$, $f_j - f_i$ is strictly decreasing on $(0, \hat{x}_{ij})$ and strictly increasing on $(\hat{x}_{ij}, \infty)$ for some $\hat{x}_{ij} \in \mathbb{R}_+$.

Economically speaking, the ordering condition $f_i \prec f_j$ means that the profit in regime $j > i$ is "better" than profit in regime i from a certain level.

Analysis of upward switching region

The main results of this section provide a qualitative description of the upward switching regions.

Proposition 4.1.1. *Let* $i \in \mathbb{I}_d$.
1) The switching region $\mathcal{S}_i^+$ *is nonempty if and only if :*

$$\cup_{j>i} Q_{ij} \; \neq \; \emptyset \quad \Longleftrightarrow \quad \exists j > i, \quad (f_j - f_i)(\infty) \; > \; r g_{ij}. \tag{4.1.8}$$

2) Suppose $\mathcal{S}_i^+ \neq \emptyset$. *Then there exists a unique* $j = j^+(i) > i$ *such that* $\sup \mathcal{S}_i^+ = \sup \mathcal{S}_{ij} = \infty$, *and we have* $\sup \mathcal{S}_{ik} < \infty$ *for all* $k > i$, $k \neq j^+(i)$. *Moreover,* $\mathcal{S}_{ij^+(i)}$ *contains an interval in the form* $[x_{ij^+(i)}, \infty)$ *for some* $x_{ij^+(i)} \in (0, \infty)$, *and* $j^+(i) = \min J(i)$ *where*

$$J(i) \quad = \quad \Big\{ j \in \mathbb{I}_d, \; j > i \; : (f_k - f_j)(\infty) \leq r(g_{ik} - g_{ij}), \; \forall k \in \mathbb{I}_d, \; k > i \Big\}. \tag{4.1.9}$$

Economic interpretation. The first assertion gives explicit necessary and sufficient conditions under which, in a given regime, it is optimal to switch up. The second assertion says that in a given regime, say i, where one has interest to switch up (under the conditions of assertion 1), there is a unique regime where one should switch up to when the state is sufficiently large. Moreover, this uniquely chosen regime is explicitly determined as the minimum of the explicitly given set $J(i)$. In the two-regime case, i.e. $d = 2$, we obviously have $j^+(1) = 2$. In the multi-regime case, it can be practically calculated.

Proposition 4.1.2. *Let* $i \in \{1, \ldots, d-1\}$ *with* $\mathcal{S}_i^+ \neq \emptyset$.
1) Suppose that

$$\sup \Big[\mathcal{S}_{ik} \setminus \mathcal{S}_{ij^+(i)} \Big] \quad \leq \quad \inf \mathcal{S}_{ij^+(i)}, \quad \forall k \neq i, j^+(i). \tag{4.1.10}$$

Then, we have

$$\mathcal{S}_{ij^+(i)} \quad = \quad [\bar{x}_{ij^+(i)}, \infty),$$

with $\bar{x}_{ij^+(i)} \in (0, \infty)$.
2) Suppose that there exists $k > i$, $k \neq j^+(i)$ *such that* $\mathcal{S}_{ik}$ *is nonempty and*

$$\sup \mathcal{S}_{ik} \quad \leq \quad \inf \mathcal{S}_{ij}, \quad \forall j \neq i, k. \tag{4.1.11}$$

Then, $\mathcal{S}_{ik}$ *is in the form*

$$\mathcal{S}_{ik} \quad = \quad [\bar{x}_{ik}, \bar{y}_{ik}],$$

with $0 < \bar{x}_{ik} \leq \bar{y}_{ik} < \infty$.

Analysis of downward switching region

The main results of this paragraph provide a qualitative description of the downward switching regions.

Proposition 4.1.3. *For all* $i = 2, \ldots, d$, *the switching region* $\mathcal{S}_i^-$ *is nonempty. Moreover,* $\inf \mathcal{S}_{i1} = 0$, $\mathcal{S}_{i1}$ *contains some interval in the form* $(0, y_{i1}]$, $y_{i1} > 0$, *and* $\inf \mathcal{S}_{ij} > 0$ *for all* $1 < j < i$.

Economic interpretation. The first assertion means that one always has interest to switch down due to the negative switching costs. Moreover, for small values of the state, one should switch down to the lowest regime $i = 1$. This is intuitively justified by the fact that for small values of the state, the running profits are close to zero, and so one chooses the regime with the largest compensation fee, i.e. regime 1.

4.1.6 The three-regime case

In the following results, we summarize our findings on the qualitative structure of the switching regions in the three-regime model.

Theorem 4.1.1. *(Switching regions in the three-regime model)*
We have the following four cases :
A) If $(f_2 - f_1)(\infty) \leq rg_{12}$, $(f_3 - f_1)(\infty) \leq rg_{13}$, *and* $(f_3 - f_2)(\infty) \leq rg_{23}$, *then*

$$\begin{aligned} \mathcal{S}_1 = \mathcal{S}_1^+ &= \emptyset, \\ \mathcal{S}_2^- &= (0, \underline{y}_{21}], \quad \mathcal{S}_2^+ = \emptyset, \\ \mathcal{S}_{31} &= (0, \underline{y}_{31}], \quad \mathcal{S}_{32} \ \textit{is either empty or} \ \mathcal{S}_{32} = [\underline{x}_{32}, \underline{y}_{32}] \end{aligned}$$

for some $0 < \underline{y}_{31} \leq \underline{x}_{32} \leq \underline{y}_{32} < \infty$, $0 < \underline{y}_{21} < \underline{x}_{32}$.
B) If $(f_2 - f_1)(\infty) > rg_{12}$ *or* $(f_3 - f_1)(\infty) > rg_{13}$, *and* $(f_3 - f_2)(\infty) \leq r(g_{13} - g_{12})$, *then*

$$\begin{aligned} \mathcal{S}_{12} &= [\bar{x}_{12}, \infty), \quad \mathcal{S}_{13} = \emptyset, \\ \mathcal{S}_2^- &= (0, \underline{y}_{21}], \quad \mathcal{S}_2^+ = \emptyset, \\ \mathcal{S}_{31} &= (0, \underline{y}_{31}], \quad \mathcal{S}_{32} \ \textit{is either empty or} \ \mathcal{S}_{32} = [\underline{x}_{32}, \underline{y}_{32}] \end{aligned}$$

for some $0 < \underline{y}_{21} \leq \bar{x}_{12}$, $0 < \underline{y}_{31} \leq \underline{x}_{32} \leq \underline{y}_{32} < \infty$, $\underline{y}_{31} < \bar{x}_{12} < \infty$, $0 < \underline{y}_{21} < \underline{x}_{32}$.
C) If $(f_3 - f_1)(\infty) > rg_{13}$, *and* $r(g_{13} - g_{12}) < (f_3 - f_2)(\infty) \leq rg_{23}$, *then*

$$\begin{aligned} \mathcal{S}_{13} &= [\bar{x}_{13}, \infty), \quad \mathcal{S}_{12} \ \textit{is either empty or} \ \mathcal{S}_{12} = [\bar{x}_{12}, \bar{y}_{12}] \\ \mathcal{S}_2^- &= (0, \underline{y}_{21}], \quad \mathcal{S}_2^+ = \emptyset, \\ \mathcal{S}_{31} &= (0, \underline{y}_{31}], \quad \mathcal{S}_{32} \ \textit{is either empty or} \ \mathcal{S}_{32} = [\underline{x}_{32}, \underline{y}_{32}], \end{aligned}$$

for some $0 < \bar{x}_{12} \leq \bar{y}_{12} \leq \bar{x}_{13} < \infty$, $0 < \underline{y}_{31} \leq \underline{x}_{32} \leq \underline{y}_{32} < \infty$, $0 < \underline{y}_{21} < \bar{x}_{12}$, $\underline{y}_{31} < \underline{x}_{13}$.
D) If $(f_3 - f_1)(\infty) > rg_{13}$, *and* $(f_3 - f_2)(\infty) > rg_{23}$, *then*

$$\begin{aligned} \mathcal{S}_{13} &= [\bar{x}_{13}, \infty), \quad \mathcal{S}_{12} \ \textit{is either empty or} \ \mathcal{S}_{12} = [\bar{x}_{12}, \bar{y}_{12}] \\ \mathcal{S}_2^- &= (0, \underline{y}_{21}], \quad \mathcal{S}_2^+ = [\bar{x}_{23}, \infty), \\ \mathcal{S}_{31} &= (0, \underline{y}_{31}], \quad \mathcal{S}_{32} \ \textit{is either empty or} \ \mathcal{S}_{32} = [\underline{x}_{32}, \underline{y}_{32}], \end{aligned}$$

for some $0 < \bar{x}_{12} \leq \bar{y}_{12} \leq \bar{x}_{13} < \infty$, $0 < \underline{y}_{31} \leq \underline{x}_{32} \leq \underline{y}_{32} < \bar{x}_{23} < \infty$, $0 < \underline{y}_{21} < \bar{x}_{12}$, $\bar{y}_{12} < \bar{x}_{23}$, $\underline{y}_{31} < \underline{x}_{13}$.

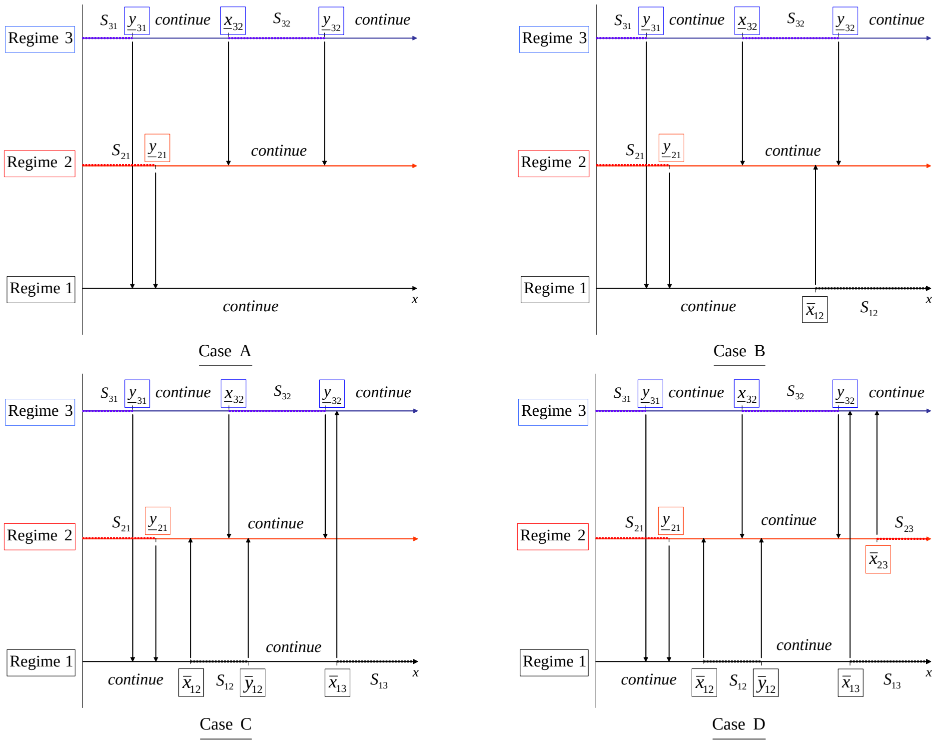

Figure 4.1: Switching regions in the three-regime model

4.1.7 Numerical procedure

The qualitative structure of optimal switching controls derived in the previous section states that the optimal sequence of stopping times is given by the hitting times of the diffusion process X at a finite number of threshold levels. This is of vital importance in eventually solving (either analytically or numerically) our problem, because it reduces the originally very complex problem into one finding a small number of critical values in state which is a finite-dimensional optimization problem. In this paper, we demonstrate how to design algorithms to find these critical values in state.

4.2 Optimal exit strategies for an investment project

4.2.1 Introduction

There is often a time when a firm is engaged in a project that does not produce to its full potential and faces the difficult dilemma of shutting it down or keeping it alive in the hope that it will become profitable once again. When an investment in not totally irreversible, assets can be sold at their scrap value minus some liquidation and project termination costs, which may include for example termination pay to workers, legal fees and a liquidity premium in the case of fire sale of the assets. Since these closing costs may be substantial, it may be worthwhile to wait for the project to be profitable again or to wait for an interested buyer that will pay the fair value of the assets and put them to better use. In this study, we give an analytical solution to this problem when the liquidation costs and the value of the assets are diffusion processes and the arrival time of a buyer is modeled by means of an intensity function depending on the current state of a Markov chain.

The firm, we consider, in this paper, must decide between liquidating the assets of an underperforming project and waiting for the project to become once again profitable, in a setting where the liquidation costs and the value of the assets are given by general diffusion processes. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting.

Amongst the large literature on optimal stopping problems, we may refer to some related works including Bouchard, El Karoui and Touzi [18], Carr [28], Dayanik and Egami [43], Dayanik and Karatzas [44], Guo and Zhang [61], Lamberton and Zervos [78]. In [44] and [78], the authors study optimal stopping problems with general 1-dimensional processes. Random maturity in optimal stopping problem was introduced in [28] and [18]. It allowed to reduce the dimension of their problems as well as addressing the numerical issues. We may refer to Dayanik and Egami [43] for a recent paper on optimal stopping time and random maturity. For optimal stopping problem with regime shifting, we may refer to Guo and Zhang [61], where an explicit optimal stopping rule and the corresponding value function in a closed form are obtained.

In this paper, our optimal stopping problem combines all the above features, i.e., random maturity and regime shifting, in the bi-dimensional framework. We are able to characterize the value function of our problem and provide explicit solution in some particular cases where we manage to reduce the dimension of our control problem.

In the general bi-dimensional framework, the main difficulty is related to the proof of the continuity property and the PDE characterization of the value function. Since it is not possible to get the smooth-fit property, the PDE characterization may be obtained only via the viscosity approach. To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.

In the particular cases where it is possible to reduce our problem to a one-dimensional problem, we are able to provide explicit solution. Our reduced one-dimensional problem is highly related to previous studies in the literature, see for instance Zervos, Johnson and Alezemi [111] and Leung, Li and Wang [79].

4.2.2 The Investment Project

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times. Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation

ρ, i.e. $d[W,B]_t = \rho dt$ for all t.

We consider a firm which owns assets that are currently locked up in an investment project which currently produces no output per unit of time. Because the firm is currently not using the assets at its full potential, it considers two possibilities. The first is to liquidate the assets in a fire sale and recover any remaining value. The cash flow obtained in the latter case is the fair value of the assets minus liquidation and project termination costs. We denote by θ the moment at which the firm decides to take this option. The second option is to wait for the project to become profitable once again, or equivalently, to wait for an investor or another firm who will purchase the assets as a whole at their fair value S_τ where τ is the moment when this happens.

The fair value of the assets are given by $S = \exp(X)$, in which

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0 \\ X_0 &= x. \end{aligned} \tag{4.2.1}$$

Assume that μ and σ are Lipschitz functions on $\mathbb{R}$ satisfying some growth conditions.

Liquidation and Termination Costs. We model the liquidation cost of the assets and terminal costs of the project as a given process $(f(Y_t))_{t\geq 0}$, where f is strictly decreasing C^2 function defined on $\mathbb{R}^+ \to [0,1]$, and satisfies some conditions.

Unlike the value of financial assets, it is natural to model liquidation costs with mean-reverting properties. As such, the costs, given by $f(Y_t)$ at time t, is defined in terms of the mean-reverting non-negative process Y which is governed by the following SDE:

$$\begin{aligned} dY_t &= \alpha(Y_t)dt + \gamma(Y_t)dW_t, \\ Y_0 &= y, \end{aligned} \tag{4.2.2}$$

where α is a Lipschitz function on $\mathbb{R}^+$ and, for any $\varepsilon > 0$, γ is a Lipschitz function on $[\varepsilon, \infty)$. We assume that α and γ satisfy linear growth conditions.

The recovery time. We model the arrival time of a buyer, denoted by τ, or equivalently the time when the project becomes profitable again, by means of an intensity function λ_i depending on the current state i of a continuous-time, time-homogenous, irreducible Markov chain L, independent of W and B, with $m+1$ states. The generator of the chain L under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=0,\dots m}$. Without loss of generality we assume $\lambda_0 > \lambda_1 > \ldots > \lambda_m > 0$.

Utility function. We let U denote the utility function of the firm. We assume that U satisfies the following assumptions.

Assumption 4.2.1. *$U : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing, concave and twice continuously differentiable, and satisfies*

$$\lim_{x\to 0} x\, U'(x) \quad < \quad +\infty. \tag{4.2.3}$$

Assumption 4.2.2. *U is supermeanvalued w.r.t. S, i.e.*

$$U(S_t) \geq \mathbb{E}[U(S_\theta)|\mathcal{F}_t] \tag{4.2.4}$$

for any stopping time $\theta \in \mathcal{T}$.

Objective function. The objective of the firm is to maximize the expected profit obtained from the sale of the illiquid asset, either through liquidation or at its fair value at the exogenously given stopping time τ. As such, we consider the following value function:

$$v(i,x,y) := \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\,\mathbf{1}_{\theta \leq \tau} + U(e^{X_\tau})\,\mathbf{1}_{\theta > \tau}\right], \quad x \in \mathbb{R},\, y \in \mathbb{R}^+,\, i \in \{0,\ldots,m\} \tag{4.2.5}$$

where $\mathbb{E}^{i,x,y}$ stands for the expectation with initial conditions $X_0 = x$, $Y_0 = y$ and $L_0 = i$, and $h(x,y) = U(e^x f(y))$. Recall that τ is defined through the Markov chain L.

4.2.3 Characterization of the value function

Before characterizing the value funcion, We first prove the continuity of the value function, which has two main difficulties that need a no-standard treatment. The first one comes from the SDE satisfied by Y (4.2.2) since we do not assume the standard hypothesis of Lipschitz coefficients. We overcome this drawback by showing that the local Lipschitz property is satisfied until the smallest optimal exit time from the investment. The second difficulty is related to the bi-dimensional setting where the classical arguments used to show the regularity of value function are not longer available. We then need to show the continuity in term of limits of sequences and to distinguish different sub-sequences with ad-hoc proofs.

The complexity of the proof of the continuity suggests that a direct proof of differentiability, i.e. smooth-fit property, of the value function is probably out of reach in our setting. We will then turn to the viscosity characterization approach to overcome the impossibility to use a verification approach.

Theorem 4.2.2. *The value functions v_i, $i \in \{0,\ldots,m\}$, are continuous on $\mathbb{R} \times \mathbb{R}^+$, and constitute the unique viscosity solution on $\mathbb{R} \times \mathbb{R}^+$ with growth condition*

$$|v_i(x,y)| \leq |U(e^x)| + |U(e^x)f(y)|,$$

and boundary condition

$$\lim_{y \downarrow 0} v_i(x,y) = U(e^x),$$

to the system of variational inequalities :

$$\min\Big[-\mathcal{L}v_i(x,y) - \mathcal{G}_i v_\cdot(x,y) - \mathcal{J}_i v_i(x,y)\,,\, v_i(x,y) - U(e^x f(y))\Big] = 0, \quad \forall\,(x,y) \in \times\mathbb{R} \times \mathbb{R}^+_*, \text{ and } i \in \{0,\ldots,n\}, \tag{4.2.6}$$

in which $\mathcal{L}$ is the second order differential operator associated to the state processes (X,Y) and $\mathcal{G}_i$ and $\mathcal{J}_i$ are defined as

$$
\begin{aligned}
\mathcal{G}_i\varphi(.,x,y) &= \sum_{j\neq i}\vartheta_{i,j}\left(\varphi(j,x,\,y)-\varphi(i,x,\,y)\right)\\
\mathcal{J}_i\varphi(i,x,y) &= \lambda_i\left(e^x-\varphi(i,x,\,y)\right).
\end{aligned}
$$

Remark 4.2.1. *To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.*

4.2.4 Liquidation and continuation regions

We now prove useful qualitative properties of the liquidation regions of the optimal stopping problem. We introduce the following liquidation and continuation regions:

$$
\begin{aligned}
\mathcal{LR} &= \left\{(i,x,y)\in\{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+\,|\,v(i,x,y)=h(x,y)\right\}\\
\mathcal{CR} &= \{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+\setminus\mathcal{LR}.
\end{aligned}
$$

Clearly, outside the liquidation region $\mathcal{LR}$, it is never optimal to liquidate the assets at the available discounted value. Moreover, the smallest optimal stopping time θ^*_{ixy} verifies

$$\theta^*_{ixy}=\inf\left\{u\geq 0\,|\,\left(L^i_u,X^x_u,Y^y_u\right)\in\mathcal{LR}\right\}.$$

We define the $(i,x)-$sections for every $(i,x)\in\{0,...,m\}\times\mathbb{R}$ by

$$\mathcal{LR}_{(i,x)}=\{y\geq 0\,|\,v(i,x,y)=h(x,y)\}\ \text{ and } \mathcal{CR}_{(i,x)}=\mathbb{R}^+\setminus\mathcal{LR}_{(i,x)}.$$

Proposition 4.2.1 (Properties of liquidation region)**.**
i) $\mathcal{E}$ is closed in $\{0,...,m\}\times\mathbb{R}\times(0,+\infty)$,
ii) Let $(i,x)\in\{0,...,m\}\times\mathbb{R}$.
- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})]=U(e^x)$, then, for all $y\in\mathbb{R}^+$, $v(i,x,y)=U(e^x)$ and $\mathcal{E}_{(i,x)}=\{0\}$.*
- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})]<U(e^x)$, then there exists $x_0\in\mathbb{R}$ such that $\mathcal{E}_{(i,x_0)}\setminus\{0\}\neq\emptyset$ and $\bar{y}^*(i,x):=\sup\mathcal{E}_{(i,x)}<+\infty$.*

4.2.5 Logarithmic utility

Throughout this section, we assume that the diffusion processes X and Y are governed by the following SDE, which are particular cases of (4.2.1) and (4.2.2)

$$
\begin{aligned}
dX_t &= \mu dt+\sigma(X_t)dB_t;\ X_0=x &\quad (4.2.7)\\
dY_t &= \kappa\left(\beta-Y_t\right)dt+\gamma\sqrt{Y_t}dW_t;\ Y_0=y. &\quad (4.2.8)
\end{aligned}
$$

The following theorem shows that in the logarithmic case, we can reduce the dimension of the problem by factoring out the x-variable. For this purpose, we define $\mathcal{T}_{L,W}$ the set of stopping times with respect to the filtration generated by (L, W), and the differential operator $\overline{\mathcal{L}}\phi(y) := \frac{1}{2}\gamma^2 y \frac{\partial^2\phi}{\partial y^2} + \kappa(\beta - y)\frac{\partial\phi}{\partial y} + \mu$, for $\phi \in C^2(\mathbb{R}^+)$.

Theorem 4.2.3. *For $(i, y) \in \{1, ..., m\} \times \mathbb{R}^+$ we define the function:*

$$w(i, y) = \sup_{\theta \in \mathcal{T}_{L,W}} \mathbb{E}^{i,y}[\mu(\theta \wedge \tau) + \ln(f(Y_\theta))\, 1\!\!1_{\{\theta \le \tau\}}].$$

Then,

$$v(i, x, y) = x + w(i, y) \ \text{ on } \{0, ..., m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with w the unique viscosity solution to the system of equations:

$$\min\Big[-\overline{\mathcal{L}}w(i, y) + \lambda_i w(i, y) - \sum_{j \ne i} \vartheta_{i,j}\,(w(j,\, y) - w(i,\, y))\ ,\ w(i, y) - g(y)\Big] = 0, \quad (4.2.9)$$

where $g(y) := \ln(f(y))$ Moreover, the functions $w(i, .)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{C}_{(i,x)} \cup \mathrm{Int}(\mathcal{E}_{(i,x)})$.

4.2.6 Liquidation region

In the logarithmic case, the liquidation region can be characterized in more details.

Proposition 4.2.2. *Let $i \in \{0, ..., m\}$ and set*

$$\hat{y}_i = \inf\{y \ge 0 :\ \mathcal{H}_i g(y) \ge 0\} \ \text{ with } \mathcal{H}_i g(y) = \overline{\mathcal{L}}g(y) - \lambda_i g(y) + \sum_{j \ne i} \vartheta_{i,j}\,(w(j,\, y) - g(y))\,.$$

There exists $y_i^* \ge 0$ such that $[0, y_i^*] = \mathcal{LR}_{(i,.)} \cap [0, \hat{y}_i]$. Moreover, $w(i, \cdot) - g(\cdot)$ is non-decreasing on $[y_i^*, \hat{y}_i]$.

Proposition 4.2.3. *Assume that the function $y \mapsto \overline{\mathcal{L}}g(y)$ is non-decreasing on $\mathbb{R}^+$, then for all $i \in \{0, ..., m\}$, $w(i, \cdot) - g(\cdot)$ is non-decreasing on $\mathbb{R}^+$ and we have $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$, with $y_i^* > 0$.*

4.2.7 Explicit solutions in Logarithmic utility in the two regime case

From the above results, we may get completely explicit solution in the two-regime case. In particular, the value function may be written in terms of the confluent hypergeometric functions.

Proposition 4.2.4. *The function w is given by*

$$w(0,y) = \begin{cases} g(y) & y \in [0, y_0^*] \\ \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) & y \in (y_0^*, y_1^*] \\ \quad +\mathcal{I}\left(\frac{2\kappa}{\gamma^2}, \beta, -2\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}, 2\frac{\vartheta_{0,1}g(\cdot)+\mu}{\gamma^2}\right)(y) & \\ p_0^0\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}x\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad +p_1^0\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}x\right) + \frac{\mu}{\widetilde{\lambda}_1}\right] & \end{cases} \tag{4.2.10}$$

$$w(1,y) = \begin{cases} g(y) & y \in [0, y_1^*] \\ p_0^1\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad +p_1^1\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_1}\right], & \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind, and $\mathcal{I}$ is a particular solution to the non-homogeneous confluent differential equation. Moreover, $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $w(0,y)$ and $w(1,y)$ belong to $C^1(\mathbb{R}^+)$.

Numerical Simulation

In Figure 4.2, we represent the value functions in the two-regime case, for the cases $\mu = -.05$ and $\mu = -0.3$. Other numerical results, in particular sensitivity analysis for the parameters μ, λ, β, and $\vartheta_{0,1}$ are equally obtained.

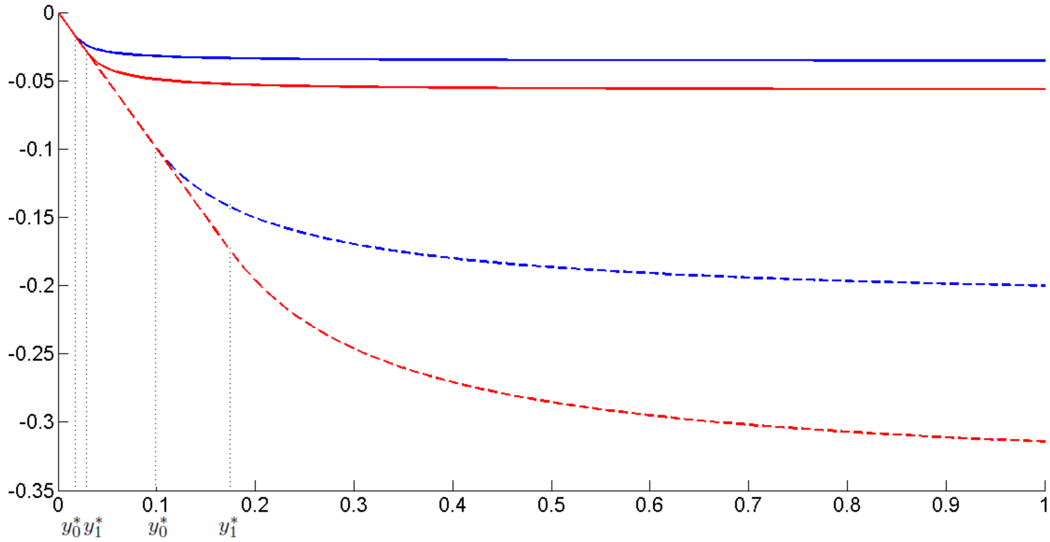

Figure 4.2: Value functions in the two-regime case, for the cases $\mu = -0.05$ (solid line) and $\mu = -0.3$ (dashed line). Regime 0 is presented in blue and regime 1 in red. The parameters used are $\lambda_0 = 2, \lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 1, \kappa = 1, \beta = 0.25, \gamma = 0.5$. The liquidation region are indicated by dashed lines. In the case $\mu = -0.5$, $y_0^* = 0.0172$ and $y_1^* = 0.0288$. In the case $\mu = -0.3$, $y_0^* = 0.0983$ and $y_1^* = 0.1742$.

Chapter 5

Optimal Dividend and investment strategies under constraints

This Chapter is based on
[A4]. "A mixed singular/switching control problem for a dividend policy with reversible technology investment", with H. Pham et S. Villeneuve, Annals of Applied Probability, 2008, 18, pp. 1164-1200.
[A7]. "An optimal dividend and investment control problem under debt constraints", with E. Chevalier and S. Scotti, 2013, SIAM J. Finan. Math., 4(1), 297 - 326.
[A11]. "Liquidity risk and optimal dividend/investment strategies", with E. Chevalier and M. Gaigi, submitted.

Summary. we investigate a number of problems related to optimal dividend distribution policy and investment decisions, which will lead us to a variety of combined singular and switching control problems. In [A4], we consider a combined stochastic control problem which studies the interactions between dividend policy and investment under uncertainty. In [A7], we introduce debt constraints in our optimal strategies while in [A11], we no longer simplify the optimal dividend and investment problem by assuming that firm's assets are infinitely liquid. The formulation of these financial problem under constraints has led to combined singular and multi-switching control problem under constraints both unidimensional and bi-dimensional settings. Studying such a combined control problem turns out to be a real challenge to us, especially when our objective is to provide quasi-explicit solutions to our problems.

5.1 Introduction

The first and natural dividend control problem was studied by [71]. They consider a firm whose cash reserve follows a drift brownian motion as follows:

$$dX_t \;=\; \mu dt + \sigma dW_t - dZ_t,\; X_{0-} = x.$$

The objective is to find the best dividend policy which maximizes shareholder's value:

$$\hat{V}_0(x) \;=\; \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t\right], \tag{5.1.1}$$

where $T_0 = \inf\{t \geq 0 \ : X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 0.

It is known that $\hat{V}_0$, as the value function of a pure singular control problem, is characterized as the unique continuous viscosity solution on $(0, \infty)$, with linear growth condition to the variational inequality :

$$\min\left[\rho\hat{V}_0 - \mathcal{L}_0\hat{V}_0 \;,\; \hat{V}_0' - 1\right] \;=\; 0, \quad x > 0, \tag{5.1.2}$$

and boundary data

$$\hat{V}_0(0) \;=\; 0.$$

Actually, $\hat{V}_0$ is C^2 on $(0, \infty)$ and explicit computations of this standard singular control problem are developed in Shreve, Lehoczky and Gaver [100], Jeanblanc and Shiryaev [71], or Radner and Shepp [96] :

$$\hat{V}_0(x) \;=\; \begin{cases} \frac{f_0(x)}{f_0'(\hat{x}_0)} & , 0 \leq x \leq \hat{x}_0 \\ x - \hat{x}_0 + \frac{\mu_0}{\rho}, & x \geq \hat{x}_0, \end{cases}$$

where

$$f_0(x) \;=\; e^{m_0^+ x} - e^{m_0^- x}, \quad \hat{x}_0 \;=\; \frac{1}{m_0^+ - m_0^-} \ln\left(\frac{(m_0^+)^2}{(m_0^-)^2}\right),$$

and $m_0^- < 0 < m_0^+$ are roots of the characteristic equation :

$$\rho - \mu_0 m - \frac{1}{2}\sigma^2 m^2 \;=\; 0.$$

In other words, this means that the optimal cash reserve process is given by the reflected diffusion process at the threshold $\hat{x}_0$ with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq \hat{x}_0$, the optimal dividend policy is to distribute immediately the amount $x - \hat{x}_0$ and then follows the dividend policy characterized by the local time.

In this section, the objective is to address related problems when we incorporate the following aspects:

- investment problems: the interaction between dividend policy and investment policy. By investment, we mean the ability of the firm to allow the company to capture growth opportunity which it self-finances on its cash reserve.

- investment under debt constraints: the firm is allowed to make investment and finance it through debt issuance/raising, which in turn would impact its capital structure and risk profile.

- dividend and investment policy under liquidity risk: the firm is allowed to make investment decisions by acquiring or selling productive assets. But we no longer assume that firm assets are either infinitely illiquid or liquid.

The formulation of these financial problems has led to different combined singular and multi-switching control problems under constraints, which turn out to be real challenges to us, especially when our objective is to provide quasi-explicit solutions to our problems.

5.2 A mixed singular/switching control problem for a dividend policy with reversible technology investment

In this paper, we consider a combined stochastic control problem that has emerged in a recent paper by Décamps and Villeneuve [45] with the study of the interactions between dividend policy and investment under uncertainty. We shall consider a firm with a technology in place that has the opportunity to invest in a new technology that increases its profitability. The firm self-finances the opportunity cost on its cash reserve. Once installed, the manager can decide to return back to the old technology by receiving some cash compensation. The mathematical formulation of this problem leads to a combined singular control/switching control for a one dimensional diffusion process. The diffusion process may take two regimes old or new that are switched at stopping times decisions. Within a regime, the manager has to choose a dividend policy that maximizes the expected value of all payouts until bankruptcy or regime transition. The transition from one regime to another incurs a cost or a benefit. The problem is to find the optimal mixed strategy that maximizes the expected returns.

Our analysis is rich enough to address several important questions that have arisen recently in the real option literature [1]. What is the effect of financing constraints on investment decision? When is it optimal to postpone dividends distribution in order to invest? Basically, two assumptions in the real option theory are that the investment decision is made independently of the financial structure of the investment firm and also that the cash process generated by the investment is independent of any managerial decision. In contrast, our model studies the investment under uncertainty with the following set of assumptions. The firm is cash constrained and must finance its investments on its cash benefits, and the cash process generated by the investment depends only on the managerial decision to pay or not dividends, to quit or not the project. Our major finding is to characterize the natural intuition that the manager will delay dividend payments if the investment is sufficiently valuable.

1. See the book of Dixit and Pyndick [47] for an overview of this literature.

5.2.1 The model formulation

The mathematical formulation of our problem has led to a mixted singular and switching control problem. The process cash reserve is assumed to follow the following s.d.e.

$$dX_t \;=\; \mu_{I_t} dt + \sigma dW_t - dZ_t - dK_t, \quad X_{0^-} \;=\; x, \tag{5.2.3}$$

where μ_{I_t} represents the quantity cash generated by the firm depending on the regime $I_t \in \{0,1\}$ under which the firm is operating. Z represents the total amount of dividends paid until time t whereas K represents the costs related to the investment or disinvestment decisions. The investment cost in upgrading from technology 0 to technology 1 is assumed to be g, whereas by selling back the technology and returning to regime 0, the firm may get back $(1-\lambda)g$ of cash, where $0 < \lambda < 1$.

The optimal firm value is

$$v_i(x) \;=\; \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right], \quad x \in \mathbb{R},\; i = 0,1. \tag{5.2.4}$$

Here, we used the notation : $\int_0^{T^-} e^{-\rho t} dZ_t = \int_{[0,T)} e^{-\rho t} dZ_t$.

5.2.2 Results

This mixed singular control problem has, via the dynamic programming principle, led to a system of variational inequalities, which may be solved with the viscosity characterization. We obtain the following results:

- The value functions $v_i, i = 0,1$, are continuous and are the unique viscosity solution to the associated system of variational inequalities.
- The value functions are of class $\mathcal{C}^1$ on $(0,+\infty)$ and $\mathcal{C}^2$ on the union of the continuation and dividend regions.

The main results of our study is the characterization of our natural intuition that the manager should always delay dividend payment if growth opportunity of an investment is deemed satisfying. Furthermore, we qualitatively identify the switching regions which may take several forms depending the profits rate generated under each technology and the investment and disinvestment costs. The results below give the complete qualitative and explicit descriptions of the solution to our control problem:

Main results. We distinguish the following different cases:

(i) If the growth opportunity is too weak, i.e. $\mu_1 \leq \text{Threshold}_m$

$\star$ in regime 0, it is optimal to never invest,

$\star$ in regime 1, it is optimal to distribute all cash reserve and disinvest.

(ii) If the growth opportunity is average, i.e. $\text{Threshold}_m < \mu_1 \leq \text{Threshold}_M$

$\star$ in regime 0, it is optimal to never invest,

$\star$ in regime 1, always stay in that regime, except when bankruptcy is reached, we disinvest.

(iii) If the growth opportunity est sufficiently strong, i.e. $\text{Threshold}_M < \mu_1$

$\star$ in regime 1, always stay in that regime, except when bankruptcy is reached, we disinvest.

$\star$ in regime 0, we need to separate two cases :

case 1.) It is optimal to invest when $x > x_{01}^*$, however, when $x < a$, it is optimal to distribute $x - x_0$ and never invest (with $x_0 < a < x_{01}^*$).

case 2.) The manager delays all dividend distribution and starts investing when the cash reserve exceeds x_{01}^*.

5.3 An optimal dividend and investment control problem under debt constraints

In this paper, we consider the problem of determining the optimal control on the dividend and investment policy of a firm under debt constraints. As in the Merton model, we consider that firm value follows a geometric Brownian process and more importantly we consider that the firm carries a debt obligation in its balance sheet. However, as in most studies, we still assume that the firm assets is either highly liquid and may be assimilated to cash equivalents or cash reserve, or infinitely illiquid except the cash reserve. We allow the company to make investment and finance it through debt issuance/raising, which would impact its capital structure and risk profile. This debt financing results therefore in higher interest rate on the firm's outstanding debts. More precisely, we model the decisions to raise or redeem some debt obligations as switching decisions controls where each regime corresponds to a specific level debt.

Furthermore, we consider that the manager of the firm works in the interest of the shareholders, but only to a certain extent. Indeed, in the objective function, we introduce a penalty cost P and assume that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy. This penalty cost could represent, for instance, an estimated cost of the negative image upon his/her own reputation due to the bankruptcy under his management leadership. Mathematically, we formulate this problem as a combined singular and multiple-regime switching control problem. Each regime corresponds to a level of debt obligation held by the firm.

The studies that are most relevant to our problem are the one investigating combined singular and switching control problems, see [60] and [A4]. However, none of the above papers on dividend and investment policies, which provides qualitative solutions, has yet moved away from the basic Bachelier model or the simplistic assumption that firms hold no debt obligations. In our model, unlike [A5], switching from one regime, i.e. debt level, to another directly impacts the state process itself. Indeed, the drift of the stochastic differential equation governing the firm value would equally switch as the results of the change in interest rate paid on the outstanding debt. A given level of debt is no other than the threshold below which the firm value should never go to avoid bankruptcy. As such, debt level switching also signifies a change of default constraints on the state process in our optimal control problem. Further original contributions in terms of financial studies of our paper include the feature of the conflicts of interest for firm manager through the presence of the penalty cost in the event of bankruptcy. Studying a mixed singular and multi-switching problem combining with the above financial features including debt constraints and penalty cost turns out to be a major mathematical challenge, especially when our objective is to provide quasi-explicit solutions. In addition, it is always tricky to overcoming the combined difficulties of the singular control with those of the switching control, especially when there are multiple regimes, for instance, building a strict supersolution to our HJB system in order to prove the comparison principle.

5.3.1 The model formulation

We assume that the cash-reserve process of the firm $X^{x,i,\alpha}$, denoted by X when there is no ambiguity and associated to a strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, is governed by the following stochastic differential equation:

$$dX_t = bX_t dt - r_{I_t} D_{I_t} dt + \sigma X_t dW_t - dZ_t + dK_t \tag{5.3.5}$$

where $I_t = \sum_{n\geq 0} k_n 1_{\tau_n \leq t < \tau_{n+1}}$, $I_{0^-} = i$ and $k_n \in \mathbb{I}_N := \{1, ..., N\}$. D_i and r_i represent respectively different levels of debt and their associated interest rate paid on those debts. The process K_t represents the cash-flow due to the change in the firm's indebtedness. More precisely $K_t = \sum_{n\geq 0} \left(D_{\kappa_{n+1}} - D_{\kappa_n} - g\right) 1_{\tau_{n+1}\leq t}$, where g represents the additional cost associated with the change of firm's level of debt.
For a given control strategy $\alpha =$, the bankruptcy time is represented by the stopping time T^α defined as

$$T^\alpha = \inf\{t \geq 0, X_t^{x,i,\alpha} \leq D_{I_t}\}. \tag{5.3.6}$$

We equally introduce a penalty cost or a liquidation cost $P > 0$, in the case of a holding company looking to liquidate one of its own affiliate or activity. In the case of the penalty, it mainly assumes that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy.

We define the value functions which the manager actually optimizes as follows

$$v_i(x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{(i,x)} \left[\int_0^{T^-} e^{-\rho t} dZ_t - Pe^{-\rho T} \right], \quad x \in \mathbb{R},\ i \in \{1, ..., N\}, \tag{5.3.7}$$

where $\mathcal{A}$ represents the set of admissible control strategies, and ρ the discount rate.

5.3.2 PDE Characterization

Using the dynamic programming principle, we obtain the associated system of variational inequalities satisfied by the value functions:

$$\begin{aligned} \min \left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x) - 1\ ,\ v_i(x) - \max_{j \neq i} v_j(x + D_j - D_i - g) \right] &= 0,\ x > D_i,\ i \in \mathbb{I}_N \\ v_i(D_i) &= -P, \end{aligned}$$

where the operator $\mathcal{A}_i$ is defined by $\mathcal{A}_i \phi = \mathcal{L}_i \phi - \rho \phi$, and $\mathcal{L}_i \varphi = [bx - r_i D_i]\varphi'(x) + \frac{1}{2}\sigma^2 x^2 \varphi''(x)$

We may obtain the following results:

Proposition 5.3.1. *The value functions v_i are continuous on (D_i, ∞) and satisfy*

$$v_i(D_i^+) := \lim_{x \downarrow D_i} v_i(x) = -P. \tag{5.3.8}$$

Theorem 5.3.1. *The value functions v_i, $i \in \mathbb{I}_N$, are continuous on (D_i, ∞), and are the unique viscosity solutions on (D_i, ∞) with linear growth condition and boundary data $v_i(D_i) = -P$, to the system of variational inequalities :*

$$\min \left[-\mathcal{A}_i v_i(x)\ ,\ v_i'(x) - 1\ ,\ v_i(x) - \max_{j \neq i} v_j(x + D_j - D_i - g) \right] = 0,\ x > D_i. \tag{5.3.9}$$

Actually, we obtain some more regularity results on the value functions.

Proposition 5.3.2. *The value functions v_i, $i \in \mathbb{I}_N$, are C^1 on (D_i, ∞). Moreover, if we set for $i \in \mathbb{I}_N$:*

$$\mathcal{S}_i = \left\{ x \geq D_i\ , v_i(x) = \max_{j \neq i} v_j(x + D_j - D_i - g), \right\} \tag{5.3.10}$$

$$\mathcal{D}_i = \overline{\text{int}\left(\{x \geq D_i\ , v_i'(x) = 1\}\right)}, \tag{5.3.11}$$

$$\mathcal{C}_i = (D_i, \infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i), \tag{5.3.12}$$

then v_i is C^2 on the open set $\mathcal{C}_i \cup \text{int}(\mathcal{D}_i) \cup \text{int}(\mathcal{S}_i)$ of (D_i, ∞), and we have in the classical sense

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i.$$

$\mathcal{S}_i$, $\mathcal{D}_i$, and $\mathcal{C}_i$ respectively represent the switching, dividend, and continuation regions when the outstanding debt is at regime i.

5.3.3 qualitative results on the switching regions

For $i,\ j \in \mathbb{I}_N$ and $x \in [D_i, +\infty)$, we introduce some notations:

$$\delta_{i,j} = D_j - D_i, \quad \Delta_{i,j} = (b - r_j)D_j - (b - r_i)D_i \quad \text{and} \quad x_{i,j} = x + \delta_{i,j} - g.$$

We set $x_i^* = \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\}$ for all $i \in \mathbb{I}_N$

We equally define $\mathcal{S}_{i,j}$ as the switching region from debt level i to j.

$$\mathcal{S}_{i,j} = \{x \in (D_i,\ +\infty), v_i(x) = v_j(x_{i,j})\}.$$

and notice that $\mathcal{S}_i \ = \ \cup_{j \neq i} \mathcal{S}_{i,j},\ i \in \mathbb{I}_N$.

We now turn to the first result which that there exists a finite level of cash such that it is optimal to distribute dividends up to this level.

Lemma 5.3.1. *For all $i \in \mathbb{I}_N$, we have $x_i^* := \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\} < +\infty$.*

In order to compute the dividend regions, we establish the following lemma.

Lemma 5.3.2. *Let $i,\ j \in \mathbb{I}_N$ such that $j \neq i$. We assume that there exists $\hat{x}_i$ a left-boundary of $\mathcal{D}_i$.*

i) Assume that $\hat{x}_i \notin \mathcal{S}_i$, then we have $(b - r_i)D_i > -\rho P$ and $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_i D_i$. As $x \to \rho v_i(x) - bx + r_i D_i$ is increasing, it implies that

$$\rho v_i(x) < bx - r_i D_i \text{ on } (D_i, \hat{x}_i) \quad \text{and} \quad \rho v_i(x) > bx - r_i D_i \text{ on } (\hat{x}_i, +\infty).$$

ii) Assume that $\hat{x}_i \in \mathcal{S}_{i,j}$ then we have

ii.a) $[\hat{x}_i,\ \hat{x}_i + \varepsilon] \subset \mathcal{S}_{i,j}$ and $\hat{x}_i + \delta_{i,j} - g$ is a left-boundary of $\mathcal{D}_j$.

ii.b) $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_i D_i + \Delta_{i,j} - bg$ and $\Delta_{i,j} > 0$.

ii.c) $\forall k \in \mathbb{I}_N - \{i, j\}, \hat{x}_i \notin \mathcal{S}_{i,k}$.

Notice that the last equality implies that $-\rho P + bg < (b - r_j)D_j$.

We now establish an important result in determining the description of the switching regions. The following Theorem states that it is never optimal to expand its operation, i.e. to make investment, through debt financing, should it result in a lower "drift" ($(b - r_i)D_i$) regime. However, when the firm's value is low, i.e. with a relatively high bankruptcy risk, it may be optimal to make some divestment, i.e. sell parts of the company, and use the proceedings to lower its debt outstanding, even if it results in a regime with lower "drift". In other words, to lower the firm's bankruptcy risk, one should try to decrease its volatility, i.e. the diffusion coefficient. In our model, this clearly means making some debt repayment in order to lower the firm's volatility, i.e. σX_t.

Theorem 5.3.2. *Let $i, j \in \mathbb{I}_N$ such that $(b - r_j)D_j > (b - r_i)D_i$. We have the following results:*

1) $x_j^* \notin \mathcal{S}_{j,i}$ *and* $\mathring{\mathcal{D}}_j = (x_j^*,\ +\infty)$.
2) $\mathring{\mathcal{S}}_{j,i} \subset (D_j + g,\ x_j^*)$. *Furthermore, if* $D_j < D_i$*, then* $\mathring{\mathcal{S}}_{j,i} = \emptyset$.

From the above Theorem, we may obtain the following Corollary and Proposition on the determination of different regions. We will in particular see in the next section how from these results, we may obtain the complete results in the two-regime case and above.

Corollary 5.3.1. *Let* $m \in \mathbb{I}_N$ *such that* $(b - r_m)D_m = \max_{i\in\mathbb{I}_N}(b - r_i)D_i$.
1) $x_m^* \notin \mathcal{S}_m$ *and* $\mathring{\mathcal{D}}_m = (x_m^*,\ +\infty)$.
2) For all $i \in \mathbb{I}_N - \{m\}$*, we have:*
i) If $D_m < D_i$, $\mathring{\mathcal{S}}_{m,i} = \emptyset$.
ii) If $D_i < D_m$, $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g,\ x_m^*)$. *Furthermore, if* $b \geq r_i$*, then* $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g,\ (a_i^* + \delta_{i,m} + g) \wedge x_m^*)$*, where* a_i^* *is the unique solution of the equation* $\rho v_i(x) = (bx - r_iD_i)v_i'(x)$. *We further have* $a_i^* \neq x_i^*$.

We now turn to the following results ordering the left-boundaries $(x_i^*)_{i\in\mathbb{I}_N}$ of the dividend regions $(\mathcal{D}_i)_{i\in\mathbb{I}_N}$.

Proposition 5.3.3. *Consider* $i, j \in \mathbb{I}_N$*, such that* $(b-r_i)D_i < (b-r_j)D_j$. *We always have* $x_i^* + \delta_{i,j} - g \leq x_j^*$ *unless there exists a regime* k *such that* $(b - r_j)D_j < (b - r_k)D_k$ *and* $x_i^* \in \mathcal{S}_{i,k}$*, then we have* $x_j^* - \delta_{i,j} + g < x_i^* < x_k^* - \delta_{i,k} + g$.

5.3.4 The two regime-case

Throughout this section, we now assume that $N = 2$, in which case, we will get a complete description of the different regions. We will see that the most important parameter to consider is the so-called "drifts" $((b - r_i)D_i)_{i=1,2}$ and in particular their relative positions. To avoid cases with trivial solution, i.e. immediate consumption, we will assume that $-\rho P < (b - r_i)D_i, i = 1, 2$. Throughout the following Theorems, we provide a complete resolution to our problem in each case.

Theorem 5.3.3. *We assume that* $(b - r_2)D_2 < (b - r_1)D_1$.
We have

$$\mathcal{C}_1 = [D_1, x_1^*),\ \mathcal{D}_1 = [x_1^*,\ +\infty),\ \textit{and}\ \mathring{\mathcal{S}}_1 = \emptyset\ \textit{where}\ \rho v_1(x_1^*) = bx_1^* - r_1D_1.$$

1) If $\mathcal{S}_2 = \emptyset$ *then we have*

$$\mathcal{C}_2 = [D_2, x_2^*),\ \textit{and}\ \mathcal{D}_2 = [x_2^*,\ +\infty)\ \textit{where}\ \rho v_2(x_2^*) = bx_2^* - r_2D_2.$$

2) If $\mathcal{S}_2 \neq \emptyset$ *then there exists* y_2^* *such that* $\mathcal{S}_2 = [y_2^*,\ +\infty)$ *and we distinguish two cases*
a) If $x_2^* + \delta_{2,1} - g < x_1^*$*, then* $y_2^* > x_2^*$, $y_2^* = x_1^* + \delta_{1,2} + g$ *and*

$$\mathcal{C}_2 = [D_2, x_2^*),\ \textit{and}\ \mathcal{D}_2 = [x_2^*,\ +\infty)\ \textit{where}\ \rho v_2(x_2^*) = bx_2^* - r_2D_2.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$ *then* $y_2^* \leq x_2^*$, $\rho v_2(x_2^*) = bx_2^* - r_2 D_2 + \Delta_{2,1} - bg$.
We define a_2^* *as the solution of* $\rho v_2(a_2^*) = ba_2^* - r_2 D_2$ *and have two cases*
i) If $a_2^* \notin \mathcal{D}_2$*, we have*

$$\mathcal{D}_2 = [x_2^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_2 = [D_2,\ y_2^*).$$

ii) If $a_2^* \in \mathcal{D}_2$*, there exists* $z_2^* \in (a_2^*, y_2^*)$ *such that*

$$\mathcal{D}_2 = [a_2^*,\ z_2^*] \cup [x_2^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_2 = [D_2,\ a_2^*) \cup (z_2^*,\ y_2^*).$$

Theorem 5.3.4. *We assume that* $(b - r_1)D_1 < (b - r_2)D_2$,
1) we have

$$\begin{aligned} \mathcal{D}_2 &= [x_2^*,\ +\infty) \textit{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2 \\ \mathring{\mathcal{S}}_2 &= \emptyset \textit{ or there exist } s_2^*, S_2^* \in (D_2 + g,\ x_2^*) \textit{ such that } \mathring{\mathcal{S}}_2 = (s_2^*, S_2^*). \end{aligned}$$

2) If $\mathring{\mathcal{S}}_1 = \emptyset$ *then we have*

$$\mathcal{C}_1 = [D_1, x_1^*),\ \textit{and } \mathcal{D}_1 = [x_1^*,\ +\infty) \textit{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

3) If $\mathring{\mathcal{S}}_1 \neq \emptyset$ *there exists* y_1^* *such that* $\mathring{\mathcal{S}}_1 = (y_1^*,\ +\infty)$
a) If $x_1^* + \delta_{1,2} - g < x_2^*$*, then* $y_1^* > x_1^*$, $y_1^* = x_2^* + \delta_{2,1} + g$ *and*

$$\mathcal{C}_1 = [D_1, x_1^*),\ \textit{and } \mathcal{D}_1 = [x_1^*,\ +\infty) \textit{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$*, then* $y_1^* \leq x_1^*$, $\rho v_1(x_1^*) = bx_1^* - r_1 D_1 + \Delta_{1,2} - bg$.
We define a_1^* *as the solution of* $\rho v_1(a_1^*) = ba_1^* - r_1 D_1$ *and have two cases.*
i) If $a_1^* \notin \mathcal{D}_1$*, we have*

$$\mathcal{D}_1 = [x_1^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_1 = [D_1,\ y_1^*).$$

ii) If $a_1^* \in \mathcal{D}_1$*, there exists* $z_1^* \in (a_1^*, y_1^*)$ *such that*

$$\mathcal{D}_1 = [a_1^*,\ z_1^*] \cup [x_1^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_1 = [D_1,\ a_1^*) \cup (z_1^*,\ y_1^*).$$

5.4 Liquidity risk and optimal dividend/investment strategies

In [71], [6], [35], [$A5$], [$A7$], the authors study an optimal dividend problem and consider a stochastic process which represents the cash reserve of the firm. The firm goes into bankruptcy when its cash reserve reaches zero. The underlying financial assumption behind the above model is to consider that the firm's assets may be separated into two types of assets, highly liquid assets which may be assimilated as cash reserve, i.e. cash & equivalents, or infinitely illiquid assets, i.e. productive assets that may not be sold.

In this paper, we no longer simplify the optimal dividend and investment problem by assuming that firm's assets are either infinitely illiquid or liquid. For the same reason as highlighted in financial market problems, it is necessary to take into account the liquidity constraints. More precisely, investment (for instance acquiring producing assets) and disinvestment (selling assets) should be possible but not necessarily at their fair value. The firm may have to face some liquidity costs when buying or selling assets. While taking into account liquidity constraints and costs has become the norm in recent financial markets problems, it is still not the case in the corporate finance, to the best of our knowledge, in particular in the studies of optimal dividend and investment strategies. In our paper, we consider the company's assets may be separated in two categories, cash & equivalents, and risky assets which are subjected to liquidity costs. The risky assets are assimilated to productive assets which may be increased when the firm decides to invest or decreased when the firm decides to disinvest. We assume that the price of the risky assets is governed by a stochastic process. The firm manager may buy or sell assets but has to bear liquidity costs. The objective of the firm manager is to find the optimal dividend and investment strategy maximizing its shareholders' value, which is defined as the expected present value of dividends. Mathematically, we formulate this problem as a combined multidimensional singular and multi-regime switching control problem.

The studies that are most relevant to our problem are the one investigating combined singular and switching control problems [60], [A4], and [A7]. By incorporating uncertainty into illiquid assets value, we no longer have to deal with a uni-dimensional control problem but a bi-dimensional singular and multi-regime switching control problem. In such a setting, it is clear that it will be no longer possible to easily get explicit or quasi-explicit optimal strategies. Consequently, to determine the four regions comprising the continuation, dividend and investment/disinvestment regions, numerical resolutions are required.

5.4.1 Problem formulation

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let W and B be two correlated $\mathbb{F}$-Brownian motions, with correlation coefficient c.
We consider a firm which has the ability to make investment or disinvestment by buying or selling productive assets, for instance, factories. We assume that these productive assets are risky assets whose value process S is solution of the following equation:

$$dS_t = S_t\left(\mu dt + \sigma dB_t\right),\; S_0 = s, \tag{5.4.13}$$

where μ and σ are positive constants.
We denote by $Q_t \in \mathbb{N}$ the number of units of producing assets owned by the company at time t.

We consider a control strategy: $\alpha = ((\tau_i, q_i)_{i \in \mathbb{N}}, Z)$ where τ_i are $\mathbb{F}$-stopping times, corresponding to the investment decision times of the manager, and q_i are $\mathcal{F}_{\tau_i}$-measurable variables valued in $\mathbb{Z}$ and representing the number of productive assets units bought (or sold if $q_i \leq 0$) at time τ_i. When q_i is positive, it means that the firm decides to make investment to increase the assets quantity. Each purchase or sale incurs a fixed cost denoted $\kappa > 0$. The non-decreasing càdlàg process Z represents the total amount of dividends distributed up to time t. Starting from an initial number of assets q and given a control α, the dynamics of the quantity of assets held by the firm is governed by:

$$\left\{ \begin{array}{lcl} dQ_t & = & 0 \quad \text{for } \tau_i \leq t < \tau_{i+1}, \\ Q_{\tau_i} & = & Q_{\tau_i^-} + q_i, \\ Q_0 & = & q, \end{array} \right. \qquad \text{for } i \in \mathbb{N}. \tag{5.4.14}$$

Similarly, starting from an initial cash value x and given a control α, the dynamics of the cash reserve (or more precisely the firm's cash and equivalents) process of the firm is governed by:

$$\left\{ \begin{array}{lcl} dX_t & = & rX_t dt + h(Q_t)(bdt + \eta dW_t) - dZ_t, \text{ for } \tau_i \leq t < \tau_{i+1} \\ X_{\tau_i} & = & X_{\tau_i^-} - S_{\tau_i} f(q_i) q_i - \kappa, \\ X_0 & = & 0, \end{array} \right. \qquad \text{for } i \in \mathbb{N}. \tag{5.4.15}$$

where b, r and η are positive constants and h a non-negative, non-decreasing and concave function satisfying $h(q) \leq H$ with $h(1) > 0$ and $H > 0$. The function f represents the liquidity cost function (or impact function with the impact being temporary) and is assumed to be non-negative, non-decreasing, such that $f(0) = 1$.
We denote by $Y_t^y = (X_t^x, S_t^s, Q_t^q)$ the solution to (5.4.13)-(5.4.15) with initial condition $(X_0^x, S_0^s, Q_0^q) = (x, s, q) := y$. At each time t, the firm's cash value and number of units of producing assets have to remain non-negative i.e. $X_t \geq 0$ and $Q_t \geq 0$, for all $t \geq 0$.

The bankruptcy time is defined as

$$T := T^{y,\alpha} := \inf\{t \geq 0, X_t < 0\}.$$

We define the liquidation value as $L(x, s, q) := x + (sf(-q)q - \kappa)^+$ and notice that $L \geq 0$ on $\mathbb{R}^+ \times (0, +\infty) \times \mathbb{N}$. We introduce the following notation

$$\mathcal{S} := \mathbb{R}^+ \times (0, +\infty) \times \mathbb{N}.$$

The optimal firm value is defined on $\mathcal{S}$, by

$$v(x, s, q) = \sup_{\alpha \in \mathcal{A}(x,s,q)} \mathbb{E}^{(x,s,q)} \left[\int_0^{T^-} e^{-\rho u} dZ_u \right] \tag{5.4.16}$$

5.4.2 Characterization of auxiliary functions

The aim of this section is to provide an implementable algorithm of our problem. To tackle the stochastic control problem as defined in (5.4.16), one usual way is to first characterize the value function as a unique solution to its associated HJB equation. The second step is to deduce the optimal strategies from smooth-fit properties and more generally from viscosity solution techniques. The optimal strategies may be characterized by different regions of the state-space, i.e. the continuation region, the dividend region as well as the Buy and Sell regions. In such cases, the solutions may be either of explicit or quasi-explicit nature. However, in a non-degenerate multidimensional setting such as in our problem, getting explicit or quasi-explicit solutions is out of reach.
As such, to solve our control problem, we characterize our value function as the limit of a sequence of auxiliary functions. The auxiliary functions are defined recursively and each one may be characterized as a unique viscosity solution to its associated HJB equation. This will allow us to get an implementable algorithm approximating our problem.

An approximating sequence of functions.

We recall the notation $y = (x, s, q) \in \mathcal{S}$. From this point, we may use alternatively y or (x, s, q). We now introduce the following subsets of $\mathcal{A}(y)$:

$$\mathcal{A}_N(y) := \{\alpha = ((\tau_k, \xi_k)_{k\in\mathbb{N}^*}, Z) \in \mathcal{A}(y) : \tau_k = +\infty \text{ a.s. for all } k \geq N+1\}$$

and the corresponding value function v_N, which describes the value function when the investor is allowed to make at most N interventions (investments or disinvestments):

$$v_N(y) = \sup_{\alpha\in\mathcal{A}_N(y)} \mathbb{E}^{(x,s,q)}\left[\int_0^{T^-} e^{-\rho u} dZ_u\right], \ \forall N \in \mathbb{N} \tag{5.4.17}$$

We shall show in Proposition 5.4.7 that the sequence $(v_N)_{N\geq 0}$ goes to v when N goes to infinity, but we first have to carefully study some properties of this sequence.

In the next Proposition, we recall explicit formulas for v_0 and the optimal strategy associated to this singular control problem. This problem is indeed very close to the one solved in the pioneering work of Jeanblanc and Shirayev (see [71]). The only difference in our framework is due to the interest $r \neq 0$ and therefore the cash process X does not follow exactly a Bachelier model. However, proofs and results can easily be adapted to obtain Proposition 5.4.4.

Proposition 5.4.4. *There exists $x^*(q) \in [0, +\infty)$ such that*

$$v_0(x, s, q) := \begin{cases} V_q(x) & \text{if} \quad 0 \leq x \leq x^*(q) \\ x - x^*(q) + V_q(x^*(q)) & \text{if} \quad x \geq x^*(q), \end{cases}$$

where V_q is the $\mathcal{C}^2$ function, solution of the following differential equation

$$\frac{\eta^2 h(q)^2}{2} y'' + (rx + bh(q))y' - \rho y = 0; \quad y(0) = 0, \ y'(x^*(q)) = 1 \text{ and } y''(x^*(q)) = 0 \quad (5.4.18)$$

Notice that $x \to v_0(x, s, q)$ is a concave and $\mathcal{C}^2$ function on $[0, +\infty)$ and that if $h(0) = 0$, it is optimal to immediately distribute dividends up to bankruptcy therefore $v_0(x, s, 0) = x$.

We now are able to characterize our impulse control problem as an optimal stopping time problem, defined through an induction on the number of interventions N.

Proposition 5.4.5. *(Optimal stopping)*
For all $(x, s, q, N) \in \mathcal{S} \times \mathbb{N}^*$, we have

$$v_N(x, s, q) = \sup_{(\tau, Z) \in \mathcal{T} \times \mathcal{Z}} \mathbb{E}[\int_0^{T \wedge \tau} e^{-\rho u} \, dZ_u + e^{-\rho \tau} G_{N-1}(X^x_{\tau^-}, S^s_\tau, q) 1\!\!1_{\{\tau < T\}}], \quad (5.4.19)$$

where $\mathcal{T}$ is the set of stopping times, $\mathcal{Z}$ the set of predictable and non-decreasing càdlàg processes, and

$$G_{N-1}(x, s, q) \quad := \quad \max_{n \in a(x,s,q)} v_{N-1}(\Gamma(y, n)) \text{ and } G_{-1} = 0, \quad (5.4.20)$$

$$\text{with} \qquad a(x, s, q) := \left\{ n \in \mathbb{Z} : n \geq -q \text{ and } nf(n) \leq \frac{x - \kappa}{s} \right\}, \quad (5.4.21)$$

$$\text{and} \qquad \Gamma(y, n) := (x - nf(n)s - \kappa, \ s, \ q + n). \quad (5.4.22)$$

Bounds and convergence of $(v_N)_{N \geq 0}$.

We begin by stating a standard result which says that any smooth function, which is supersolution to the HJB equation, is a majorant of the value function.

Proposition 5.4.6. *Let $N \in \mathbb{N}$ and $\phi = (\phi_q)_{q \in \mathbb{N}}$ be a family of non-negative $\mathcal{C}^2$ functions on $\mathbb{R}^+ \times (0, +\infty)$ such that $\forall q \in \mathbb{N}$ (we may use both notations $\phi(x, s, q) := \phi_q(x, s)$), $\phi_q(0, s) \geq 0$ for all $s \in (0, \infty)$ and*

$$\min \left[\rho\phi(y) - \mathcal{L}^N \phi(y), \phi(y) - G_{N-1}(y), \frac{\partial \phi}{\partial x}(y) - 1 \right] \ \geq \ 0 \quad (5.4.23)$$

for all $y \in (0, +\infty) \times (0, +\infty) \times \mathbb{N}$, where we have set

$$\begin{aligned} \mathcal{L}^N \varphi \quad = \quad & \frac{\eta^2 h(q)^2}{2} \frac{\partial^2 \varphi}{\partial x^2} + (rx + bh(q)) \frac{\partial \varphi}{\partial x} \\ & + 1\!\!1_{\{N > 0\}} \left[\frac{\sigma^2 s^2}{2} \frac{\partial^2 \varphi}{\partial s^2} + c\sigma\eta s h(q) \frac{\partial^2 \varphi}{\partial s \partial x} + \mu s \frac{\partial \varphi}{\partial s} \right]. \end{aligned}$$

then we have $v_N \leq \phi$.

Corollary 5.4.2. Bounds*:*
For all $N \in \mathbb{N}^*$ and $(x, s, q) \in \mathcal{S}$, we have

$$L(x, s, q) \leq v_N(x, s, q) \leq x + sq + K \qquad \text{where } \rho K = bH.$$

We are able to conclude on the asymptotic behavior of our approximating sequence of functions. The next Proposition shows that this sequence of functions goes to our value function v when N goes to infinity.

Proposition 5.4.7. *(Convergence) For all $y \in \mathcal{S}$, we have*

$$\lim_{N \to +\infty} v_N(y) = v(y).$$

5.4.3 Characterization of the value functions and numerical results

Let $N > 0$. This subsection is devoted to the characterization of the function v_N as the unique function which satisfies the boundary condition

$$v_N(y) = G_{N-1}(y) \text{ on } \{0\} \times (0, +\infty) \times \mathbb{N}. \tag{5.4.24}$$

Below is the associated HJB equation:

$$\min\{\rho v_{_N}(y) - \mathcal{L}v_{_N}(y); \frac{\partial v_{_N}}{\partial x}(y) - 1; v_{_N}(y) - G_{N-1}(y)\} = 0 \text{ on } (0, +\infty)^2 \times \mathbb{N}, \tag{5.4.25}$$

where we have set

$$\mathcal{L}\varphi \quad = \quad \frac{\eta^2 h(q)^2}{2}\frac{\partial^2\varphi}{\partial x^2} + \frac{\sigma^2 s^2}{2}\frac{\partial^2\varphi}{\partial s^2} + c\sigma\eta s h(q)\frac{\partial^2\varphi}{\partial s \partial x} + (rx + bh(q))\frac{\partial\varphi}{\partial x} + \mu s\frac{\partial\varphi}{\partial s}.$$

It relies on the following Dynamic Programming Principle. Let $\theta \in \mathcal{T}$, $y := (x, s, q) \in \mathcal{S}$ and set $\nu = T \wedge \theta$, we have

$$v_N(y) = \sup_{(\tau,Z)\in\mathcal{T}\times\mathcal{Z}} \mathbb{E}[\int_0^{(\nu\wedge\tau)^-} e^{-\rho s}\, dZ_s + e^{-\rho(\nu\wedge\tau)} v_N\left(X^x_{(\nu\wedge\tau)^-}, S^s_{\nu\wedge\tau}, q\right) \mathbb{1}_{\{\tau<\nu\}}] \tag{5.4.26}$$

We are now able to establish the main results of this section.

Theorem 5.4.5. *For all $(N, q) \in \mathbb{N}^* \times \mathbb{N}$, the value function $v_N(\cdot, \cdot, q)$ is continuous on $(0, +\infty)^2$. Moreover v_N is the unique viscosity solution on $(0, +\infty)^2 \times \mathbb{N}$ of the HJB equation (5.4.25) satisfying the boundary condition (5.4.24) and the following growth condition*

$$|v_N(x, s, q)| \leq C_1 + C_2 x + C_3 sq, \quad \forall (x, s, q) \in \mathcal{S},$$

for some positive constants C_1, C_2 and C_3.

Below, we present some numerical results by approximating the solution of the HJB equation. To solve the HJB equation arising from the stochastic control problem (5.4.17), we choose to use a finite difference scheme which leads to the construction of an approximating Markov chain. The convergence of the scheme can be shown using standard arguments as in [76]. We may equally refer to [24], [65], and [72] for numerical schemes involving singular control problems.

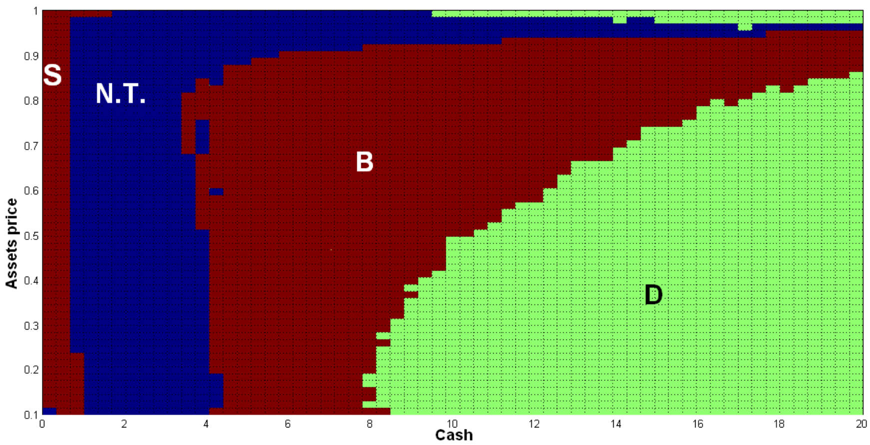

Figure 5.1: *Description of different regions, in* (x, s) *for a fixed* q_0.

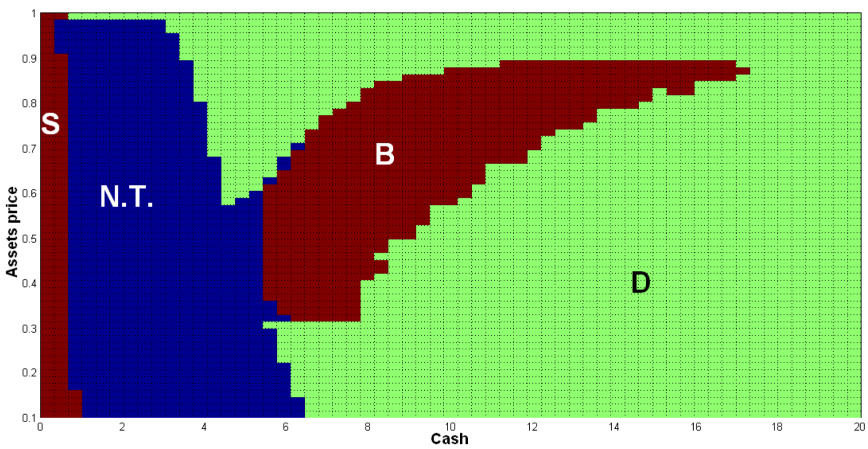

Figure 5.2: *Description of different regions, in* (x, s) *for* $q_1 > q_0$.

Part III

Ongoing research and projects

Chapter 6

Ongoing research and projects

In this Chapter, I present some on-going and future research projects. At the moment, my future research projects concerns in particular corporate finance problems and problems related to liquidity risk modelling.

5.1 Optimal dividend and capital injection policy under audit

Joint project with E. Chevalier and A. Roch.

We consider that the firm holds a fixed amount of debt D. Unlike study in [A7], we do not assume that the firm goes into bankruptcy when its cash reserves are below D, but it is only in financial difficulty. The firm can inject capital at any time, and can pay out dividends when it is not in financial difficulty. When the firm is in financial difficulty, it can be audited at any time. The probability of being audited in the time interval $[t, t + dt)$ is λdt. When the firm is being audited, bankruptcy is defined in terms of a grace period, denoted δ. In other words, the firm is declared bankrupt when it has spent a continual period of time δ in financial distress from the start of the audit period or if its cash reserves hit zero at any time.

5.2 An optimal capital structure control problem under uncertainty

Working paper, with E. Chevalier and E. Bayraktar

This paper concerns with the problem of determining an optimal control on the capital structure, dividend and investment policy of a bank operating under solvability constraints. We assume that the bank's assets consist of both clients' deposits and shareholders' equity. The managers of the bank may invest in either risky assets or in risk-free assets. The objective of the manager is to optimize the bank shareholders' value, ie. the cumulative dividend distributed over the life time of the bank while controlling its solvability. Indeed, the bank is considered to operate under an uncertain environment and is obliged to respect a number of constraints, in particular solvency ratio constraints as defined under the Basle frameworks. We allow the bank to seek recapitalization or to issue new capital should they fall under financial difficulties.

We formulate this problem as a combined impulse control, regular and singular control problem. We will see how this bi-dimensional control problem may be reduced to a one-dimensional one and how quasi-explicit solution may be obtained. We further enrich our studies with some numerical illustrations.

5.3 Wages and Employment in Economies with Multi-Worker Firms, Uncertainty and Labor Turnover Costs

Working paper, with S. Scotti and A. Vidigni

We present a dynamic general equilibrium model of the labor market where multi-worker firms, producing with decreasing returns to scale technology subject to a number of different productivity shocks, bargain à la Stole-Zwiebel (a generalization of Nash bargaining) over wages, in presence of hiring and dismissal costs. We show that the optimal employment policy of firms lets the marginal value of labor fluctuate persistently in an interval, defined as the inaction range, and hirings or dismissals take place only when the two reflecting barriers characterizing it are hit. We prove that the uncertainty generated by random shocks which directly affect the size of the firm, increases the size of the inaction range by making firms more cautious in both hiring and dismissal, and decreases job creation and employment. Higher uncertainty generated by shocks to the productivity of firms, also reduces unambiguously long run aggregate employment, consistently with recently provided empirical evidence, but has no effect on the employment policy of each particular firm. Additionally, we provide formal proofs for a number of well-established empirical regularities, such as the existence of wage dispersion across observationally equivalent workers, and the fact that larger firms tend to pay higher wages. We also account for the fact that the differential growth rate of employment in large vs. small firms appears to be strongly procyclical, along many dimensions. Furthermore, we demonstrate that Gibrat's law holds on and off the stationary equilibrium, if idiosyncratic productivity follows a particular diffusion process. The causal mechanism at work in our theory does not rely on search frictions and convex vacancy creation costs (which are intentionally ignored), but only on the interaction between labor turnover costs, the existence of firms of (endogenously) variable size due to stochastic shocks, and to the relatively standard production technology and wage setting rule assumed. Methodologically, our problem is formulated as a bi-dimensional singular control problem and we use the viscosity theory to characterize and solve explicitly or quasi-explicitly our problem.

5.4 Optimal trading strategies in a market with partial information

joint project with G. Bernis (Researcher at Natixis) and S. Scotti

We consider an impulse control problem under constraints and partial information. We consider a market model in which over-reaction and under-reaction to market news is taken into account as in [26]. As usual, jumps are used to model the arrival of important (positive or negative) news about the firm. An extensive literature focuses on empirical studies on cross-section average stock returns and shows the presence of anomalies that are classified as "Under-reaction and Over-reaction to information". Market data will be used for calibration purposes.

Bibliography

[A1] Competitive market equilibrium under asymmetric information, Decisions in Economics and Finance, 2007, 30, pp. 79-94.

[A2] Explicit solution to an optimal switching problem in the two regime case, with H. Pham, Siam Journal on Control and Optim., 2007, 46, pp. 395-426.

[A3] A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact, with M. Mnif and H. Pham, Finance and Stochastics, 2007, 11, pp. 51-90.

[A4] A mixed singular/switching control problem for a dividend policy with reversible technology investment, with H. Pham and S. Villeneuve, Annals of Applied Probability, 2008, 18, pp. 1164-1200.

[A5] Optimal switching over multiple regimes, with H. Pham and X.Y. Zhou, Siam Journal on Control and Optim., 2009, 48, pp. 2217-2253.

[A6] Bid-Ask Spread modelling, a perturbation approach, with T. Lim, JM. Sahut, and S. Scotti, 2011, Seminar on Stochastic Analysis, Random Fields and Applications VII (Ascona 2011).

[A7] An Optimal Dividend and Investment Control Problem under Debt Constraints, with E. Chevalier, and S. Scotti, 2013, SIAM J. Finan. Math., 4(1), 297 - 326.

[A8] Exit Optimal exit strategies for investment projects, with E. Chevalier, A. Roch and S. Scotti, 2015, Journal of Mathematical Analysis and Applications, Vol.425(2), pp.666-694.

[A9] Optimal execution cost for liquidation through a limit order market, with E. Chevalier, A. Roch and S. Scotti, 2014, to appear in International Journal of Theoretical and Applied Finance.

[A10] Numerical approximation for a portfolio optimization problem under liquidity risk and costs, with M. Gaigi, M. Mnif, and S. Toumi, 2014, to appear in Applied Mathematics and Optimization.

Submitted papers

[S11] Liquidity risk and optimal dividend/investment strategies, with E. Chevalier et M. Gaigi, 2015.

[S12] Optimal market dealing under constraints, with E. Chevalier, M. Gaigi, and M. Mnif, 2013.

Working papers and works in progress

[W13] Wages and Employment in Economies with Multi-Worker Firms, Uncertainty and Labor Turnover Costs, with S. Scotti, A. Vindigni, 2015, working paper.

[W14] An optimal capital structure control problem under uncertainty, with E. Bayraktar and E. Chevalier, work in progress.

[W15] Optimal execution in a one-sided order book with stochastic resilience, with E. Chevalier and S. Pulido, work in progress.

[W16] Optimal dividend and capital injection policy with external audit, with E. Chevalier and A. Roch, work in progress.

[1] Abergel F., Jedidi A. (2013) : A mathematical approach to order book modelling. Int. J. Theor. Appl. Finan, **16**, 5, 1-40.

[2] Akian M., Sulem A. and M. Taksar (2001) : "Dynamic optimization of long term growth rate for a portfolio with transaction costs and logarithm utility", *Math. Finance*, **11**, 153-188.

[3] Alfonsi, A., A. Fruth and A. Schied (2010): "Optimal execution strategies in limit order books with general shape functions", *Quantitative Finance*, **10**, 143-157.

[4] Almgren, R., and N. Chriss (1999) "Value under liquidation", *Risk*, December 1999.

[5] Almgren R. and N. Chriss (2001) : "Optimal execution of portfolio transactions", *Journal of Risk*, 3, 5-39.

[6] Asmussen S., Taksar M. (1997): Controlled diffusion models for optimal dividend payout. Insurance Math. Econom., 20, pp. 1-15.

[7] Avellaneda, M. and S. Stoikov (2008): "High-frequency trading in a limit order book", *Quantitative Finance*, **8**, 217-224.

[8] Back K. (1992) : "Insider trading in continuous time", *Review of Financial Studies*, **5**, 387-409.

[9] Bank P. and D. Baum (2004) : "Hedging and portfolio optimization in illiquid financial markets with a large trader", *Mathematical Finance*, **14**, 1-18.

[10] Barles G. (1994) : Solutions de viscosité des équations d'Hamilton-Jacobi, *Math. et Appli.*, Springer Verlag.

[11] Barles G., Souganidis P.E. (1991): Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic analysis, 4, 271-283.

[12] Barndorff-Nielsen, O., Hansen, P.R., Lnde, A. and N. Shephard (2008): "Designing Realised Kernels to Measure the Ex-Post Variation of Equity Prices in the Presence of Noise", Econometrica, 76(6) p. 1481-1536.

[13] Bayraktar E., M. Egami (2010): "On the One-Dimensional Optimal Switching Problem", Mathematics of Operations Research, 2010, 35 (1), 140-159.

[14] Bayraktar E., Ludkovski M. (2014): Liquidation in Limit Order Books with Controlled Intensity. Mathematical Finance, **24** (4), 627-650.

[15] Bensoussan A. and J.L. Lions (1982) : Impulse control and quasi-variational inequalities, Gauthiers-Villars.

[16] Blais, M. and P. Protter (2010): "An Analysis of the Supply Curve for Liquidity Risk through Book Data", *International Journal of Theoretical and Applied Finance*, **13**, 821-838.

[17] Blanchet-Scalliet C., Gibson R., de Saporta B., Talay D., Tanre E. (2009): Viscosity solutions to optimal portfolio allocation problems in models with random time changes and transaction costs. Advanced Financial Modelling. H. Albrecher, W. Runggaldier, W. Schachermayer (Eds). Radon Series on Computational and Applied Mathematics 8. De Gruyter.

[18] Bouchard B., N. El Karoui and N. Touzi (2005): "Maturity randomization for stochastic control problems", Ann. Appl. Probab. Volume 15, Number 4, 2575-2605.

[19] Bouchaud J.P. (2010): Price Impact. Encyclopedia of Quantitative Finance.

[20] Bouchaud, J.P., Farmer, J.D. and F. Lillo (2008): "How markets slowly digest changes in supply and demand", Thorsten Hens, and Klaus Schenk-Hoppe, ed. : Handbook of Financial Markets : Dynamics and Evolution.

[21] Bouchaud, J.-P., M. Mézard and M. Potters (2002). Statistical properties of stock order books : empirical results and models. Mathematics ArXiv, http : //xxx.lanl.gov, cond-mat/0203511.

[22] Brekke K. and B. Oksendal (1994) : "Optimal switching in an economic activity under uncertainty", *SIAM J. Cont. Optim.*, **32**, 1021-1036.

[23] Brennan M. and E. Schwartz (1985): *Evaluating natural resource extraction*, J. Business, 58, 135-137.

[24] Budhiraja A., Ross K. (2007): Convergent Numerical Scheme for Singular Stochastic Control with State Constraints in a Portfolio Selection Problem. SIAM J. Control Optim., 45(6), 2169-2206.

[25] Cadenillas A. and F. Zapatero (1999) : "Optimal central bank intervention in the foreign exchange market", *Journal of Econ. Theory*, **97**, 218-242.

[26] Callegaro G., M. Gaigi, S. Scotti and C. Sgarra (2015): "Optimal Investment in Markets with Over and Under-Reaction to Information", preprint.

[27] Carmona R., Ludkovski M. (2010): Valuation of Energy Storage: An Optimal Switching Approach. Quantitative Finance, 10(4), 359-374.

[28] Carr P. (1998): "Randomization and the American put", Rev. Financ. Stud. (1998) 11 (3): 597-626.

[29] Cartea, A. and S. Jaimungal (2013): "Optimal Execution with Limit and Market Orders", *Quantitative Finance*, Forthcoming.

[30] Cartea, A and S. Jaimungal (2013): "Risk Metrics and Fine Tuning of High Frequency Trading Strategies", *Mathematical Finance*, Forthcoming

[31] Cartea, A., S. Jaimungal and J. Ricci (2014). "Buy low, sell high: A high frequency trading perspective." *SIAM Journal on Financial Mathematics,* **5**, 415-444.

[32] Cetin U., Jarrow R. and P. Protter (2004) : "Liquidity risk and arbitrage pricing theory", *Finance and Stochastics*, **8**, 311-341.

[33] Cetin U. and C. Rogers (2005) : "Modelling liquidity effects in discrete-time", to appear in *Mathematical Finance*.

[34] Chancelier J.P., Oksendal B. and A. Sulem (2001) : "Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control", *Stochastic Financial Mathematics*, Proc. Steklov Math. Inst. Moscou, 149-175, ed. A. Shiryaev.

[35] Choulli T. , Taksar M. and X.Y. Zhou (2003) : "A diffusion model for optimal dividend distribution for a company with constraints on risk control", *SIAM J. Cont. Optim.*, **41**, 1946-1979.

[36] Cont R. (2001) "Empirical properties of asset returns: stylized facts and statistical issues", *Quantitative Finance*, 1(2) p.223-236.

[37] Cont, R., Stoikov, S. and R. Talreja (2009): "A stochastic model for order book dynamics", Working paper.

[38] Crandall M., Ishii H. and P.L. Lions (1992) : "User's guide to viscosity solutions of second order partial differential equations", *Bull. Amer. Math. Soc.*, **27**, 1-67.

[39] Cuoco D. and J. Cvitanic (1998) : "Optimal consumption choice for a large investor", *Journal of Economic Dynamics and Control*, **22**, 401-436.

[40] Dai M., Zhang Q., Zhu Q. (2010): Trend Following Trading under a Regime Switching Model. SIAM Journal on Financial Mathematics 1, 780-810.

[41] Davis M. and A. Norman (1990) : "Portfolio selection with transaction costs", *Math. of Oper. Research*, **15**, 676-713.

[42] Davis M. and M. Zervos (1994) : "A Problem of Singular Stochastic Control with Discretionary Stopping", *Annals of Applied Probability*, **4**, pp.226-240.

[43] Dayanik S. and M. Egami (2012): "Optimal stopping problems for asset management", Adv. in Appl. Probab. Volume 44, Number 3, 655-677.

[44] Dayanik S. and I. Karatzas (2003) : "On the optimal stopping problem for one-dimensional diffusions", *Stochastic Process. Appl.*, **107**, 173–212.

[45] Dćamps J.P. and S. Villeneuve (2007): "Optimal dividend policy and growth option", *Finance and Stochastics*, **11**(1), 3-27.

[46] Dixit A. (1989) : "Entry and exit decisions under uncertainty", *J. Political Economy*, **97**, 620-638.

[47] Dixit A. and R. Pindick (1994) : Investment under uncertainty, Princeton University Press.

[48] Djehiche B., Hamadène S. and A. Popier (2007) : "A finite horizon optimal switching problem", preprint.

[49] Duckworth K. and M. Zervos (2001) : "A model for investment decisions with switching costs", *Annals of Applied Probability*, **11**, 239-250.

[50] Dynkin E. B. (1965), Markov Processes, Vol. II, Springer Verlag.

[51] El Karoui N. (1981) : Les aspects probabilistes du contrôle stochastique, Lect. Notes in Math., Springer Verlag.

[52] Fleming W. and M. Soner (1993) : Controlled Markov processes and viscosity solutions, Springer Verlag, New York.

[53] Frey R. (1998) : "Perfect option hedging for a large trader", *Finance and Stochastics*, **2**, 115-141.

[54] Gilbarg D. and N. Trudinger (1977) : Elliptic partial differential equations of second order, Springer Verlag, Berlin.

[55] Glosten L., Milgrom P. (1985): Bid, Ask and transaction prices in a specialist market with heterogeneously informed traders. Journal of Financial Economics, **14**, 71-100.

[56] Guéant, O., and C. Lehalle (2013): "General Intensity Shapes In Optimal Liquidation", *Mathematical Finance*, Forthcoming.

[57] Guéant, O., C. Lehalle, and J. Tapia (2011): "Optimal portfolio liquidation with limit orders", *SIAM J. Finan. Math.*, **3**, 740-764.

[58] Guéant O., Lehalle C., Tapia J. (2013): Dealing with the Inventory Risk A solution to the market making problem. Mathematics and Financial Economics, Volume 7, Issue 4, pp 477-507.

[59] Guilbaud, F. and H. Pham (2013): "Optimal high frequency trading with limit and market orders", *Quantitative Finance*, **13**, 79-94.

[60] Guo X. and P. Tomecek (2007) : "Connections between singular control and optimal switching", to appear in *SIAM J. Cont. Optim.*

[61] Guo X. and Q. Zhang (2004): "Closed-form solutions for perpetual american put options with regime switching", SIAM J. APPL. Math., Vol. 64, No. 9, pp. 2034-2049.

[62] Hamadène S. and M. Jeanblanc (2007) : "On the starting and stopping problem : application in reversible investment", *Math. Oper. Res.*, **32**, 182-192.

[63] Hansen, P.R. and A. Lunde (2006): "Realized Variance and Market Microstructure Noise", *Journal of Business and Economics Statistics*, 24, p.127-161.

[64] He H. and H. Mamaysky (2005) : "Dynamic trading policies with price impact", *Journal of Economic Dynamics and Control*, **29**, 891-930.

[65] Hindy A., Huang C., Zhu H. (1993): Numerical Analysis of a Free-Boundary Singular Control Problem in Financial Economics. J. Econom. Dynam. Control, 21, 297-327.

[66] Ho T., Stoll H.R. (1981): Optimal dealer pricing under transactions and return uncertainty. Journal of Financial Economics, **9(1)**, 47-73.

[67] Hu Y. and S. Tang (2007) : "Multi-dimensional BSDE with oblique reflection and optimal switching", preprint.

[68] Ishii K. (1993) : "Viscosity solutions of nonlinear second order elliptic PDEs associated with impluse control problems", *Funkcial. Ekvac.*, **36**, 123-141.

[69] Ishii H. and P.L. Lions (1990) : "Viscosity solutions of fully nonlinear second order elliptic partial differential equations", *Journal of Differential equations*, 83, 26-78.

[70] Jacod J., Shiryaev A. N. (2003): Limit Theorems for Stochastic Processes. Springer-Verlag.

[71] Jeanblanc M. and A. Shiryaev (1995) : "Optimization of the flow of dividends", *Russian Math Surveys*, **50**, 257-277.

[72] Jin Z., G. Yin, Zhu C. (2012): Numerical solutions of optimal risk control and dividend optimization policies under a generalized singular control formulation. Automatica, 48(8), 1489-1501 .

[73] Jouini E. and H. Kallal (1995) : "Martingale and arbitrage in securities markets with transaction costs", *Journal of Econ. Theory*, **66**, 178-197.

[74] Korn R. (1998) : "Portfolio optimization with strictly positive transaction costs and impulse control", *Finance and Stochastics*, **2**, 85-114.

[75] Kühn, C., and Muhle-Karbe, J. (2014). Optimal liquidity provision in limit order markets. Swiss Finance Institute Research Paper, No. 13-71.

[76] Kushner, H., Dupuis, P. (2001): Numerical Methods for Stochstic Control Problems in Continuous Time. Stochastic Modelling and Applied Probability, Volume 24, Springer, New York second edition.

[77] Kyle A. (1985) : "Continuous auctions and insider trading", *Econometrica*, **53**, 1315-1335.

[78] Lamberton D. and M. Zervos (2013): "On the optimal stopping of a one-dimensional diffusion", Electron. J. Probab. 18, No. 34, 1-49.

[79] Leung T., Li X., Wang Z. (2014): "Optimal Multiple Trading Times Under the Exponential OU Model withTransaction Costs" (September 8, 2014). Available at SSRN: http://ssrn.com/abstract=2493492.

[80] Lo A., Mamayski H. and J. Wang (2004) : "Asset prices and trading volume under fixed transcation costs", *J. Political Economy*, 112, 1054-1090.

[81] Lokka A. and M. Zervos (2013): "Long-term optimal investment strategies in the presence of adjustment costs." *SIAM Journal on Control and Optimization*, **51**, 996-1034.

[82] Lon P. C., Zervos M. (2011): A model for optimally advertising and launching a product. Mathematics of Operations Research, 36, 363-376.

[83] Lumley R. and M. Zervos (2001): "A model for investments in the natural resource industry with switching costs." *Mathematics of Operations Research* **26**, 637-653.

[84] McDonald R. and D. Siegel (1986): "The Value of Waiting to Invest." *The Quarterly Journal of Economics* **101**, 707-727.

[85] Mildenstein E., Schleef H. (1983): The optimal pricing of a monopolistic marketmaker in the equity market. The Journal of Finance, **38-1**, 218-231.

[86] Morton A. and S. Pliska (1995) : "Optimal portfolio management with fixed transaction costs", *Mathematical Finance*, 5, 337-356.

[87] Nyström, K., S. M. Ould Aly, and C. Zhang (2014): "Market Making And Portfolio Liquidation Under Uncertainty", *International Journal of Theoretical and Applied Finance*, **17**, 1450034 (33 pages).

[88] Obizhaeva, A. and J. Wang (2013): "Optimal Trading Strategy and Supply/Demand Dynamics", *Journal of Financial Markets*, **16**, 1-32.

[89] Oksendal B. (2000): "Irreversible investment problems" *Finance and Stochastics*, **4**, 223-250.

[90] Oksendal B. (2003) Stochastic Differential Equations, 6th ed., Springer, New York.

[91] Oksendal, B. and A. Sulem. (2007): Applied Stochastic Control of Jump Diffusions, Universitext, Springer Verlag, Berlin.

[92] Oksendal B. and A. Sulem (2002) : "Optimal consumption and portfolio with both fixed and proportional transaction costs", *SIAM J. Cont. Optim.*, **40**, 1765-1790.

[93] Papanicolaou G. and R. Sircar (1998) : "General Black-Scholes models accounting for increased market volatility from hedging strategies", *Applied Mathematical Finance*, **5**, 45-82.

[94] Platen E. and M. Schweizer (1998) : "On feedback effects from hedging derivatives", *Mathematical Finance*, 8, 67-84.

[95] Pham H. (2007) : "On the smooth-fit property for one-dimensional optimal switching problem", *Séminaire de Probabilités*, Vol. XL, 187-201.

[96] Radner R. and L. Shepp (1996) : "Risk vs. profit potential : a model of corporate strategy", *Journal of economic dynamics and Control*, **20**, 1373-1393.

[97] Roch, A. (2011): "Liquidity risk, price impacts and the replication problem", *Finance and Stochastics*, **15**, 399-419.

[98] Roch, A. and M. Soner (2013): "Resilient price impact of trading and the cost of illiquidity", *International Journal of Theoretical and Applied Finance* **16**, 1-27.

[99] Rosenbaum, M. (2010): "A new microstructure noise index", Quantitative Finance, to appear.

[100] Shreve S., Lehoczky J.P. and D. Gaver (1984) : "Optimal consumption for general diffusions with absorbing and reflecting barriers", *SIAM J. Cont. Optim.* , **22**, 55-75.

[101] Shreve, S. E., and H. M. Soner (1994): "Optimal investment and consumption with transaction costs" *The Annals of Applied Probability* **4**, 609-692.

[102] Soner H. (1986) : "Optimal control with state-space constraint, I and II", *SIAM J. Cont. Optim.*, **24**, 552-561, and 1110-1122.

[103] Subramanian A. and R. Jarrow (2001) : "The liquidity discount", *Mathematical Finance*, **11**, 447-474.

[104] Tang S. and J. Yong (1993) : "Finite horizon stochastic optimal switching and impulse controls with a viscosity solution approach", *Stoch. and Stoch. Reports*, 45, 145-176.

[105] Tricomi F. G. (1960): Fonctions hypergéométriques confluentes, Gauthier-Villars, Paris, France.

[106] Vayanos D. (1998) : "Transaction costs and asset prices : a dynamic equilibrium model", *Rev. Fin. Studies*, 11, 1-58.

[107] Villeneuve S. (2007): "On the Threshold Strategies and Smooth-Fit Principle for Optimal Stopping Problems", *Journal of Applied Probability*, **44**, 181-198.

[108] Villeneuve S. and X. Warin (2014): "Optimal Liquidity management and Hedging in the presence of a non-predictable investment opportunity." *Mathematics and Financial Economics* **8**, 193-227.

[109] Zariphopoulou T. (1988) : Optimal investment-consumption models with constraints, Phd Thesis, Brown University.

[110] Zervos M. (2003) : "A problem of sequential entry and exit decisions combined with discretionary stopping", *SIAM J. Cont. Optim.*, 42, 397-421.

[111] Zervos M. , T. Johnson, and Alazemi (2013): "Buy-low and sell-high investment strategies", Mathematical Finance, 23: 560-578. doi: 10.1111/j.1467-9965.2011.00508.x.