A Model of Optimal Portfolio Selection under Liquidity Risk and Price Impact*

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Abstract

We study a financial model with one risk-free and one risky asset subject to liquidity risk and price impact. In this market, an investor may transfer funds between the two assets at any discrete time. Each purchase or sale policy decision affects the price of the risky asset and incurs some fixed transaction cost. The objective is to maximize the expected utility from terminal liquidation value over a finite horizon and subject to a solvency constraint. This is formulated as an impulse control problem under state constraint and we characterize the value function as the unique constrained viscosity solution to the associated quasi-variational Hamilton-Jacobi-Bellman inequality.

Key words : portfolio selection, liquidity risk, impulse control, state constraint, discontinuous viscosity solutions.

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1 Introduction

Classical market models in mathematical finance assume perfect elasticity of traded assets : traders act as price takers, so that they buy and sell with arbitrary size without changing the price. However, the market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying assets. Moreover, in practice, investors face trading strategies constraints, typically of finite variation, and they cannot rebalance them continuously. We then usually speak about liquidity risk or illiquid markets. While the assumption of perfect liquidity market may not be practically important over a very long term horizon, price impact can have a significant difference over a short time horizon.

Several suggestions have been proposed to formalize the liquidity risk. In [25] and [3], the impact of trading strategies on prices is explained by the presence of an insider. In the market manipulation literature, prices are assumed to depend directly on the trading strategies. For instance, the paper [12] considers a diffusion model for the price dynamics with coefficients depending on the large investor's strategy, while [17], [30], [29], [6] or [8] develop a continuous-time model where prices depend on strategies via a reaction function. While the assumption of price-taker may not be practically important for investors making allocation decision over a very long time horizon, price impact can make a significant difference when investors execute large trades over a short time of horizon. The market microstructure literature has shown both theoretically and empirically that large trades move the price of the underlying securities. Moreover, it is also well established that transaction costs in asset markets are an important factor in determining the trading behavior of market participants; we mention among others [14] and [23] for the literature on arbitrage and optimal trading policies, and [34], [26] for the literature on the impact of transaction costs on agents' economic behavior. Consequently, transaction costs should affect market liquidity and asset prices. This is the point of view in the academic literature where liquidity is defined in terms of the bid-ask spread and/or transaction costs associated with a trading strategy. On the other hand, in the practitioner literature, illiquidity is often viewed as the risk that a trader may not be able to extricate himself from a position quickly when need arises. Such a situation occurs when continuous trading is not permitted, for instance, because of fixed transaction costs.

Of course, in actual markets, both aspects of market manipulation and transaction costs are correct and occur simultaneously. In this paper, we propose a model of liquidity risk and price impact that adopts both these perspectives. Our model is inspired from the recent papers [32] and [19], and may be described roughly as follows. Trading on illiquid assets is not allowed continuously due to some fixed costs but only at any discrete times. These liquidity constraints on strategies are in accordance with practitioner literature and consistent with the academic literature on fixed transaction costs, see e.g. [27]. There is an investor, who is large in the sense that his strategies affect asset prices : prices are pushed up when buying stock shares and moved down when selling shares. In this context, we study an optimal portfolio choice problem over a finite horizon : the investor maximizes his expected utility from terminal liquidation wealth and under a natural economic solvency constraint. In some sense, our problem may be viewed as a continuous-time version of the recent discrete-time one proposed in [9] . We mention also the paper [2], which studies an optimal trade execution problem in a discrete time setting with permanent and temporary market impact.

Our optimization problem is formulated as a parabolic impulse control problem with three variables (besides time variable) related to the cash holdings, number of stock shares and price. This problem is known to be associated by the dynamic programming principle to a Hamilton-Jacobi-Bellman (HJB) quasi-variational inequality, see [5]. We refer to [22], [24], [7] or [28] for some recent papers involving applications of impulse controls in finance, mostly over an infinite horizon and in dimension 1, except [24] and [28] in dimension 2. There is in addition, in our context, an important aspect related to the economic solvency condition requiring that liquidation wealth is nonnegative, which is translated into a state constraint involving a nonsmooth boundary domain. The model and the detailed description of the liquidation value and solvency region, and its formulation as an impulse control problem are exposed in Section 2. Our main goal is to obtain a rigorous characterization result on the value function through the associated HJB quasi-variational inequality. The main result is formulated in Section 3.

The features of our stochastic control problem make appear several technical difficulties related to the nonlinearity of the impulse transaction function and the solvency constraint. In particular, the liquidation net wealth may grow after transaction, which makes nontrivial the finiteness of the value function. Hence, the Merton bound does not provide as e.g. in transaction cost models, a natural upper bound on the value function. Instead, we provide a suitable "linearization" of the liquidation value that provides a sharp upper bound of the value function. The solvency region (or state domain) is not convex and its boundary even not smooth, in contrast with transaction cost model (see [14]), so that continuity of the value function is not direct. Moreover, the boundary of the solvency region is not absorbing as in transaction cost models and singular control problems, and the value function may be discontinuous on some parts of the boundary. Singularity of our impulse control problem appears also at the liquidation date, which translates into discontinuity of the value function at the terminal date. These properties of the value function are studied in Section 4.

In our general set-up, it is then natural to consider the HJB equation with the concept of (discontinuous) viscosity solutions, which provides by now a well established method for dealing with stochastic control problems, see e.g. the book [16]. More precisely, we need to consider constrained viscosity solutions to handle the state constraints. Our first main result is to prove that the value function is a constrained viscosity solution to its associated HJB quasi-variational inequality. Our second main result is a new comparison principle for the state constraint HJB quasi-variational inequality, which ensures a PDE characterization for the value function of our problem. Previous comparison results derived for variational inequality (see [20], [33]) associated to impulse problem do not apply here. In our context, we prove that one can compare a subsolution with a supersolution to the HJB quasi-variational inequality provided that one can compare them at the terminal date (as usual in parabolic problems) but also on some part D_0 of the solvency boundary, which represents an original point in comparison principle for state-constraint problem. Section

5 is devoted to the PDE viscosity characterization of the value function. We conclude in Section 6 with some remarks.

2 The Model

This section presents the details of the model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ supporting an one-dimensional Brownian motion W on a finite horizon $[0,T]$, $T < \infty$. We consider a continuous time financial market model consisting of a money market account yielding a constant interest rate $r \geq 0$ and a risky asset (or stock) of price process $P = (P_t)$. We denote by X_t the amount of money (or cash holdings) and by Y_t the number of shares in the stock held by the investor at time t.

Liquidity constraints. We assume that the investor can only trade discretely on $[0, T)$. This is modelled through an impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n \geq 1}$: $\tau_1 \leq \ldots \tau_n \leq \ldots < T$ are stopping times representing the intervention times of the investor and ζ_n, $n \geq 1$, are $\mathcal{F}_{\tau_n}$-measurable random variables valued in $\mathbb{R}$ and giving the number of stock purchased if $\zeta_n \geq 0$ or selled if $\zeta_n < 0$ at these times. The sequence (τ_n, ζ_n) may be a priori finite or infinite. The dynamics of Y is then given by :

$$Y_s = Y_{\tau_n}, \quad \tau_n \leq s < \tau_{n+1} \tag{2.1}$$

$$Y_{\tau_{n+1}} = Y_{\tau_n} + \zeta_{n+1}. \tag{2.2}$$

Notice that we do not allow trade at the terminal date T, which is the liquidation date.

Price impact. The large investor affects the price of the risky stock P by his purchases and sales : the stock price goes up when the trader buys and goes down when he sells and the impact is increasing with the size of the order. We then introduce a price impact positive function $Q(\zeta, p)$ which indicates the post-trade price when the large investor trades a position of ζ shares of stock at a pre-trade price p. In absence of price impact, we have $Q(\zeta, p) = p$. Here, we have $Q(0, p) = p$ meaning that no trading incurs no impact and Q is nondecreasing in ζ with $Q(\zeta, p) \geq$ (resp. $\leq$) p for $\zeta \geq$ (resp. $\leq$) 0. Actually, in the rest of the paper, we consider a price impact function in the form

$$Q(\zeta, p) = pe^{\lambda\zeta}, \qquad \text{where } \lambda > 0. \tag{2.3}$$

The proportionality factor $e^{\lambda\zeta}$ represents the price increase (resp. discount) due to the ζ shares bought (resp. sold). The positive constant λ measures the fact that larger trades generate larger quantity impact, everything else constant. This form of price impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see [25]).

We then model the dynamics of the price impact as follows. In the absence of trading, the price process is governed by

$$dP_s = P_s(bds + \sigma dW_s), \quad \tau_n \leq s < \tau_{n+1}, \tag{2.4}$$

where b, σ are constants with $\sigma > 0$. When a discrete trading $\Delta Y_s := Y_s - Y_{s^-} = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$, the price jumps to $P_s = Q(\Delta Y_s, P_{s^-})$, i.e.

$$P_{\tau_{n+1}} = Q(\zeta_{n+1}, P_{\tau_{n+1}^-}). \tag{2.5}$$

Notice that with this modelling of price impact, the price process P is always strictly positive, i.e. valued in $\mathbb{R}_+^* = (0, \infty)$.

Cash holdings. We denote by $\theta(\zeta, p)$ the cost function, which indicates the amount for a (large) investor to buy or sell ζ shares of stock when the pre-trade price is p :

$$\theta(\zeta, p) \quad = \quad \zeta Q(\zeta, p).$$

In absence of transactions, the process X grows deterministically at exponential rate r :

$$dX_s \quad = \quad rX_s ds, \qquad \tau_n \leq s < \tau_{n+1}. \tag{2.6}$$

When a discrete trading $\Delta Y_s = \zeta_{n+1}$ occurs at time $s = \tau_{n+1}$ with pretrade price $P_{s^-} = P_{\tau_{n+1}^-}$, we assume that in addition to the amount of stocks $\theta(\Delta Y_s, P_{s^-}) = \theta(\zeta_{n+1}, P_{\tau_{n+1}^-})$, there is a fixed cost $k > 0$ to be paid. This results in a variation of cash holdings by $\Delta X_s := X_s - X_{s^-} = -\theta(\Delta Y_s, P_{s^-}) - k$, i.e.

$$X_{\tau_{n+1}} \quad = \quad X_{\tau_{n+1}^-} - \theta(\zeta_{n+1}, P_{\tau_{n+1}^-}) - k. \tag{2.7}$$

The assumption that any trading incurs a fixed cost of money to be paid will rule out continuous trading, i.e. optimally, the sequence (τ_n, ζ_n) is not degenerate in the sense that for all n, $\tau_n < \tau_{n+1}$ and $\zeta_n \neq 0$ a.s. A similar modelling of fixed transaction costs is considered in [27] and [24].

Liquidation value and solvency constraint. The solvency constraint is a key issue in portfolio/consumption choice problem. The point is to define in an economically meaningful way what is the portfolio value of a position in cash and stocks. In our context, we introduce the liquidation function $\ell(y, p)$ representing the value that an investor would obtained by liquidating immediately his stock position y by a single block trade, when the pre-trade price is p. It is given by :

$$\ell(y, p) \quad = \quad -\theta(-y, p).$$

If the agent has the amount x in the bank account, the number of shares y of stocks at the pre-trade price p, i.e. a state value $z = (x, y, p)$, his net wealth or liquidation value is given by :

$$L(z) \quad = \quad \max[L_0(z), L_1(z)] 1_{y \geq 0} + L_0(z) 1_{y < 0}, \tag{2.8}$$

where

$$L_0(z) \ = \ x + \ell(y, p) - k, \qquad L_1(z) \ = \ x.$$

The interpretation is the following. $L_0(z)$ corresponds to the net wealth of the agent when he liquidates his position in stock. Moreover, if he has a long position in stock, i.e. $y \geq 0$, he can also choose to bin his stock shares, by keeping only his cash amount, which leads to a net wealth $L_1(z)$. This last possibility may be advantageous, i.e. $L_1(z) \geq L_0(z)$, due to the fixed cost k. Hence, globally, his net wealth is given by (2.8). In the absence of liquidity risk, i.e $\lambda = 0$, and fixed transaction cost, i.e. $k = 0$, we recover the usual definition of wealth $L(z) = x + py$. Our definition (2.8) of liquidation value is also consistent with the one in transaction costs models where portfolio value is measured after stock position is liquidated and rebalanced in cash, see e.g. [13] and [28]. Another alternative would be to measure the portfolio value separately in cash and stock as in [15] for transaction costs models. This study would lead to multidimensional utility functions and is left for future research.

We then naturally introduce the liquidation solvency region (see Figure 1) :

$$\mathcal{S} \;=\; \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: L(z) > 0\big\},$$

and we denote its boundary and its closure by

$$\partial\mathcal{S} \;=\; \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: L(z) = 0\big\} \quad \text{and} \quad \bar{\mathcal{S}} \;=\; \mathcal{S} \cup \partial\mathcal{S}.$$

Remark 2.1 The function L is clearly continuous on $\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: y \neq 0\}$. It is discontinuous on $z_0 = (x, 0, p) \in \bar{\mathcal{S}}$, but it is easy to check that it is upper-semicontinuous on z_0, so that globally L is upper-semicontinuous. Hence $\bar{\mathcal{S}}$ is closed in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$. We also notice that L is nonlinear in the state variables, which contrasts with transaction costs models.

Remark 2.2 For any $p > 0$, the function $y \mapsto \ell(y, p) = pye^{-\lambda y}$ is increasing on $[0, 1/\lambda]$, decreasing on $[1/\lambda, \infty)$ with $l(0, p) = \lim_{y\to\infty} l(y, p) = 0$ and $l(1/\lambda, p) = pe^{-1}/\lambda$. We then distinguish the two cases :

⋆ if $p < k\lambda e$, then $l(y, p) < k$ for all $y \geq 0$.

⋆ if $p \geq k\lambda e$, then there exists an unique $y_1(p) \in (0, 1/\lambda]$ and $y_2(p) \in [1/\lambda, \infty)$ such that $l(y_1(p), p) = l(y_2(p), p) = k$ with $l(y, p) < k$ for all $y \in [0, y_1(p)) \cup (y_2(p), \infty)$. Moreover, $y_1(p)$ (resp. $y_2(p)$) decreases to 0 (resp. increases to ∞) when p goes to infinity, while $y_1(p)$ (resp.$y_2(p)$) increases (resp. decreases) to $1/\lambda$ when p decreases to $k\lambda e$.

The boundary of the solvency region may then be explicited as follows (see Figures 2 and 3) :

$$\partial\mathcal{S} \;=\; \partial_\ell^-\mathcal{S} \cup \partial^y\mathcal{S} \cup \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S} \cup \partial_2^x\mathcal{S} \cup \partial_\ell^+\mathcal{S},$$

where

$$\begin{aligned}
\partial_\ell^-\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: x + \ell(y, p) = k,\ y \leq 0\big\} \\
\partial^y\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: 0 \leq x < k,\ y = 0\big\} \\
\partial_0^x\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: x = 0,\ y > 0,\ p < k\lambda e\big\} \\
\partial_1^x\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: x = 0,\ 0 < y < y_1(p)),\ p \geq k\lambda e\big\} \\
\partial_2^x\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: x = 0,\ y > y_2(p),\ p \geq k\lambda e\big\} \\
\partial_\ell^+\mathcal{S} &= \big\{z = (x, y, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^* \;: x + \ell(y, p) = k,\ y_1(p) \leq y \leq y_2(p),\ p \geq k\lambda e\big\}.
\end{aligned}$$

In the sequel, we also introduce the corner lines in $\partial\mathcal{S}$:

$$\begin{aligned}
D_0 &= \{(0, 0)\} \times \mathbb{R}_+^* \;\subset\; \partial^y\mathcal{S}, \qquad D_k = \{(k, 0)\} \times \mathbb{R}_+^* \;\subset\; \partial_\ell^-\mathcal{S} \\
C_1 &= \{(0, y_1(p), p) \;: p \in \mathbb{R}_+^*\} \;\subset\; \partial_\ell^+\mathcal{S}, \qquad C_2 = \{(0, y_2(p), p) \;: p \in \mathbb{R}_+^*\} \;\subset\; \partial_\ell^+\mathcal{S}.
\end{aligned}$$

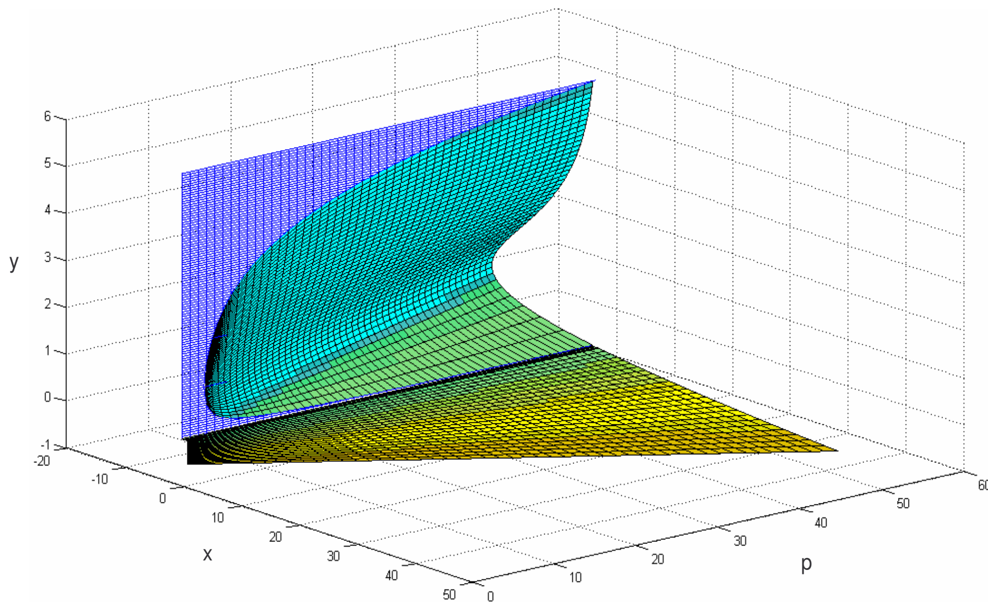

Figure 1: The solvency region when k = 1, λ = 1

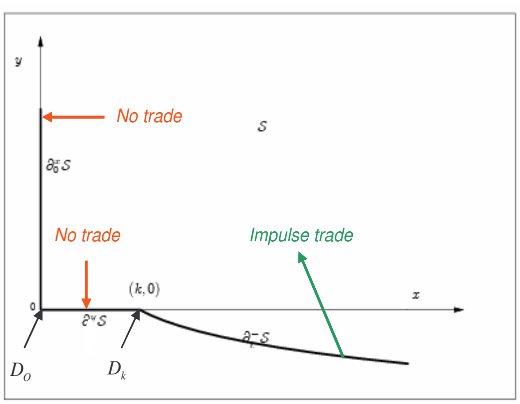

Figure 2: The solvency region when p < kλe

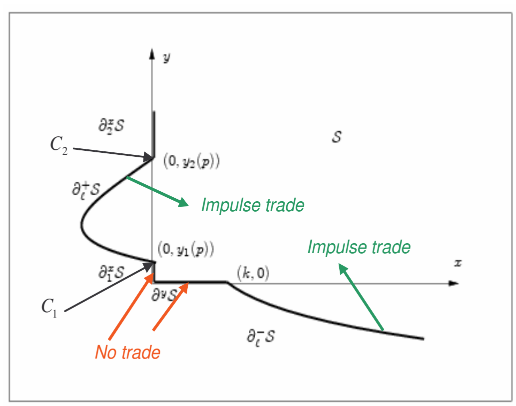

Figure 3: The solvency region when p > kλe

Admissible controls. Given $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$ and an initial state $Z_{t^-} = z$, we say that the impulse control strategy $\alpha = (\tau_n, \zeta_n)_{n\geq 1}$ is admissible if the process $Z_s = (X_s, Y_s, P_s)$ given by (2.1)-(2.2)-(2.4)-(2.5)-(2.6)-(2.7) (with the convention $\tau_0 = t$) lies in $\bar{\mathcal{S}}$ for all $s \in [t,T]$. We denote by $\mathcal{A}(t,z)$ the set of all such policies. We shall see later that this set of admissible controls is nonempty for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$.

Remark 2.3 We recall that we do not allow intervention time at T, which is the liquidation date. This means that for all $\alpha \in \mathcal{A}(t,z)$, the associated state process Z is continuous at T, i.e. $Z_{T^-} = Z_T$.

In the sequel, for $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we also denote $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$, $t \leq s \leq T$, the state process when no transaction (i.e. no impulse control) is applied between t and T, i.e. the solution to :

$$dZ_s^0 \quad = \quad \begin{pmatrix} rX_s^0 \\ 0 \\ bP_s^0 \end{pmatrix} ds + \begin{pmatrix} 0 \\ 0 \\ \sigma P_s^0 \end{pmatrix} dW_s, \tag{2.9}$$

starting from z at time t.

Investment problem. We consider an utility function U from $\mathbb{R}_+$ into $\mathbb{R}$, strictly increasing, concave and w.l.o.g. $U(0) = 0$, and s.t. there exist $K \geq 0$, $\gamma \in [0,1)$:

$$U(w) \quad \leq \quad Kw^\gamma, \quad \forall w \geq 0, \tag{2.10}$$

We denote U_L the function defined on $\bar{\mathcal{S}}$ by :

$$U_L(z) \quad = \quad U(L(z)).$$

We study the problem of maximizing the expected utility from terminal liquidation wealth and we then consider the value function :

$$v(t,z) \quad = \quad \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[U_L(Z_T)\right], \qquad (t,z) \in [0,T] \times \bar{\mathcal{S}}. \tag{2.11}$$

Remark 2.4 We shall see later that for all $\alpha \in \mathcal{A}(t,z) \neq \emptyset$, $U_L(Z_T)$ is integrable so that the expectation in (2.11) is well-defined. Since U is nonnegative and nondecreasing, we immediately get a lower bound for the value function :

$$v(t,z) \quad \geq \quad U(0) = 0, \qquad \forall t \in [0,T],\ z = (x,y,p) \in \bar{\mathcal{S}}.$$

We shall also see later that the value function v is finite in $[0,T] \times \bar{\mathcal{S}}$ by providing a sharp upper bound.

Notice that in contrast to financial models without frictions or with proportional transaction costs, the dynamics of the state process $Z = (X,Y,P)$ is nonlinear and then the value function v does not inherit the concavity property of the utility function. The solvency region is even not convex. In particular, one cannot derive as usual the continuity of the value function as a consequence of the concavity property. Moreover, for power-utility functions $U(w) = Kw^\gamma$, the value function does not inherit the homogeneity property of the utility function.

We shall adopt a dynamic programming approach to study this utility maximization problem. We end this section by recalling the dynamic programming principle for our stochastic control problem.

Dynamic programming principle (DPP). For all $(t, z) \in [0, T) \times \bar{\mathcal{S}}$, we have

$$v(t,z) = \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[v(\tau, Z_\tau)\right], \tag{2.12}$$

where $\tau = \tau(\alpha)$ is any stopping time valued in $[t, T]$ depending on α in (2.12). The precise meaning is :

(i) for all $\alpha \in \mathcal{A}(t, z)$, for all $\tau \in \mathcal{T}_{t,T}$, set of stopping times valued in $[t, T]$:

$$\mathbb{E}[v(\tau, Z_\tau)] \leq v(t,z) \tag{2.13}$$

(ii) for all $\varepsilon > 0$, there exists $\hat{\alpha}^\varepsilon \in \mathcal{A}(t, z)$ s.t. for all $\tau \in \mathcal{T}_{t,T}$:

$$v(t,z) \leq \mathbb{E}[v(\tau, \hat{Z}^\varepsilon_\tau)] + \varepsilon. \tag{2.14}$$

Here $\hat{Z}^\varepsilon$ denotes the state process starting from z at t and controlled by $\hat{\alpha}^\varepsilon$.

3 Quasi-variational Hamilton-Jacobi-Bellman inequality and the main result

In this section, we introduce some notations, recall the dynamic programming quasi-variational inequality associated to the impulse control problem (2.11) and formulate the main result.

We define the impulse transaction function from $\bar{\mathcal{S}} \times \mathbb{R}$ into $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^*_+$:

$$\Gamma(z,\zeta) = (x - \theta(\zeta,p) - k, y + \zeta, Q(\zeta,p)), \quad z = (x,y,p) \in \bar{\mathcal{S}},\ \zeta \in \mathbb{R},$$

This corresponds to an immediate trading at time t of ζ shares of stock, so that from (2.2)-(2.5)-(2.7) the state process jumps from $Z_{t^-} = z \in \bar{\mathcal{S}}$ to $Z_t = \Gamma(z, \zeta)$. We then consider the set of admissible transactions :

$$\mathcal{C}(z) = \left\{\zeta \in \mathbb{R} \ : \Gamma(z,\zeta) \in \bar{\mathcal{S}}\right\} = \left\{\zeta \in \mathbb{R} \ : L(\Gamma(z,\zeta)) \geq 0\right\},$$

in accordance with the solvency constraint and the set of admissibles controls $\mathcal{A}(t, z)$. We introduce the impulse operator $\mathcal{H}$ defined by :

$$\mathcal{H}\varphi(t,z) = \sup_{\zeta \in \mathcal{C}(z)} \varphi(t, \Gamma(z,\zeta)), \quad (t,z) \in [0,T] \times \bar{\mathcal{S}},$$

for any measurable function φ on $[0, T] \times \bar{\mathcal{S}}$. If for some $z \in \bar{\mathcal{S}}$, the set $\mathcal{C}(z)$ is empty, we denote by convention $\mathcal{H}\varphi(t, z) = -\infty$.

We also define $\mathcal{L}$ as the infinitesimal generator associated to the system (2.9) corresponding to a no-trading period :

$$\mathcal{L}\varphi = rx\frac{\partial \varphi}{\partial x} + bp\frac{\partial \varphi}{\partial p} + \frac{1}{2}\sigma^2 p^2 \frac{\partial^2 \varphi}{\partial p^2}.$$

The HJB quasi-variational inequality arising from the dynamic programming principle (2.12) is then written as :

$$\min\left[-\frac{\partial v}{\partial t} - \mathcal{L}v\ ,\ v - \mathcal{H}v\right] = 0, \quad \text{on} \quad [0,T)\times\mathcal{S}. \tag{3.1}$$

This divides the time-space liquidation solvency region $[0,T)\times\mathcal{S}$ into a *no-trade region*

$$\mathbf{NT} = \{(t,z)\in[0,T)\times\mathcal{S}\ :\ v(t,z) > \mathcal{H}v(t,z)\},$$

and a *trade region*

$$\mathbf{T} = \{(t,z)\in[0,T)\times\mathcal{S}\ : v(t,z) = \mathcal{H}v(t,z)\}.$$

The rigorous characterization of the value function through the quasi-variational inequality (3.1) together with the boundary and terminal conditions is stated by means of constrained viscosity solutions. Our main result is the following theorem, which follows from the results proved in Sections 4 and 5.

Theorem 3.1 *The value function v is continuous on $[0,T)\times\mathcal{S}$ and is the unique (in $[0,T)\times\mathcal{S}$) constrained viscosity solution to* (3.1) *satisfying the boundary and terminal condition :*

$$\lim_{\substack{(t',z')\to(t,z)\\ z'\in\mathcal{S}}} v(t',z') = 0, \qquad \forall (t,z)\in[0,T)\times D_0 \tag{3.2}$$

$$\lim_{\substack{(t,z')\to(T,z)\\ t<T,\, z'\in\mathcal{S}}} v(t,z') = \max[U_L(z),\mathcal{H}U_L(z)], \qquad \forall z\in\bar{\mathcal{S}}, \tag{3.3}$$

and the growth condition :

$$|v(t,z)| \le K\left(1+\left(x+\frac{p}{\lambda}\right)\right)^{\gamma}, \quad \forall (t,z)\in[0,T)\times\mathcal{S} \tag{3.4}$$

for some positive constant $K<\infty$.

Remark 3.1 Continuity and uniqueness of the value function for the HJBQVI (3.1) hold true in $[0,T)\times\mathcal{S}$ in the class of functions satisfying the growth condition (3.4), associated to the terminal condition (3.3) (as usual in parabolic problems) but also to some specific boundary condition (3.2). This last point is nonstandard in constrained control problems, where one gets usually an uniqueness result for constrained viscosity solutions to the corresponding Bellman equation without any additional boundary condition, see e.g. [35] or [28]. Here, we have to impose a boundary condition on the nonsmooth part D_0 of the solvency boundary. Notice also that the terminal condition is not given by U_L. Actually, it takes into account the fact that just before the liquidation date T, one can do an impulse transaction : the effect is to lift-up the utility function U_L through the impulse transaction operator $\mathcal{H}$.

4 Properties of the value function

4.1 Some properties on the impulse transactions set

In order to show that the value function of problem (2.11) is finite, which is not trivial a priori, we need to derive some preliminary properties on the set of admissible transactions $\mathcal{C}(z)$. Starting from a current state $z = (x, y, p) \in \bar{\mathcal{S}}$, an immediate transaction of size ζ leads to a new state $z' = (x', y', p') = \Gamma(z, \zeta)$. Recalling the expression (2.3) of the price impact function, we then have :

$$
\begin{aligned}
L_0(\Gamma(z,\zeta)) \;=\; x' + \ell(y',p') - k &\;=\; x + \ell(y,p) - k + p\zeta(e^{-\lambda y} - e^{\lambda\zeta}) - k \\
&\;=\; L_0(z) + pg(y,\zeta) - k, \qquad (4.1)
\end{aligned}
$$

with

$$
g(y,\zeta) \;=\; \zeta(e^{-\lambda y} - e^{\lambda\zeta}). \qquad (4.2)
$$

It then appears that due to the nonlinearity of the price impact function, and in contrast with transaction costs models, the net wealth may grow after some transaction : $L(\Gamma(z,\zeta))$ $> L(z)$ for some $z \in \bar{\mathcal{S}}$ and $\zeta \in \mathcal{C}(z)$. We first state the following useful result.

Lemma 4.1 *For all $z \in \bar{\mathcal{S}}$, the set $\mathcal{C}(z)$ is compact, eventually empty. We have :*

$$
\begin{aligned}
\mathcal{C}(z) &\;=\; \emptyset \quad \text{if } z \in \partial^y\mathcal{S} \cup \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S}, \\
-\frac{1}{\lambda} \in \mathcal{C}(z) &\;\subset\; (-y, 0) \quad \text{if } z \in \partial_2^x\mathcal{S}, \\
-y \in \mathcal{C}(z) &\;\subset\; \begin{cases} [0, -y] & \text{if } z \in \partial_\ell^-\mathcal{S} \\ [-y, 0) & \text{if } z \in \partial_\ell^+\mathcal{S} \end{cases}
\end{aligned}
$$

Moreover,

$$
\mathcal{C}(z) \;=\; \{-y\} \quad \text{if } z \in (\partial_\ell^-\mathcal{S} \cup \partial_\ell^{+,\lambda}\mathcal{S}) \cap \mathcal{N}_\ell
$$

where

$$
\partial_\ell^{+,\lambda}\mathcal{S} \;=\; \partial_\ell^+\mathcal{S} \cap \left\{ z \in \bar{\mathcal{S}} \;: y \leq \frac{1}{\lambda} \right\}, \qquad \mathcal{N}_\ell \;=\; \left\{ z \in \bar{\mathcal{S}} \;: p\bar{g}(y) < k \right\},
$$

and $\bar{g}(y) = \max_{\zeta\in\mathbb{R}} g(y,\zeta)$.

The proof is based on detailed and long but elementary calculations on the liquidation net wealth $L(\Gamma(z,\zeta)) = \max\left[L_0(\Gamma(z,\zeta)), L_1(\Gamma(z,\zeta))\right] 1_{y+\zeta\geq 0} + L_0(\Gamma(z,\zeta))1_{y+\zeta<0}$ and is rejected in Appendix.

Remark 4.1 Actually, we have a more precise result on the compactness result of $\mathcal{C}(z)$. Let $z \in \bar{\mathcal{S}}$ and $(z_n)_n$ be a sequence in $\bar{\mathcal{S}}$ converging to z. Consider any sequence $(\zeta_n)_n$ with $\zeta_n \in \mathcal{C}(z_n)$, i.e. $L(\Gamma(z_n, \zeta_n)) \geq 0$:

$$
\begin{aligned}
&\max\left[L_0(z_n) + p_n g(y_n,\zeta_n) - k, x - \theta(\zeta_n, p_n) - k\right] 1_{y_n+\zeta_n\geq 0} \\
&\qquad + \left[L_0(z_n) + p_n g(y_n,\zeta_n) - k\right] 1_{y_n+\zeta_n<0} \;\geq\; 0.
\end{aligned}
$$

Since $g(y,\zeta)$ and $-\theta(\zeta,p)$ goes to $-\infty$ as ζ goes to infinity, and $g(y,\zeta)$ goes to $-\infty$ as ζ goes to $-\infty$, this proves that the sequence (ζ_n) is bounded. Hence, up to a subsequence, (ζ_n) converges to some $\zeta \in \mathbb{R}$. Since the function L is uppersemicontinuous, we see that the limit ζ satisfies : $L(\Gamma(z,\zeta)) \geq 0$, i.e. ζ lies in $\mathcal{C}(z)$.

We can now check that the set of admissible controls is not empty.

Corollary 4.1 *For all $(t,z) \in [0,T) \times \bar{\mathcal{S}}$, we have $\mathcal{A}(t,z) \neq \emptyset$.*

Proof. By continuity of the process $Z_s^{0,t,z}$, $t \leq s \leq T$, it is clear that it suffices to prove $\mathcal{A}(t,z) \neq \emptyset$ for any $t \in [0,T) \times \partial\mathcal{S}$. Fix now some arbitrary $t \in [0,T)$. From Lemma 4.1, the set of admissible transactions $\mathcal{C}(z)$ contains at least one nonzero element for any $z \in \partial_2^x\mathcal{S} \cup \partial_\ell^+\mathcal{S} \cup \partial_\ell^-\mathcal{S} \setminus D_k$. So once the state process reaches this boundary part, it is possible to jump inside the open solvency region $\mathcal{S}$. Hence, we only have to check that $\mathcal{A}(t,z)$ is nonempty when $z \in \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S} \cup \partial^y\mathcal{S} \cup D_k$. This is clear when $z \in \partial^y\mathcal{S} \cup D_k$: indeed, by doing nothing the state process $Z_s = Z_s^{0,t,z} = (xe^{r(s-t)}, 0, P_s^{0,t,p})$, $t \leq s \leq T$, obviously stays in $\bar{\mathcal{S}}$, since $x \geq 0$ and so $L_1(Z_s) \geq 0$ for all $t \leq s \leq T$. Similarly, when $z \in \partial_0^x\mathcal{S} \cup \partial_1^x\mathcal{S}$, by doing nothing the state process $Z_s = Z_s^{0,t,z} = (0, y, P_s^{0,t,p})$, $t \leq s \leq T$, also stays in $\bar{\mathcal{S}}$ since $y \geq 0$ and so $L_1(Z_s) \geq 0$ for all $t \leq s \leq T$. □

We next turn to the finiteness of the value function, which is not trivial due to the impulse control. As mentioned above, since the net wealth may grow after transaction due to the nonlinearity of the liquidation function, one cannot bound the value function v by the value function of the Merton problem with liquidated net wealth. We then introduce a suitable "linearization" of the net wealth by defining the following functions on $\bar{\mathcal{S}}$:

$$\tilde{L}(z) \;=\; x + \frac{p}{\lambda}(1 - e^{-\lambda y}), \quad \text{and} \quad \bar{L}(z) \;=\; x + \frac{p}{\lambda}, \quad z = (x,y,p) \in \bar{\mathcal{S}}.$$

Lemma 4.2 *For all $z = (x,y,p) \in \bar{\mathcal{S}}$, we have :*

$$0 \;\leq\; L(z) \;\;\leq\;\; \tilde{L}(z) \;\leq\; \bar{L}(z) \tag{4.3}$$

and for all $\zeta \in \mathcal{C}(z)$

$$\tilde{L}(\Gamma(z,\zeta)) \;\;\leq\;\; \tilde{L}(z) - k \tag{4.4}$$

$$\bar{L}(\Gamma(z,\zeta)) \;\;\leq\;\; \bar{L}(z) - k. \tag{4.5}$$

In particular, we have $\mathcal{C}(z) = \emptyset$ for all $z \in \tilde{\mathcal{N}} := \{z \in \mathcal{S} \; : \tilde{L}(z) < k\}$.

Proof. 1) The inequality $\tilde{L} \leq \bar{L}$ is clear. Notice that for all $y \in \mathbb{R}$, we have

$$0 \;\;\leq\;\; 1 - e^{-\lambda y} - \lambda y e^{-\lambda y}. \tag{4.6}$$

This immediately implies for all $z = (x,y,p) \in \bar{\mathcal{S}}$,

$$L_0(z) \;\;\leq\;\; \tilde{L}(z). \tag{4.7}$$

If $y \geq 0$, we obviously have $L_1(z) = x \leq \tilde{L}(z)$ and so $L(z) \leq \tilde{L}(z)$. If $y < 0$, then $L(z) = L_0(z) \leq \tilde{L}(z)$ by (4.7).
2) For any $z = (x, y, p) \in \bar{\mathcal{S}}$ and $\zeta \in \mathbb{R}$, a straightforward computation shows that

$$\tilde{L}(\Gamma(z,\zeta)) \;=\; \tilde{L}(z) - k + \frac{p}{\lambda}(e^{\lambda\zeta} - 1 - \lambda\zeta e^{\lambda\zeta}) \;\leq\; \tilde{L}(z) - k,$$

from (4.6). Similarly, we show (4.5). Finally, if $z \in \tilde{\mathcal{N}}$, we have from (4.5), $\tilde{L}(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathcal{C}(z)$, which shows with (4.3) that $\mathcal{C}(z) = \emptyset$. □

As a first direct corollary, we check that the no-trade region is not empty.

Corollary 4.2 *We have* $\mathbf{NT} \neq \emptyset$. *More precisely, for each* $t \in [0,T)$, *the* t*-section of* $\mathbf{NT}$, *i.e.* $\mathbf{NT}(t) = \{z \in \mathcal{S} \ : (t,z) \in \mathbf{NT}\}$ *contains the nonempty subset* $\tilde{\mathcal{N}}$ *of* $\mathcal{S}$.

Proof. This follows from the fact that for any z lying in the nonempty set $\tilde{\mathcal{N}}$ of $\mathcal{S}$, we have $\mathcal{C}(z) = \emptyset$. In particular, $\mathcal{H}v(t,z) = -\infty < v(t,z)$ for $(t,z) \in [0,T) \times \tilde{\mathcal{N}}$. □

As a second corollary, we have the following uniform bound on the controlled state process.

Corollary 4.3 *For any* $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, *we have almost surely for all* $t \leq s \leq T$ *:*

$$\sup_{\alpha\in\mathcal{A}(t,z)} L(Z_s) \;\leq\; \sup_{\alpha\in\mathcal{A}(t,z)} \tilde{L}(Z_s) \;\leq\; \tilde{L}(Z_s^{0,t,z}) \;=\; X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}(1 - e^{-\lambda y}), \tag{4.8}$$

$$\sup_{\alpha\in\mathcal{A}(t,z)} L(Z_s) \;\leq\; \sup_{\alpha\in\mathcal{A}(t,z)} \bar{L}(Z_s) \;\leq\; \bar{L}(Z_s^{0,t,z}) \;=\; X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}, \tag{4.9}$$

$$\sup_{\alpha\in\mathcal{A}(t,z)} |X_s| \;\leq\; \frac{e}{e-1}\bar{L}(Z_s^{0,t,z}), \tag{4.10}$$

$$\sup_{\alpha\in\mathcal{A}(t,z)} P_s \;\leq\; \frac{\lambda e}{e-1}\bar{L}(Z_s^{0,t,z}). \tag{4.11}$$

Proof. Fix $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and consider for any $\alpha \in \mathcal{A}(t,z)$, the process $\tilde{L}(Z_s)$, $t \leq s \leq T$, which is nonnegative by (4.3). When a transaction occurs at time s, we deduce from (4.4) that the variation $\Delta\tilde{L}(Z_s) = \tilde{L}(Z_s) - \tilde{L}(Z_{s^-})$ is always negative : $\Delta\tilde{L}(Z_s) \leq -k \leq 0$. Therefore, the process $\tilde{L}(Z_s)$ is smaller than its continuous part :

$$L(Z_s) \;\leq\; \tilde{L}(Z_s) \;\leq\; \tilde{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \tag{4.12}$$

which proves (4.8) from the arbitrariness of α. Relation (4.9) is proved similarly.

From the second inequality in (4.9), we have for all $\alpha \in \mathcal{A}(t,z)$:

$$X_s \;\leq\; \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}, \quad t \leq s \leq T, \ a.s. \tag{4.13}$$

$$\;\leq\; \bar{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \tag{4.14}$$

By definition of L and using (4.13), we have :

$$\begin{aligned} 0 \;\leq\; L(Z_s) \;&\leq\; \max\left(\bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}(1 - \lambda Y_s e^{-\lambda Y_s}), \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}\right) \\ &\leq\; \bar{L}(Z_s^{0,t,z}) - \frac{P_s}{\lambda}\left(1 - \frac{1}{e}\right), \quad t \leq s \leq T, \ a.s. \end{aligned}$$

since the function $y \mapsto \lambda y e^{-\lambda y}$ is upper bounded by $1/e$. We then deduce

$$P_s \quad \leq \quad \frac{\lambda e}{e-1}\bar{L}(Z_s^{0,t,z}), \quad t \leq s \leq T, \ a.s. \tag{4.15}$$

and so (4.11) from the arbitrariness of α. By recalling that $X_s + P_s/\lambda \geq 0$ and using (4.15), we get

$$-\frac{e}{e-1}\bar{L}(Z_s^{0,t,z}) \quad \leq \quad X_s, \quad t \leq s \leq T, \ a.s.$$

By combining with (4.14) and from the arbitrariness of α, we obtain (4.10). □

As a third direct corollary, we state that the number of intervention times is finite. More precisely, we have the following result :

Corollary 4.4 *For any $(t,z) \in [0,T]\times\bar{\mathcal{S}}$, $\alpha = (\tau_n, \zeta_n) \in \mathcal{A}(t,z)$, the number of intervertion times strictly between t and T is finite a.s. :*

$$\begin{aligned} N_t(\alpha) &:= \text{Card}\{n \ : t < \tau_n < T\} \\ &\leq \frac{1}{k}\left[\bar{L}(Z_t) - \bar{L}(Z_{T^-}) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s\right] < \infty \ a.s. \end{aligned} \tag{4.16}$$

Proof. Fix some $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and $\alpha \in \mathcal{A}(t,z)$, and consider $Z_s = (X_s, Y_s, P_s)$, $t \leq s \leq T$, the associated controlled state process. By applying Itô's formula to $\bar{L}(Z_s) = X_s + P_s/\lambda$ between t and T, we have :

$$\begin{aligned} 0 \ \leq \ \bar{L}(Z_{T^-}) &= \bar{L}(Z_t) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s + \sum_{t<s<T} \Delta L(Z_s) \\ &\leq \bar{L}(Z_t) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s - kN_t(\alpha), \end{aligned}$$

by (4.5). We deduce the required result :

$$N_t(\alpha) \quad \leq \quad \frac{1}{k}\left[\bar{L}(Z_t) - \bar{L}(Z_{T^-}) + \int_t^T \left(rX_s + \frac{P_s}{\lambda}\right)ds + \int_t^T \frac{\sigma}{\lambda}P_s dW_s\right] \ < \ \infty \quad a.s.$$

□

4.2 Bound on the value function

We can now give a sharp upper bound on the value function.

Proposition 4.1 *For all $t \in [0,T]$, $z = (x,y,p) \in \bar{\mathcal{S}}$, we have*

$$\sup_{\alpha\in\mathcal{A}(t,z)} U_L\left(Z_T\right) \quad \leq \quad U\left(\tilde{L}\left(Z_T^{0,t,z}\right)\right) \quad \in \ L^1(\mathbb{P}). \tag{4.17}$$

In particular, the family $\{U_L(Z_T), \alpha \in \mathcal{A}(t,z)\}$ is uniformly integrable and we have

$$v(t,z) \quad \leq \quad v_0(t,z) \ := \ \mathbb{E}\left[U\left(\tilde{L}\left(Z_T^{0,t,z}\right)\right)\right], \quad (t,z) \in [0,T]\times\bar{\mathcal{S}}, \tag{4.18}$$

with

$$v_0(t,z) \quad \leq \quad K e^{\rho(T-t)} \tilde{L}(z)^\gamma, \tag{4.19}$$

where ρ is a positive constant s.t.

$$\rho \quad > \quad \frac{\gamma}{1-\gamma} \frac{b^2 + r^2 + \sigma^2 r(1-\gamma)}{\sigma^2}. \tag{4.20}$$

Proof. Fix $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and consider for some arbitrary $\alpha \in \mathcal{A}(t,z)$, the process $\tilde{L}(Z_s)$, $t \leq s \leq T$, which is nonnegative by (4.3). By (4.8), we have :

$$L(Z_s) \ \leq \ \tilde{L}(Z_s) \quad \leq \quad \tilde{L}(Z_s^{0,t,z}) \ = \ X_s^{0,t,x} + \frac{P_s^{0,t,p}}{\lambda}(1 - e^{-\lambda y}), \quad t \leq s \leq T. \tag{4.21}$$

From the arbitrariness of α and the nondecreasing property of U, we get the inequality in (4.17). From the growth condition (2.10) on the nonnegative function U and since clearly $|X_T^{0,t,x}|^\gamma$ and $(P_T^{0,t,p})^\gamma$ are integrable, i.e. in $L^1(\mathbb{P})$, we have $U\left(X_T^{0,t,x} + \frac{P_T^{0,t,p}}{\lambda}(1-e^{-\lambda y})\right)$ $\in L^1(\mathbb{P})$. This clearly implies (4.18).

Consider now the nonnegative function :

$$\varphi(t,z) \quad = \quad e^{\rho(T-t)} \tilde{L}(z)^\gamma \ = \ e^{\rho(T-t)} \left(x + \frac{p}{\lambda}(1 - e^{-\lambda y}) \right)^\gamma$$

and notice that φ is smooth C^2 on $[0,T] \times (\bar{\mathcal{S}} \setminus D_0)$. We claim that for ρ large enough, the function φ satisfies :

$$-\frac{\partial \varphi}{\partial t}(t,z) - \mathcal{L}\varphi(t,z) \quad \geq \quad 0, \quad \forall\, (t,z) \in [0,T] \times \bar{\mathcal{S}} \setminus D_0. \tag{4.22}$$

Indeed, a straightforward calculation shows that for all $t \in [0,T)$, $z = (x,y,p) \in \bar{\mathcal{S}} \setminus D_0$:

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(t,z) - \mathcal{L}\varphi(t,z) \\ = \quad & e^{\rho(T-t)} \tilde{L}(z)^{\gamma-2} \left[A x^2 + B \left(\frac{p}{\lambda}(1 - e^{-\lambda y}) \right)^2 + 2Cx \frac{p}{\lambda}(1 - e^{-\lambda y}) \right], \end{aligned} \tag{4.23}$$

where

$$A \ = \ \rho - r\gamma, \qquad B \ = \ \rho - b\gamma + \frac{1}{2}\sigma^2 \gamma(1-\gamma), \quad C \ = \ \rho - \frac{(b+r)\gamma}{2}.$$

Hence, (4.22) is satisfied whenever $A > 0$ and $BC - A^2 > 0$, which is the case for ρ larger than the constant in the r.h.s. of (4.20).

Fix some $(t,z) \in [0,T) \times \bar{\mathcal{S}}$. If $z = (0,0,p)$ then we clearly have $v_0(t,z) = U(0)$ and so inequality (4.19) follows from $U(0) \leq K_1$ (see (2.10)). Consider now the case where $z \in \bar{\mathcal{S}} \setminus D_0$ and notice that the process $Z_s^{0,t,z} = (X_s^{0,t,x}, y, P_s^{0,t,p})$ never reaches $\{(0,0)\} \times \mathbb{R}_+^*$. Consider the stopping time

$$T_R \quad = \quad \inf\left\{ s \geq t \ : |Z_s^{0,t,z}| > R \right\} \wedge T$$

so that the stopped process $(Z^{0,t,z}_{s\wedge T_R})_{t\leq s\leq T}$ stays in the bounded set $\{z=(x,y,p)\in\bar{\mathcal{S}}\setminus D_0 \;:\; |z|\leq R\}$ on which $\varphi(t,.)$ is smooth C^2 and its derivative in p, $\frac{\partial\varphi}{\partial p}$ is bounded. By applying Itô's formula to $\varphi(s,Z^{0,t,z}_s)$ between $s=t$ and $s=T_R$, we have :

$$\varphi(T_R,Z^{0,t,z}_{T_R}) \;=\; \varphi(t,z)+\int_t^{T_R}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(s,Z^{0,t,z}_s)ds+\int_t^{T_R}\frac{\partial\varphi}{\partial p}(s,Z^{0,t,z}_s)\sigma P^{0,t,p}_s dW_s.$$

Since the integrand in the stochastic integral is bounded, we get by taking expectation in the last relation :

$$\mathbb{E}[\varphi(T_R,Z^{0,t,z}_{T_R})] \;=\; \varphi(t,z)+\mathbb{E}\left[\int_t^{T_R}\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}\varphi\right)(s,Z^{0,t,z}_s)ds\right] \;\leq\; \varphi(t,z),$$

where we used in the last inequality (4.22). Now, for almost $\omega\in\Omega$, for R large enough ($\geq\bar{R}(\omega)$), we have $T_R=T$ so that $\varphi(T_R,Z_{T_R})$ converges a.s. to $\varphi(T,Z_T)$. By Fatou's lemma, we deduce that $\mathbb{E}[\varphi(T,Z_T)]\leq\varphi(t,z)$. Since $\varphi(T,z)=\tilde{L}(z)^\gamma$, this yields

$$\mathbb{E}\left[\tilde{L}\left(Z^{0,t,z}_T\right)^\gamma\right] \;\leq\; \varphi(t,z). \tag{4.24}$$

Finally, by the growth condition (2.10), this proves the required upper bound on the value function v. □

Remark 4.2 The upper bound of the last proposition shows that the value function lies in the set of functions satisfying the growth condition :

$$\mathcal{G}_\gamma([0,T]\times\bar{\mathcal{S}}) \;=\; \left\{u:[0,T]\times\bar{\mathcal{S}}\longrightarrow\mathbb{R},\quad \sup_{[0,T]\times\bar{\mathcal{S}}}\frac{|u(t,z)|}{1+\left(x+\frac{p}{\lambda}\right)^\gamma}<\infty\right\}.$$

Remark 4.3 The upper bound (4.18) is sharp in the sense that when λ goes to zero (no price impact), we find the usual Merton bound :

$$v(t,z) \;\leq\; \mathbb{E}[U(X^{0,t,x}_T+yP^{0,t,p}_T)] \;\leq\; Ke^{\rho(T-t)}(x+py)^\gamma.$$

As a corollary, we can explicit the value function on the hyperplane of $\bar{S}$:

$$\bar{S}^y \;=\; \mathbb{R}_+\times\{0\}\times\mathbb{R}^*_+ \;\subset\; \bar{S},$$

where the agent does not hold any stock shares.

Corollary 4.5 *For any $t\in[0,T)$, $z=(x,0,p)\in\bar{S}^y$, the investor optimally does not transact during $[t,T]$, i.e.*

$$v(t,z) \;=\; \mathbb{E}\left[U\left(X^{0,t,x}_T\right)\right] \;=\; U\left(xe^{r(T-t)}\right).$$

Proof. For $z=(x,0,p)\in\bar{S}^y$, let us consider the no impulse control strategy starting from z at t which leads at the terminal date to a net wealth $L(Z^{0,t,z}_T)=X^{0,t,x}_T=xe^{r(T-t)}$. We then have $v(t,z)\geq\mathbb{E}[U(X^{0,t,x}_T)]=U(xe^{r(T-t)})$. On the other hand, we have from (4.18) : $v(t,z)\leq v_0(t,z)=\mathbb{E}[U(X^{0,t,x}_T)]$. This proves the required result. □

4.3 Boundary properties

We now turn to the behavior of the value function on the boundary of the solvency region. The situation is more complex than in models with proportional transaction costs where the boundary of the solvency region is an absorbing barrier and all transactions are stopped. Here, the behavior depends on which part of the boundary is the state, as showed in the following proposition.

Proposition 4.2 *1) We have*

$$v \;=\; \mathcal{H}v \quad on \;\; [0,T) \times (\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S}) \tag{4.25}$$

and

$$\mathcal{H}v \;=\; 0 \quad on \;\; [0,T) \times (\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell. \tag{4.26}$$

2) We have

$$v \;>\; \mathcal{H}v \quad on \;\; [0,T) \times \partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S} \cup D_k. \tag{4.27}$$

and

$$v \;=\; 0 \quad on \;\; [0,T) \times D_0, \tag{4.28}$$

$$v(t,z) \;=\; U(ke^{r(T-t)}), \quad (t,z) \in [0,T) \in D_k. \tag{4.29}$$

Proof. 1. a) Fix some $(t,z) \in [0,T) \times (\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S})$ and consider an arbitrary $\alpha = (\tau_n, \zeta_n)_{n\geq 1} \in \mathcal{A}(t,z)$. We claim that $\tau_1 = t$ a.s. i.e. one has to transact immediately at time t in order to satisfy the solvency constraint.

⋆ Consider first the case where $z \in \partial_\ell^- \mathcal{S} \setminus D_k$. Then on $[t,\tau_1]$, $X_s = xe^{r(s-t)}$, $Y_s = y < 0$, $P_s = pP_s^0$, and so $L(Z_s) = L_0(Z_s^{0,t,z})$. Hence, by integrating between t and τ_1, we get :

$$0 \;\leq\; e^{-r(\tau_1 - t)} L_0(Z_{\tau_1}^{0,t,z}) \;=\; \int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \left[(b-r)du + \sigma dW_u\right]. \tag{4.30}$$

By Girsanov's theorem, one can define a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which $\hat{W}_s = W_s + (b-r)s/\sigma$ is a Brownian motion. Under this measure, the stochastic integral $\int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \sigma d\hat{W}_u$ has zero expectation from which we deduce with (4.30) that

$$\int_t^{\tau_1} e^{-r(u-t)} P_u^0 y e^{-\lambda y} \sigma d\hat{W}_u \;=\; 0 \quad a.s.$$

Since $y \neq 0$ and $P_s^0 > 0$ a.s., this implies $\tau_1 = t$ a.s.

⋆ Consider the case where $z \in \partial_\ell^+ \mathcal{S}$. Then on $[t,\tau_1]$, $X_s = xe^{r(s-t)} < 0$, $Y_s = y$, $P_s = pP_s^0$, and so $L(Z_s) = L_0(Z_s)$. By the same argument as above, we deduce $\tau_1 = t$. Applying the dynamic programming principle (2.12) for $\tau = \tau_1$, we clearly deduce (4.25).

b) Fix some $(t, z = (x,y,p)) \in [0,T) \times (\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell$. Then, from Lemma 4.1, $\mathcal{C}(z)$ $= \{-y\}$ and so $\mathcal{H}v(t,z) = v(t, \Gamma(z,-y)) = v(t,0,0,p)$. Now, from Corollary 4.5, we have for all $z_0 = (0,0,p) \in D_0$, $v(t,z_0) = U(0) = 0$, which proves (4.28) and so (4.26).

2. Fix some $t \in [0,T)$ and $z \in \partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S}$. Then by Lemma 4.1, $\mathcal{C}(z) = \emptyset$, hence $\mathcal{H}v(t,z) = -\infty$ and so (4.27) is trivial. For $z = (k,0,p) \in D_k$, we have by Lemma 4.1, $\mathcal{C}(z)$ $= \{0\}$ and so $\mathcal{H}v(t,z) = v(t,\Gamma(z,0)) = v(t,0,0,p) = 0$ by (4.28). Therefore, from Corollary 4.5, we have for $z = (k,0,p) \in D_k$: $v(t,z) = U(ke^{r(T-t)}) > 0 = \mathcal{H}v(t,z)$. □

Remark 4.4 The last proposition and its proof means that when the state reaches $\partial_\ell^- \mathcal{S} \setminus D_k \cup \partial_\ell^+ \mathcal{S}$, one has to transact immediately since the no transaction strategy is not admissible. Moreover, if one is in $(\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell$, one jumps directly to D_0 where all transactions are stopped. On the other hand, if the state is in $\partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S} \cup D_k$, one should do not transact : admissible transaction does not exist on $\partial^y \mathcal{S} \cup \partial_0^x \mathcal{S} \cup \partial_1^x \mathcal{S}$ while the only zero admissible transaction on D_k is suboptimal with respect to the no transaction control. In the remaining part $\partial_2^x \mathcal{S}$ of the boundary, both decisions, transaction and no-transaction, are admissible : we only know that one of these decisions should be chosen optimally but we are not able to be explicit about which one is optimal. A representation of the behavior of the optimal strategy on the boundary of the solvency region is depicted in Figures 2 and 3.

The next result states the continuity of the value function on the part D_0 of the solvency boundary, as a direct consequence of (4.18) and (4.28).

Corollary 4.6 *The value function v is continuous on $[0, T) \times D_0$:*

$$\lim_{(t',z') \to (t,z)} v(t', z') \;=\; v(t, z) \;=\; 0, \quad \forall (t, z) \in [0, T) \times D_0.$$

Remark 4.5 Notice that except on D_0, the value function is in general discontinuous on the boundary of the solvency region. More precisely, for any $t \in [0, T)$, $z \in D_k$, we have from (4.25)-(4.26) :

$$\lim_{\substack{z' \to z \\ z' \in \partial_\ell^- \mathcal{S} \setminus D_k}} v(t, z') \;=\; 0,$$

while from Corollary 4.5 :

$$\lim_{\substack{z' \to z \\ z' \in \bar{S}^y}} v(t, z') \;=\; U(k e^{r(T-t)}).$$

This shows that v is discontinuous on $[0, T) \times D_k$. Similarly, one can show that v is discontinuous on $[0, T) \times (\partial_1^x \mathcal{S} \cap \partial_\ell^+ \mathcal{S})$.

4.4 Terminal condition

We end this section by determining the right terminal condition of the value function. We set

$$v^*(T, z) := \limsup_{\substack{(t, z') \to (T, z) \\ t < T,\, z' \in \mathcal{S}}} v(t, z'), \qquad v_*(T, z) := \liminf_{\substack{(t, z') \to (T, z) \\ t < T,\, z' \in \mathcal{S}}} v(t, z')$$

Proposition 4.3 *We have*

$$v_*(T, z) \;=\; v^*(T, z) \;=\; \bar{U}(z), \quad \forall z \in \bar{\mathcal{S}},$$

where

$$\bar{U}(z) \;:=\; \max\left[U_L(z), \mathcal{H} U_L(z)\right].$$

Proof. 1) Fix some $z \in \bar{\mathcal{S}}$ and consider some sequence $(t_m, z_m)_m \in [0,T) \times \mathcal{S}$ converging to (T, z) and s.t. $\lim_m v(t_m, z_m) = v_*(T, z)$. By taking the no impulse control strategy on $[t_m, T]$, we have

$$v(t_m, z_m) \quad \geq \quad \mathbb{E}\left[U_L(Z_T^{0,t_m,z_m})\right].$$

Since Z_T^{0,t_m,z_m} converges a.s. to z when m goes to infinity by continuity of the diffusion $Z^{0,t,z}$ in its initial conditions (t, z), we deduce by Fatou's lemma that :

$$v_*(T, z) \quad \geq \quad U_L(z). \tag{4.31}$$

Take now some arbitrary $\zeta \in \mathcal{C}(z)$. Consider first the case where $L(\Gamma(z, \zeta)) > 0$. We claim that for m large enough, $\zeta \in \mathcal{C}(z_m)$. Indeed,

$\star$ suppose that $\zeta \neq -y$. Then, by continuity of the function $z' \mapsto L(\Gamma(z', \zeta))$ on $\{z' = (x', y', p') : y' \neq \zeta\}$, we deduce that $L(\Gamma(z_m, \zeta))$ converges to $L(\Gamma(z, \zeta)) > 0$ and so for m large enough, $\zeta \in \mathcal{C}(z_m)$.

$\star$ Suppose that $\zeta = -y$, i.e. $L(\Gamma(z, \zeta)) = x + \ell(y, p) - k > 0$. Notice that

$$\begin{aligned} L(\Gamma(z_m, \zeta)) \quad &= \quad \max\left[L_0(z_m) - k + pg(-y, y_m), x_m + ye^{-\lambda y}p_m - k\right] 1_{y_m - y \geq 0} \\ &\quad + \; L_0(z_m) 1_{y_m - y < 0}. \end{aligned}$$

We then see that $\liminf_{m \to \infty} L(\Gamma(z_m, \zeta)) \geq L(\Gamma(z, \zeta))$, and so for m large enough, $\zeta \in \mathcal{C}(z_m)$.

One may then consider the admissible control with immediate impulse at t_m with size ζ and no other impulse until T so that the associated state process is $Z^{t_m,z_m} = Z^{0,t_m,\Gamma(z_m,\zeta)}$ and thus

$$v(t_m, z_m) \quad \geq \quad \mathbb{E}\left[U_L\left(Z_T^{0,t_m,\Gamma(z_m,\zeta)}\right)\right].$$

Sending m to infinity, we obtain :

$$v_*(T, z) \quad \geq \quad U_L(\Gamma(z, \zeta)), \tag{4.32}$$

for all ζ in $\mathcal{C}(z)$ s.t. $L(\Gamma(z, \zeta)) > 0$. This last inequality (4.32) holds obviously true when $L(\Gamma(z, \zeta)) = 0$ since in this case $U_L(\Gamma(z, \zeta)) = 0 \leq v_*(T, z)$. By combining with (4.31), we get $v_*(T, z) \geq \bar{U}(z)$.

2) Fix some $z \in \bar{\mathcal{S}}$ and consider some sequence $(t_m, z_m)_m \in [0,T) \times \mathcal{S}$ converging to (T, z) and s.t. $\lim_m v(t_m, z_m) = v^*(T, z)$. For any m, one can find $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_n \in \mathcal{A}(t_m, z_m)$ s.t.

$$v(t_m, z_m) \quad \leq \quad \mathbb{E}\left[U_L(\hat{Z}_T^m)\right] + \frac{1}{m} \tag{4.33}$$

where $\hat{Z}^m = (\hat{X}^m, \hat{Y}^m, \hat{P}^m)$ denotes the state process controlled by $\hat{\alpha}^m$ and given in T by :

$$\begin{aligned} \hat{Z}_T^m \;=\; \hat{Z}_{T^-}^m \quad &= \quad z_m + \int_{t_m}^T B(\hat{Z}_s^m) ds + \int_{t_m}^T \Sigma(\hat{Z}_s^m) dW_s + \sum_{t_m \leq u < T} \Delta \hat{Z}_s^m \\ &= \quad z_m + (\Gamma(z_m, \zeta_1^m) - z_m) 1_{\tau_1^m = t_m} + R_T^m \end{aligned} \tag{4.34}$$

with $B(z) = (rx, 0, bp)$ and $\Sigma(z) = (0, 0, \sigma p)$ and

$$R_T^m = \int_{t_m}^T B(\hat{Z}_s^m)ds + \int_{t_m}^T \Sigma(\hat{Z}_s^m)dW_s + \sum_{t_m<s<T} \Delta\hat{Z}_s^m. \tag{4.35}$$

We rewrite (4.33) as

$$\begin{aligned} v(t_m, z_m) \leq \; & \mathbb{E}\left[\{U_L(\Gamma(z_m, \zeta_1^m) + R_T^m) - U_L(z_m + R_T^m)\} 1_{\tau_1^m = t_m} \right. \\ & \left. + U_L(z_m + R_T^m)\right] + \frac{1}{m} \end{aligned} \tag{4.36}$$

We claim that R_T^m converges a.s. to 0 as m goes to infinity. Indeed, from the uniform bounds (4.10)-(4.11), we have

$$\begin{aligned} |B(\hat{Z}_s^m)| + |\Sigma(\hat{Z}_s^m)| & \leq (r + (b+\sigma)\lambda)\frac{e}{e-1} L(Z_s^{0,t,z_m}) \\ & \leq \text{Cte}\, L(Z_s^{0,t,z}), \quad t_m \leq s \leq T, \ a.s., \end{aligned}$$

for some positive Cte independent of m. We then deduce that the Lebesgue and stochastic integral in (4.35) converge a.s. to zero as m goes to infinity, i.e. t_m goes to T. On the other hand, by same argument as in Remark 4.1, we see that for each $t_m < s < T$, the jump ΔZ_s^m is uniformly bounded in m. Moreover, by (4.16), we have

$$N_{t_m}(\hat{\alpha}^m) \leq \frac{1}{k}\left[\bar{L}(\hat{Z}_{t_m}^m) - \bar{L}(\hat{Z}_{T^-}^m) + \int_{t_m}^T \left(r\hat{X}_s^m + \frac{\hat{P}_s^m}{\lambda}\right)ds + \int_{t_m}^T \frac{\sigma}{\lambda}\hat{P}_s^m dW_s\right] \tag{4.37}$$

Similarly as above, by the uniform bounds in (4.10)-(4.11), the integrals in (4.37) converge to zero as m goes to infinity. From the left-continuity of the state process $\hat{Z}^m$ and the continuity of $\bar{L}$, we deduce that $\bar{L}(\hat{Z}_{t^m}^m) - \bar{L}(Z_{T_-}^m)$ converge to zero as m goes to infinity. Therefore, $\sum_{t_m<s<T}\Delta\hat{Z}_s^m$ goes to zero as m goes to infinity, which proves the required zero convergence of R_T^m.

By Remark 4.1, the sequence of jump size $(\zeta_1^m)_m$ is bounded, and up to a subsequence, converges, as m goes to infinity, to some $\zeta \in \mathcal{C}(z)$. Moreover, it is easy to check that the family $\{U(X_T^{0,t_m,x_m} + \frac{P_T^{0,t_m,p_m}}{\lambda}(1 - e^{-\lambda y_m})), m \geq 1\}$ is uniformly integrable so that from (4.17), the family $\{U_L(\hat{Z}_T^m), m \geq 1\}$ is also uniformly integrable. Therefore, we can send m to infinity into (4.33) (or (4.36)) by the dominated convergence theorem and get :

$$\begin{aligned} v^*(T, z) & \leq \mathbb{E}\left[\{U_L(\Gamma(z,\zeta)) - U_L(z)\} \limsup_{m\to\infty} 1_{\tau_1^m = t_m} + U_L(z)\right] \\ & \leq \max\left\{U_L(z), \sup_{\zeta\in\mathcal{C}(z)} U_L(\Gamma(z,\zeta))\right\}. \end{aligned}$$

By completing with (4.31), this proves $v_*(T, z) = v^*(T, z) = \bar{U}(z)$. □

Remark 4.6 The previous result shows in particular that the value function is discontinuous on T. Indeed, recalling that we do not allow any impulse transaction at T, we have $v(T, z) = U_L(z)$ for all $z \in \bar{\mathcal{S}}$. Moreover, by Proposition 4.3, we have $v(T^-, z) = \bar{U}(z)$, hence $v(., z)$ is discontinuous on T for all $z \in \{z \in \bar{\mathcal{S}} \ : \mathcal{H}U_L(z) > U_L(z)\} \neq \emptyset$.

5 Viscosity characterization

In this section, we intend to provide a rigorous characterization of the value function by means of (constrained) viscosity solution to the quasi-variational inequality :

$$\min\left[-\frac{\partial v}{\partial t}-\mathcal{L}v, v-\mathcal{H}v\right] \;=\; 0, \tag{5.1}$$

together with appropriate boundary and terminal conditions.

As mentioned previously, the value function is not known to be continuous a priori and so we shall work with the notion of discontinuous viscosity solutions. For a locally bounded function u on $[0,T)\times\bar{\mathcal{S}}$ (which is the case of the value function v), we denote by u_* (resp. u^*) the lower semi-continuous (lsc) (resp. upper semi-continuous (usc)) envelope of u. We recall that in general, $u_*\leq u\leq u^*$, and that u is lsc iff $u=u_*$, u is scs iff $u=u^*$, and u is continuous iff $u_*=u^*$ $(=u)$. We denote by $LSC([0,T)\times\bar{\mathcal{S}})$ (resp. $USC([0,T)\times\bar{\mathcal{S}})$) the set of lsc (resp. usc) functions on $[0,T)\times\bar{\mathcal{S}}$.

We work with the suitable notion of constrained viscosity solutions, introduced in [31] for first-order equations, for taking into account boundary conditions arising in state constraints. The use of constrained viscosity solutions was initiated in [35] for stochastic control problems arising in optimal investment problems. The definition is given as follows :

Definition 5.1 *(i) Let $\mathcal{O}\subset\bar{\mathcal{S}}$. A locally bounded function u on $[0,T)\times\bar{\mathcal{S}}$ is a viscosity subsolution (resp. supersolution) of* (5.1) *in $[0,T)\times\mathcal{O}$ if for all $(\bar t,\bar z)\in[0,T)\times\mathcal{O}$ and $\varphi\in C^{1,2}([0,T)\times\bar{\mathcal{S}})$ s.t. $(u^*-\varphi)(\bar t,\bar z)=0$ (resp. $(u_*-\varphi)(\bar t,\bar z)=0$) and $(\bar t,\bar z)$ is a maximum of $u^*-\varphi$ (resp. minimum of $u_*-\varphi$) on $[0,T)\times\mathcal{O}$, we have*

$$\min\left[-\frac{\partial\varphi}{\partial t}(\bar t,\bar z)-\mathcal{L}\varphi(\bar t,\bar z), u^*(\bar t,\bar z)-\mathcal{H}u^*(\bar t,\bar z)\right] \;\leq\; 0 \tag{5.2}$$

$$(\;\textit{resp.}\quad\geq\; 0). \tag{5.3}$$

(ii) A locally bounded function u on $[0,T)\times\bar{\mathcal{S}}$ is a constrained viscosity solution of (5.1) *in $[0,T)\times\mathcal{S}$ if u is a viscosity subsolution of* (5.1) *in $[0,T)\times\bar{\mathcal{S}}$ and a viscosity supersolution of* (5.1) *in $[0,T)\times\mathcal{S}$.*

Remark 5.1 There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [11] :
(i) Let $\mathcal{O}\subset\bar{\mathcal{S}}$. A function $u\in USC([0,T)\times\bar{\mathcal{S}})$ is a viscosity subsolution (resp. supersolution) of (5.1) in $[0,T)\times\mathcal{O}$ if

$$\min\left[-q_0-rxq_1-bpq_3-\frac{1}{2}\sigma^2p^2M_{33}, u(t,z)-\mathcal{H}u(t,z)\right] \;\leq\; 0 \tag{5.4}$$

$$(\;\text{resp.}\quad\geq\; 0) \tag{5.5}$$

for all $(t,z=(x,y,p))\in[0,T)\times\mathcal{O}$, $(q_0,q=(q_i)_{1\leq i\leq 3},M=(M_{ij})_{1\leq i,j\leq 3})\in\bar{J}^{2,+}u(t,z)$ (resp. $\bar{J}^{2,-}u(t,z)$).

(ii) A locally bounded function u on $[0,T) \times \bar{\mathcal{S}}$ is a constrained viscosity solution to (5.1) if u^* satisfies (5.4) for all $(t,z) \in [0,T) \times \bar{\mathcal{S}}$, $(q_0, q, M) \in \bar{J}^{2,+}u^*(t,z)$, and u_* satisfies (5.5) for all $(t,z) \in [0,T) \times \mathcal{S}$, $(q_0,q,M) \in \bar{J}^{2,-}u_*(t,z)$.

Here $J^{2,+}u(t,z)$ is the parabolic second order superjet defined by :

$$J^{2,+}u(t,z) \;=\; \Big\{(q_0,q,M) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{S}^3 \;:\; \limsup_{\substack{(t',z') \to (t,z) \\ (t',z') \in [0,T) \times \mathcal{S}}} \frac{u(t',z') - u(t,z) - q_0(t'-t) - q.(z'-z) - \frac{1}{2}(z'-z).M(z'-z)}{|t'-t| + |z'-z|^2} \;\leq\; 0 \Big\},$$

$\mathbb{S}^3$ is the set of symmetric 3×3 matrices, $\bar{J}^{2,+}u(t,z)$ is its closure :

$$\bar{J}^{2,+}u(t,x) \;=\; \Big\{(q_0,q,M) \;=\; \lim_{m\to\infty}(q_0^m, q^m, M^m) \quad \text{with } (q_0^m,q^m,M^m) \in J^{2,+}u(t_m,z_m) \;\; \text{and} \;\; \lim_{m\to\infty}(t_m, z_m, u(t_m,z_m)) \;=\; (t,z,u(t,z))\Big\},$$

and $J^{2,-}u(t,x) = -J^{2,+}(-u)(t,x)$, $\bar{J}^{2,-}u(t,x) = -\bar{J}^{2,+}(-u)(t,x)$.

5.1 Viscosity property

Our first main result of this section is the following.

Theorem 5.1 *The value function v is a constrained viscosity solution to* (5.1) *in* $[0,T)\times\mathcal{S}$.

Remark 5.2 The state constraint and the boundary conditions is translated through the PDE characterization via the subsolution property, which has to hold true on the whole closed region $\bar{\mathcal{S}}$. This formalizes the property that on the boundary of the solvency region, one of the two possible decisions, immediate impulse transaction or no-transaction, should be chosen optimally.

We need some auxiliary results on the impulse operator $\mathcal{H}$.

Lemma 5.1 *Let u be a locally bounded function on $[0,T) \times \bar{\mathcal{S}}$.*
(i) $\mathcal{H}u_* \leq (\mathcal{H}u)_*$. Moreover, if u is lsc then $\mathcal{H}u$ is also lsc.
(ii) $\mathcal{H}u^*$ is usc and $(\mathcal{H}u)^* \leq \mathcal{H}u^*$.

Proof. (i) Let (t_n,z_n) be a sequence in $[0,T) \times \bar{\mathcal{S}}$ converging to (t,z) and s.t. $\mathcal{H}u(t_n,z_n)$ converges to $(\mathcal{H}u)_*(t,z)$. Then, using also the lowersemicontinuity of u_* and the continuity of Γ, we have :

$$\begin{aligned}\mathcal{H}u_*(t,z) &= \sup_{\zeta\in\mathcal{C}(z)} u_*(t,\Gamma(z,\zeta)) \;\leq\; \sup_{\zeta\in\mathcal{C}(z)} \liminf_{n\to\infty} u_*(t_n,\Gamma(z_n,\zeta)) \\ &\leq \liminf_{n\to\infty} \sup_{\zeta\in\mathcal{C}(z)} u_*(t_n,\Gamma(z_n,\zeta)) \;\leq\; \lim_{n\to\infty} \mathcal{H}u(t_n,z_n) \;=\; (\mathcal{H}u)_*(t,z).\end{aligned}$$

Suppose now that u is lsc and let $(t,z) \in [0,T) \times \bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq1}$ be a sequence in $[0,T)\times\bar{\mathcal{S}}$ converging to (t,z) (as n goes to infinity). By definition of the lsc envelope $(\mathcal{H}u)_*$, we then have :

$$\mathcal{H}u(t,z) \;=\; \mathcal{H}u_*(t,z) \;\leq\; (\mathcal{H}u)_*(t,z) \;\;\leq\;\; \liminf_{n\to\infty} \mathcal{H}u(t_n,z_n),$$

which shows the lower-semicontinuity of $\mathcal{H}u$.
(ii) Fix some $(t,z) \in [0,T) \times \bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq 1}$ be a sequence in $[0,T) \times \bar{\mathcal{S}}$ converging to (t,z) (as n goes to infinity). Since u^* is usc, Γ is continuous, and $\mathcal{C}(z_n)$ is compact for each $n \geq 1$, there exists a sequence $(\hat{\zeta}_n)_{n\geq 1}$ with $\hat{\zeta}_n \in \mathcal{C}(z_n)$ such that :

$$\mathcal{H}u^*(t_n,z_n) \;\; = \;\; u^*(t_n,\Gamma(z_n,\hat{\zeta}_n)), \quad \forall n \geq 1.$$

By Remark 4.1, the sequence $(\hat{\zeta}_n)_{n\geq 1}$ converges, up to a subsequence, to some $\hat{\zeta} \in \mathcal{C}(z)$. Therefore, we get :

$$\mathcal{H}u^*(t,z) \;\geq\; u^*(t,\Gamma(z,\hat{\zeta})) \;\;\geq\;\; \limsup_{n\to\infty} u^*(t_n,\Gamma(z_n,\hat{\zeta}_n)) \;=\; \limsup_{n\to\infty} \mathcal{H}u^*(t_n,z_n),$$

which proves that $\mathcal{H}u^*$ is usc.

On the other hand, fix some $(t,z) \in [0,T] \times \bar{\mathcal{S}}$ and let $(t_n,z_n)_{n\geq 1}$ be a sequence in $[0,T] \times \bar{\mathcal{S}}$ converging to (t,z) and s.t. $\mathcal{H}u(t_n,z_n)$ converges to $(\mathcal{H}u)^*(t,z)$. Then, we have

$$(\mathcal{H}u)^*(t,z) \;=\; \lim_{n\to\infty} \mathcal{H}u(t_n,z_n) \;\;\leq\;\; \limsup_{n\to\infty} \mathcal{H}u^*(t_n,z_n) \;\leq\; \mathcal{H}u^*(t,z),$$

which shows that $(\mathcal{H}u)^* \leq \mathcal{H}u^*$. □

We may then prove by standard arguments, using DPP (2.13), the supersolution property.

Proof of supersolution property on $[0,T) \times \mathcal{S}$.
First, for any $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, we see, as a consequence of (DPP) (2.13) applied to $\tau = t$, and by choosing any admissible control $\alpha \in \mathcal{A}(t,z)$ with immediate impulse at t of arbitrary size $\zeta \in \mathcal{C}(z)$, that $v(t,z) \geq \mathcal{H}v(t,z)$. Now, let $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{S}$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $v_*(\bar{t},\bar{z}) = \varphi(\bar{t},\bar{z})$ and $\varphi \leq v_*$ on $[0,T) \times \mathcal{S}$. Since $v \geq \mathcal{H}v$ on $[0,T] \times \bar{\mathcal{S}}$, we obtain by combining with Lemma 5.1 (i) that $\mathcal{H}v_*(\bar{t},\bar{z}) \leq (\mathcal{H}v)_*(\bar{t},\bar{z}) \leq v_*(\bar{t},\bar{z})$, and so it remains to show that

$$-\frac{\partial \varphi}{\partial t}(\bar{t},\bar{z}) - \mathcal{L}\varphi(\bar{t},\bar{z}) \;\;\geq\;\; 0. \tag{5.6}$$

From the definition of v_*, there exists a sequence $(t_m,z_m)_{m\geq 1} \in [0,T) \times \mathcal{S}$ s.t. (t_m,z_m) and $v(t_m,z_m)$ converge respectively to $(\bar{t},\bar{z})$ and $v_*(\bar{t},\bar{z})$ as m goes to infinity. By continuity of φ, we also have that $\gamma_m := v(t_m,z_m) - \varphi(t_m,z_m)$ converges to 0 as m goes to infinity. Since $(\bar{t},\bar{z}) \in [0,T) \times \mathcal{S}$, there exists $\eta > 0$ s.t. for m large enough, $t_m < T$ and $B(z_m,\eta/2)$ $\subset B(\bar{z},\eta) := \{|z-\bar{z}| < \eta\} \subset \mathcal{S}$. Let us then consider the admissible control in $\mathcal{A}(t_m,z_m)$ with no impulse until the first exit time τ_m before T of the associated state process $Z_s = Z_s^{0,t_m,z_m}$ from $B(z_m,\eta/2)$:

$$\tau_m \;\;=\;\; \inf\left\{s \geq t_m \;:|Z_s^{0,t_m,z_m} - z_m| \geq \eta/2\right\} \wedge T.$$

Consider also a strictly positive sequence $(h_m)_m$ s.t. h_m and γ_m/h_m converge to zero as m goes to infinity. By using the dynamic programming principle (2.13) for $v(t_m,z_m)$ and $\hat{\tau}_m$ $:= \tau_m \wedge (t_m + h_m)$, we get :

$$v(t_m,z_m) \;=\; \gamma_m + \varphi(t_m,z_m) \;\;\geq\;\; \mathbb{E}[v(\hat{\tau}_m, Z_{\hat{\tau}_m}^{0,t_m,z_m})] \;\geq\; E[\varphi(\hat{\tau}_m, Z_{\hat{\tau}_m}^{0,t_m,z_m}),$$

since $\varphi \leq v_* \leq v$ on $[0,T) \times \mathcal{S}$. Now, by applying Itô's formula to $\varphi(s, Z_s^{0,t_m,z_m})$ between t_m and $\hat{\tau}_m$ and noting that the integrand of the stochastic integral term is bounded, we obtain by taking expectation :

$$\frac{\gamma_m}{h_m} + \mathbb{E}\left[\frac{1}{h_m}\int_{t_m}^{\hat{\tau}_m}\left(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi\right)(s, Z_s^{0,t_m,z_m})ds\right] \geq 0. \tag{5.7}$$

By continuity a.s. of Z_s^{0,t_m,z_m}, we have for m large enough, $\hat{\tau}_m = t_m + h_m$, and so by the mean-value theorem, the random variable inside the expectation in (5.7) converges a.s. to $(-\frac{\partial \varphi}{\partial t} - \mathcal{L}\varphi)(\bar{t}, \bar{z})$ as m goes to infinity. Since this random variable is also bounded by a constant independent of m, we conclude by the dominated convergence theorem and obtain (5.6).

We next prove the subsolution property, by using DPP (2.14) and contraposition argument.

Proof of subsolution property on $[0,T) \times \bar{\mathcal{S}}$.

Let $(\bar{t}, \bar{z}) \in [0,T) \times \bar{\mathcal{S}}$ and $\varphi \in C^{1,2}([0,T) \times \bar{\mathcal{S}})$ s.t. $v^*(\bar{t}, \bar{z}) = \varphi(\bar{t}, \bar{z})$ and $\varphi \geq v^*$ on $[0,T) \times \bar{\mathcal{S}}$. If $v^*(\bar{t}, \bar{z}) \leq \mathcal{H}v^*(\bar{t}, \bar{z})$ then the subsolution inequality holds trivially. Consider now the case where $v^*(\bar{t}, \bar{z}) > \mathcal{H}v^*(\bar{t}, \bar{z})$ and argue by contradiction by assuming on the contrary that

$$\eta := -\frac{\partial \varphi}{\partial t}(\bar{t}, \bar{z}) - \mathcal{L}\varphi(\bar{t}, \bar{z}) > 0.$$

By continuity of φ and its derivatives, there exists some $\delta_0 > 0$ s.t. $\bar{t} + \delta_0 < T$ and for all $0 < \delta \leq \delta_0$:

$$-\frac{\partial \varphi}{\partial t}(t, z) - \mathcal{L}\varphi(t, z) > \frac{\eta}{2}, \quad \forall\, (t, z) \in ((\bar{t} - \delta)_+, \bar{t} + \delta) \times B(\bar{z}, \delta) \cap \bar{\mathcal{S}}. \tag{5.8}$$

From the definition of v^*, there exists a sequence $(t_m, z_m)_{m \geq 1} \in ((\bar{t} - \delta/2)_+, \bar{t} + \delta/2) \times B(\bar{z}, \delta/2) \cap \bar{\mathcal{S}}$ s.t. (t_m, z_m) and $v(t_m, z_m)$ converge respectively to $(\bar{t}, \bar{z})$ and $v^*(\bar{t}, \bar{z})$ as m goes to infinity. By continuity of φ, we also have that $\gamma_m := v(t_m, z_m) - \varphi(t_m, z_m)$ converges to 0 as m goes to infinity. By the dynamic programming principle (2.14), given $m \geq 1$, there exists $\hat{\alpha}^m = (\hat{\tau}_n^m, \hat{\zeta}_n^m)_{n \geq 1}$ s.t. for any stopping time τ valued in $[t_m, T]$, we have

$$v(t_m, z_m) \leq \mathbb{E}[v(\tau, \hat{Z}_\tau^m)] + \frac{1}{m}.$$

Here $\hat{Z}^m$ is the state process, starting from z_m at t_m, and controlled by $\hat{\alpha}^m$. By choosing $\tau = \bar{\tau}^m := \hat{\tau}_1^m \wedge \tau_\delta^m$ where

$$\tau_\delta^m = \inf\left\{s \geq t_m : \hat{Z}_s^m \notin B(z_m, \delta/2)\right\} \wedge (t_m + \delta/2)$$

is the first exit time before $t_m + \delta/2$ of $\hat{Z}^m$ from the open ball $B(z_m, \delta/2)$, we then get :

$$\begin{aligned} v(t_m, z_m) &\leq \mathbb{E}[v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\tau_\delta^m < \hat{\tau}_1^m}] + \mathbb{E}[v(\bar{\tau}^m, \Gamma(\hat{Z}_{\bar{\tau}^m,-}, \hat{\zeta}_1^m)) 1_{\hat{\tau}_1^m \leq \tau_\delta^m}] + \frac{1}{m} \\ &\leq \mathbb{E}[v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\tau_\delta^m < \hat{\tau}_1^m}] + \mathbb{E}[\mathcal{H}v(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-}) 1_{\hat{\tau}_1^m \leq \tau_\delta^m}] + \frac{1}{m}. \end{aligned} \tag{5.9}$$

Now, since $\mathcal{H}v \leq v \leq v^* \leq \varphi$ on $[0,T) \times \bar{\mathcal{S}}$, we obtain :

$$\varphi(t_m, z_m) + \gamma_m \quad \leq \quad \mathbb{E}[\varphi(\bar{\tau}^m, \hat{Z}_{\bar{\tau}^m,-})] + \frac{1}{m}.$$

By applying Itô's formula to $\varphi(s, \hat{Z}_s^m)$ between t_m and $\bar{\tau}_m$, we then get :

$$\gamma_m \quad \leq \quad \mathbb{E}\left[\int_{t_m}^{\bar{\tau}_m} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(s, \hat{Z}_s^m) ds\right] + \frac{1}{m} \;\leq\; -\frac{\eta}{2}\mathbb{E}[\bar{\tau}_m - t_m] + \frac{1}{m},$$

from (5.8). This implies

$$\lim_{m \to \infty} \mathbb{E}[\bar{\tau}_m] \quad = \quad \bar{t}. \tag{5.10}$$

On the other hand, we have by (5.9)

$$v(t_m, z_m) \quad \leq \quad \sup_{\substack{|t'-t|<\delta \\ |z'-z|<\delta}} v(t', z')\mathbb{P}[\tau_\delta^m < \hat{\tau}_1^m] + \sup_{\substack{|t'-t|<\delta \\ |z'-z|<\delta}} \mathcal{H}v(t', z')\mathbb{P}[\hat{\tau}_1^m \leq \tau_\delta^m] + \frac{1}{m}.$$

From (5.10), we then get by sending m to infinity :

$$v^*(\bar{t}, \bar{z}) \quad \leq \quad \sup_{\substack{|t'-t|<\delta \\ |z'-z|<\delta}} \mathcal{H}v(t', z').$$

Hence, sending δ to zero and by Lemma 5.1 (ii), we have

$$v^*(\bar{t}, \bar{z}) \quad \leq \quad \lim_{\delta \downarrow 0} \sup_{\substack{|t'-t|<\delta \\ |z'-z|<\delta}} \mathcal{H}v(t', z') \;=\; (\mathcal{H}v)^*(\bar{t}, \bar{z}) \;\leq\; \mathcal{H}^*v(\bar{t}, \bar{z}),$$

which is the required contradiction.

5.2 Comparison principle

We finally turn to uniqueness question. Our next main result is a comparison principle for constrained (discontinuous) viscosity solutions to the quasi-variational inequality (5.1). It states that we can compare a viscosity subsolution to (5.1) on $[0,T) \times \bar{\mathcal{S}}$ and a viscosity supersolution to (5.1) on $[0,T) \times \mathcal{S}$, provided that we can compare them at the terminal date (as usual in parabolic problems) but also on the part D_0 of the solvency boundary.

Theorem 5.2 *Suppose* $u \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}) \cap USC([0,T) \times \bar{\mathcal{S}})$ *is a viscosity subsolution to* (5.1) *in* $[0,T) \times \bar{\mathcal{S}}$ *and* $w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}}) \cap LSC([0,T) \times \bar{\mathcal{S}})$ *is a viscosity supersolution to* (5.1) *in* $[0,T) \times \mathcal{S}$ *such that :*

$$u(t,z) \quad \leq \quad \liminf_{(t',z') \to (t,z)} w(t', z'), \;\forall\, (t,z) \in [0,T) \times D_0, \tag{5.11}$$

$$u(T,z) := \limsup_{\substack{(t,z') \to (T,z) \\ t<T,\, z' \in \mathcal{S}}} u(t,z') \quad \leq \quad w(T,z) := \liminf_{\substack{(t,z') \to (T,z) \\ t<T,\, z' \in \mathcal{S}}} w(t,z'), \quad \forall\, z \in \bar{\mathcal{S}}. \tag{5.12}$$

Then,

$$u \quad \leq \quad w \quad \text{on} \;\; [0,T] \times \mathcal{S}.$$

Remark 5.3 Notice that one cannot hope to derive a comparison principle in the whole closed region $\bar{\mathcal{S}}$ since it would imply the continuity of the value function on $\bar{\mathcal{S}}$, which is not true, see Remark 4.5.

In order to deal with the impulse obstacle in the comparison principle, we first produce some suitable perturbation of viscosity supersolutions. This strict viscosity supersolution argument was introduced by [21], and used e.g. in [1] for dealing with gradient constraints in singular control problem.

Lemma 5.2 *Let* $\gamma' \in (0,1)$ *and choose* ρ' *s.t.*

$$\rho' \quad > \quad \frac{\gamma'}{1-\gamma'} \frac{b^2 + r^2 + \sigma^2 r(1-\gamma')}{\sigma^2} \vee b \vee (\sigma^2 - b)$$

Given $\nu \geq 0$*, consider the perturbation smooth function on* $[0,T] \times \bar{\mathcal{S}}$ *:*

$$\phi_\nu(t,z) \quad = \quad e^{\rho'(T-t)} \left[\tilde{L}(z)^{\gamma'} + \nu \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right) \right]. \tag{5.13}$$

Let $w \in LSC([0,T) \times \bar{\mathcal{S}})$ *be a viscosity supersolution to* (5.1) *in* $[0,T) \times \mathcal{S}$*. Then for any* $m \geq 1$*, any compact set* $\mathcal{K}$ *of* $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$*, the usc function*

$$w_m \quad = \quad w + \frac{1}{m} \phi_\nu$$

is a strict viscosity supersolution to (5.1) *in* $[0,T) \times \mathcal{S} \cap \mathcal{K}$ *: there exists some constant* δ *(depending on* $\mathcal{K}$*) s.t.*

$$\min \left[-q_0 - rxq_1 - bpq_3 - \frac{1}{2}\sigma^2 p^2 M_{33}, w_m(t,z) - \mathcal{H} w_m(t,z) \right] \quad \geq \quad \frac{\delta}{m}, \tag{5.14}$$

for all $(t, z = (x,y,p)) \in [0,T) \times \mathcal{S} \cap \mathcal{K}$*,* $(q_0, q = (q_i)_{1 \leq i \leq 3}, M = (M_{ij})_{1 \leq i,j \leq 3}) \in \bar{J}^{2,-} w_m(t,z)$*. Moreover, for* $\gamma \in (0,\gamma')$ *and* $\nu > 0$*, if* $w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$*, and* u *is also a function in* $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$*, then for any* $t \in [0,T]$*,* $m \geq 1$*,*

$$\lim_{|z| \to \infty} (u - w_m)(t,z) \quad = \quad -\infty. \tag{5.15}$$

Proof. We set

$$f_1(t,z) \;=\; e^{\rho'(T-t)} \tilde{L}(z)^{\gamma'}, \qquad f_2(t,z) \;=\; e^{\rho'(T-t)} \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right).$$

From (4.4), we have for all $t \in [0,T)$, $z \in \mathcal{S} \setminus \tilde{\mathcal{N}} = \{z \in \mathcal{S} \,:\, \tilde{L}(z) \geq k\}$:

$$f_1(t, \Gamma(z,\zeta)) \quad \leq \quad e^{\rho'(T-t)} (\tilde{L}(z) - k)^{\gamma'}, \quad \forall \zeta \in \mathcal{C}(z),$$

and so

$$(f_1 - \mathcal{H} f_1)(t,z) \quad \geq \quad e^{\rho'(T-t)} \left[\tilde{L}(z)^{\gamma'} - (\tilde{L}(z) - k)^{\gamma'} \right] \;>\; 0 \tag{5.16}$$

Notice that relation (5.16) holds trivially true when $z \in \mathcal{N}$ since in this case $\mathcal{C}(z) = \emptyset$ and so $\mathcal{H}f(t,z) = -\infty$. We then deduce that for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_0 > 0$ s.t.

$$f_1 - \mathcal{H}f_1 \quad \geq \quad \delta_0, \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}.$$

Moreover, a direct calculation shows that for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$, $\zeta \in \mathcal{C}(z)$, $f_2(t, \Gamma(z,\zeta))$ $= f_2(t,z)$, and so

$$f_2 - \mathcal{H}f_2 \quad = \quad 0.$$

This implies

$$\begin{aligned} \phi_\nu - \mathcal{H}\phi_\nu &= f_1 + \nu f_2 - \mathcal{H}(f_1 + \nu f_2) \geq (f_1 - \mathcal{H}f_1) + \nu(f_2 - \mathcal{H}f_2) \\ &\geq \delta_0, \quad \text{on } [0,T) \times \mathcal{S} \cap \mathcal{K}. \end{aligned} \tag{5.17}$$

On the other hand, the same calculation as in (4.23) shows that for ρ' large enough, actually strictly larger than $\frac{\gamma'}{1-\gamma'} \frac{b^2 + r^2 + \sigma^2 r(1-\gamma')}{\sigma^2}$, we have $-\frac{\partial f_1}{\partial t} - \mathcal{L}f_1 > 0$ on $[0,T) \times \mathcal{S}$. Hence, for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_1 > 0$ s.t.

$$-\frac{\partial f_1}{\partial t} - \mathcal{L}f_1 \quad \geq \quad \delta_1 \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}.$$

A direct calculation also shows that for all $(t,z) \in [0,T] \times \bar{\mathcal{S}}$:

$$-\frac{\partial f_2}{\partial t}(t,z) - \mathcal{L}f_2(t,z) \quad = \quad e^{\rho'(T-t)} \left[(\rho' + b - \sigma^2) \frac{e^{\lambda y}}{p} + (\rho' - b) p e^{-\lambda y} \right] \geq 0,$$

since $\rho' \geq (\sigma^2 - b) \vee b$. This implies that for any compact set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$, there exists some constant $\delta_1 > 0$ s.t.

$$\begin{aligned} -\frac{\partial \phi_\nu}{\partial t} - \mathcal{L}\phi_\nu &= -\frac{\partial f_1}{\partial t} - \mathcal{L}f_1 + \nu \left(-\frac{\partial f_2}{\partial t} - \mathcal{L}f_2 \right) \\ &\geq \delta_1 \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K}. \end{aligned} \tag{5.18}$$

By writing the viscosity supersolution property of w, we deduce from the inequalities (5.17)-(5.18) the viscosity supersolution of w_m to

$$\min \left[-\frac{\partial w_m}{\partial t} - \mathcal{L}w_m, w_m - \mathcal{H}w_m \right] \quad \geq \quad \frac{\delta}{m} \quad \text{on} \quad [0,T) \times \mathcal{S} \cap \mathcal{K},$$

and so (5.14), where we set $\delta = \delta_0 \wedge \delta_1$. Finally, since $u, w \in \mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$, we have for some positive constant K :

$$\begin{aligned} (u - w_m)(t,z) &\leq K \left[1 + \left(x + \frac{p}{\lambda} \right)^\gamma \right] - \frac{1}{m} \left[\left(x + \frac{p}{\lambda} \right)^{\gamma'} + \nu \left(\frac{e^{\lambda y}}{p} + p e^{-\lambda y} \right) \right] \\ &\longrightarrow -\infty, \quad \text{as } |z| \to \infty, \end{aligned}$$

since $\gamma' > \gamma$ and $\nu > 0$. □

We now follow general viscosity solution technique, based on the Ishii technique (see [11]) and adapt arguments from [20], [28] for handling with specificities coming from the nonlocal intervention operator $\mathcal{H}$ and [4], [1] for the boundary conditions. The general idea is to build a test function so that the minimum associated with the (strict) supersolution cannot be on the boundary. However, the usual method in [31] does not apply here since it requires the continuity of the supersolution on the boundary, which is precisely not the case in our model. Instead, we adapt a method in [4], which requires the smoothness of the boundary. This is the case here except on the part D_0 of the boundary, but for which one has proved directly in Corollary 4.6 the continuity of the value function.

Proof of Theorem 5.2

Let u and w as in Theorem 5.2. We (re)define w on $[0,T) \times \partial\mathcal{S}$ by :

$$w(t,z) \quad = \quad \liminf_{\substack{(t',z') \to (t,z) \\ (t',z') \in [0,T) \times \mathcal{S}}} w(t',z'), \quad \forall (t,z) \in [0,T) \times \partial\mathcal{S}, \tag{5.19}$$

and construct a strict viscosity supersolution to (5.1) according to Lemma 5.2, by considering for $m \geq 1$, the usc function on $[0,T) \times \bar{\mathcal{S}}$:

$$w_m \quad = \quad w + \frac{1}{m}\phi_\nu, \tag{5.20}$$

where ϕ_ν is given in (5.13) for some given $\nu > 0$. Recalling the definitions (5.12) of u and w on $\{T\} \times \bar{\mathcal{S}}$, we have then an extension of u and w_m, which are usc and lsc on $[0,T] \times \bar{\mathcal{S}}$. In order to prove the comparison principle, it is sufficient to show that $\sup_{[0,T]\times\bar{\mathcal{S}}}(u - w_m)$ ≤ 0 for all $m \geq 1$, since the required result is obtained by letting m to infinity. We argue by contradiction and suppose that there exists some $m \geq 1$ s.t.

$$\mu \; := \sup_{[0,T]\times\bar{\mathcal{S}}} (u - w_m) \quad > \quad 0.$$

Since $u - w_m$ is usc on $[0,T] \times \bar{\mathcal{S}}$, $\lim_{|z|\to\infty}(u - w_m)(z) = -\infty$ by (5.15), $(u - w_m)(T,.) \leq$ 0 by (5.12), and $(u - w_m)(t,z) \leq 0$ for $(t,z) \in [0,T) \times D_0$ by (5.11), there exists a an open set $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+^*$ with closure $\bar{\mathcal{K}}$ compact s.t.

$$\text{Arg}\max_{[0,T]\times\bar{\mathcal{S}}}(u - w_m) \;\neq\; \emptyset \quad \subset \quad [0,T) \times \bar{\mathcal{S}} \setminus D_0 \cap \mathcal{K}.$$

Take then some $(t_0, z_0) \in [0,T) \times \bar{\mathcal{S}} \setminus D_0 \cap \mathcal{K}$ s.t. $\mu = (u - w_m)(t_0, z_0)$ and distinguish the two cases :

• *Case 1.* : $z_0 \in \partial\mathcal{S} \setminus D_0 \cap \mathcal{K}$.

$\star$ From (5.19), there exists a sequence $(t_i, z_i)_{i\geq 1}$ in $[0,T) \times \mathcal{S} \cap \mathcal{K}$ converging to (t_0, z_0) s.t. $w_m(t_i, z_i)$ tends to $w_m(t_0, z_0)$ when i goes to infinity. We then set $\beta_i = |t_i - t_0|$, $\varepsilon_i = |z_i - z_0|$ and consider the function Φ_i defined on $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$ by :

$$\begin{aligned} \Phi_i(t,t',z,z') &= u(t,z) - w_m(t',z') - \varphi_i(t,t',z,z') && (5.21) \\ \varphi_i(t,t',z,z') &= |t - t_0|^2 + |z - z_0|^4 + \frac{|t - t'|^2}{2\beta_i} + \frac{|z - z'|^2}{2\varepsilon_i} + \left(\frac{d(z')}{d(z_i)} - 1\right)^4. \end{aligned}$$

Here $d(z)$ denotes the distance from z to $\partial\mathcal{S}$. We claim that for $z_0 \notin D_0$, there exists an open neighborhood $\mathcal{V}_0 \subset \mathcal{K}$ of z_0 in which this distance function $d(.)$ is twice continuously differentiable with bounded derivatives. This is well-known (see e.g. [18]) when z_0 lies in the smooth parts $\partial\mathcal{S} \setminus (D_k \cup C_1 \cup C_2)$ of the boundary $\partial\mathcal{S}$. This is also true for $z_0 \in D_k \cup C_1 \cup C_2$. Indeed, at these corner lines of the boundary, the inner normal vectors form an acute angle (positive scalar product) and thus one can extend from z_0 the boundary to a smooth boundary so that the distance d is equal, locally on a neighborhood of z_0, to the distance to this smooth boundary. Notice that this is not true when $z_0 \in D_0$, which forms a right angle. Now, since Φ_i is usc on the compact set $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$, there exists $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \in [0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$ that attains its maximum on $[0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$:

$$\mu_i \;:=\; \sup_{[0,T]^2 \times (\bar{\mathcal{S}} \cap \mathcal{K})^2} \Phi_i(t, t', z, z') \;=\; \Phi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i).$$

Moreover, there exists a subsequence, still denoted $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i)_{i \geq 1}$, converging to some $(\hat{t}_0, \hat{t}'_0, \hat{z}_0, \hat{z}'_0) \in [0,T]^2 \times (\bar{\mathcal{S}} \cap \bar{\mathcal{K}})^2$. By writing that $\Phi_i(t_0, t_i, z_0, z_i) \leq \Phi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i)$, we have :

$$u(t_0, z_0) - w_m(t_i, z_i) - \frac{1}{2}\left(|t_i - t_0| + |z_i - z_0|\right) \tag{5.22}$$

$$\leq \;\; \mu_i \;=\; u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) - \left(|\hat{t}_i - t_0|^2 + |\hat{z}_i - z_0|^4\right) - R_i \tag{5.23}$$

$$\leq \;\; u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) - \left(|\hat{t}_i - t_0|^2 + |\hat{z}_i - z_0|^4\right), \tag{5.24}$$

where we set

$$R_i \;=\; \frac{|\hat{t}_i - \hat{t}'_i|^2}{2\beta_i} + \frac{|\hat{z}_i - \hat{z}'_i|^2}{2\varepsilon_i} + \left(\frac{d(\hat{z}'_i)}{d(z_i)} - 1\right)^4.$$

From the boundedness of u, w_m on $[0,T] \times \bar{\mathcal{S}} \cap \bar{\mathcal{K}}$, we deduce by inequality (5.23) the boundedness of the sequence $(R_i)_{i \geq 1}$, which implies $\hat{t}_0 = \hat{t}'_0$ and $\hat{z}_0 = \hat{z}'_0$. Then, by sending i to infinity into (5.22) and (5.24), with the uppersemicontinuity (resp. lowersemicontinuity) of u (resp. w_m), we obtain $\mu = u(t_0, z_0) - w_m(t_0, z_0) \leq u(\hat{t}_0, \hat{z}_0) - w_m(\hat{t}_0, \hat{z}_0) - |\hat{t}_0 - t_0|^2 - |\hat{z}_0 - z_0|^4$. By the definition of μ, this shows :

$$\hat{t}_0 \;= \hat{t}'_0 \;=\; t_0, \qquad \hat{z}_0 \;= \hat{z}'_0 \;=\; z_0. \tag{5.25}$$

Sending again i to infinity into (5.22)-(5.23)-(5.24), we thus derive that $\mu \leq \lim_i \mu_i = \mu - \lim_i R_i \leq \mu$, and so

$$\mu_i \;\longrightarrow\; \mu \tag{5.26}$$

$$\frac{|\hat{t}_i - \hat{t}'_i|^2}{2\beta_i} + \frac{|\hat{z}_i - \hat{z}'_i|^2}{2\varepsilon_i} + \left(\frac{d(\hat{z}'_i)}{d(z_i)} - 1\right)^4 \;\to\; 0, \tag{5.27}$$

as i goes to infinity. In particular, for i large enough , we have $\hat{t}_i$, $\hat{t}'_i < T$ (since $t_0 < T$), $d(\hat{z}'_i) \geq d(z_i)/2 > 0$, and so $\hat{z}'_i \in \mathcal{S}$. For i large enough, we may also assume that $\hat{z}_i$, $\hat{z}'_i$ lie in the neighborhood $\mathcal{V}_0$ of z_0 so that the derivatives upon order 2 of d at $\hat{z}_i$ and $\hat{z}'_i$ exist and are bounded.

$\star$ We may then apply Ishii's lemma (see Theorem 8.3 in [11]) to $(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \in [0,T) \times [0,T) \times \bar{\mathcal{S}} \cap \mathcal{V}_0 \times \mathcal{S} \cap \mathcal{V}_0$ that attains the maximum of Φ_i in (5.21). Hence, there exist 3×3 matrices $M = (M_{jl})_{1 \leq j,l \leq 3}$ and $M' = (M'_{jl})_{1 \leq j,l \leq 3}$ s.t. :

$$\begin{aligned} (q_0, q, M) &\in \bar{J}^{2,+} u(\hat{t}_i, \hat{z}_i), \\ \left(q'_0, q', M'\right) &\in \bar{J}^{2,-} w_m(\hat{t}'_i, \hat{z}'_i) \end{aligned}$$

where

$$q_0 = \frac{\partial \varphi_i}{\partial t}(\hat{t}_i, \hat{t}'_i, \hat{z}, \hat{t}'_i), \qquad q = (q_j)_{1 \leq j \leq 3} = D_z \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \tag{5.28}$$

$$q'_0 = -\frac{\partial \varphi_i}{\partial t}(\hat{t}_i, \hat{t}'_i, \hat{z}, \hat{t}'_i), \qquad q' = (q'_j)_{1 \leq j \leq 3} = -D_{z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i). \tag{5.29}$$

and

$$\begin{pmatrix} M & 0 \\ 0 & -M' \end{pmatrix} \leq D^2_{z,z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) + \varepsilon_i \left(D^2_{z,z'} \varphi_i(\hat{t}_i, \hat{t}'_i, \hat{z}_i, \hat{z}'_i) \right)^2 \tag{5.30}$$

By writing the viscosity subsolution property (5.4) of u and the strict viscosity supersolution (5.14) of w_m, we have :

$$\min\left[-q_0 - r\hat{x}_i q_1 - b\hat{p}_i q_3 - \frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33}, u(\hat{t}_i, \hat{z}_i) - \mathcal{H}u(\hat{t}_i, \hat{z}_i) \right] \leq 0 \tag{5.31}$$

$$\min\left[-q'_0 - r\hat{x}'_i q'_1 - b\hat{p}'_i q'_3 - \frac{1}{2}\sigma^2 \hat{p}'^2_i M'_{33}, w_m(\hat{t}'_i, \hat{z}'_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) \right] \geq \frac{\delta}{m}. \tag{5.32}$$

We then distinguish the following two possibilities in (5.31) :

1. $u(\hat{t}_i, \hat{z}_i) - \mathcal{H}u(\hat{t}_i, \hat{z}_i) \leq 0$.

Since, from (5.32), we also have: $w_m(\hat{t}'_i, \hat{z}'_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) \geq \frac{\delta}{m}$, we obtain by combining these two inequalities :

$$\mu_i \leq u(\hat{t}_i, \hat{z}_i) - w_m(\hat{t}'_i, \hat{z}'_i) \leq \mathcal{H}u(\hat{t}_i, \hat{z}_i) - \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) - \frac{\delta}{m}$$

Sending i to ∞, and by (5.26), we obtain :

$$\begin{aligned} \mu &\leq \limsup_{i \to \infty} \mathcal{H}u(\hat{t}_i, \hat{z}_i) - \liminf_{i \to \infty} \mathcal{H}w_m(\hat{t}'_i, \hat{z}'_i) - \frac{\delta}{m} \\ &\leq \mathcal{H}u(t_0, z_0) - \mathcal{H}w_m(t_0, z_0) - \frac{\delta}{m}, \end{aligned}$$

from (5.25) and where we used the upper-semicontinuity of $\mathcal{H}u$ and the lower-semicontinuity of $\mathcal{H}w_m$ (see Lemma 5.1). By compactness of $\mathcal{C}(z_0)$, and since u is usc, there exists some $\zeta_0 \in \mathcal{C}(z_0)$ s.t. $\mathcal{H}u(t_0, z_0) = u(t_0, \Gamma(z_0, \zeta_0))$. We then get the desired contradiction :

$$\begin{aligned} \mu &\leq \mathcal{H}u(t_0, z_0) - \mathcal{H}w_m(t_0, z_0) - \frac{\delta}{m} \\ &\leq u(t_0, \Gamma(z_0, \zeta_0)) - w_m(t_0, \Gamma(z_0, \zeta_0)) - \frac{\delta}{m} \leq \mu - \frac{\delta}{m}. \end{aligned}$$

2. $-q_0 - r\hat{x}_i q_1 - b\hat{p}_i q_3 - \frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33} \leq 0$.

Since, from (5.32), we also have: $-q_0' - r\hat{x}_i' q_1' - b\hat{p}_i' q_3' - \frac{1}{2}\sigma^2 \hat{p}_i'^2 M_{33}' \geq \frac{\delta}{m}$, we obtain by combining these two inequalities :

$$-(q_0 - q_0') - r(\hat{x}_i q_1 - \hat{x}_i' q_1') - b(\hat{p}_i q_3 - \hat{p}_i' q_3') - \frac{1}{2}\sigma^2(\hat{p}_i^2 M_{33} - \hat{p}_i'^2 M_{33}') \quad \leq \quad -\frac{\delta}{m}. \tag{5.33}$$

Now, from (5.28)-(5.29), we explicit :

$$\begin{aligned} q_0 &= 2(\hat{t}_i - t_0) + \frac{(\hat{t}_i - \hat{t}_i')}{\beta_i}, \qquad q = 4(\hat{z}_i - z_0)|\hat{z}_i - z_0|^2 + \frac{(\hat{z}_i - \hat{z}_i')}{\varepsilon_i} \\ q_0' &= \frac{(\hat{t}_i - \hat{t}_i')}{\beta_i}, \qquad q' = \frac{(\hat{z}_i - \hat{z}_i')}{\varepsilon_i} - 4Dd(\hat{z}_i')\left(\frac{d(\hat{z}_i')}{d(z_i)} - 1\right)^3 \end{aligned}$$

and we see by (5.25) and (5.27) that $q_0 - q_0'$, $\hat{x}_i q_1 - \hat{x}_i' q_1'$ and $\hat{p}_i q_3 - \hat{p}_i' q_3'$ converge to zero when i goes to infinity. Moreover, from (5.30), we have :

$$\frac{1}{2}\sigma^2 \hat{p}_i^2 M_{33} - \frac{1}{2}\sigma^2 \hat{p}_i'^2 M_{33}' \quad \leq \quad \mathcal{E}_i, \tag{5.34}$$

where

$$\begin{aligned} \mathcal{E}_i &= A_i \left(D^2_{z,z'}\varphi_i(\hat{t}_i, \hat{t}_i', \hat{z}_i, \hat{z}_i') + \varepsilon_i \left(D^2_{z,z'}\varphi_i(\hat{t}_i, \hat{t}_i', \hat{z}_i, \hat{z}_i') \right)^2 \right) A_i^{\intercal} \\ &= A_i \left(\begin{pmatrix} \frac{1}{\varepsilon_i} I_3 + P_i & -\frac{1}{\varepsilon_i} I_3 \\ -\frac{1}{\varepsilon_i} I_3 & \frac{1}{\varepsilon_i} I_3 + Q_i \end{pmatrix} + \varepsilon_i \begin{pmatrix} \frac{1}{\varepsilon_i} I_3 + P_i & -\frac{1}{\varepsilon_i} I_3 \\ -\frac{1}{\varepsilon_i} I_3 & \frac{1}{\varepsilon_i} I_3 + Q_i \end{pmatrix}^2 \right) A_i^{\intercal} \end{aligned}$$

with

$$\begin{aligned} A_i &= \left(0, 0, \hat{p}_i, 0, 0, \hat{p}_i'\right), \quad P_i = 4|\hat{z}_i - z_0|^2 I_3 + 8(\hat{z}_i - z_0)(\hat{z}_i - z_0)^{\intercal} \\ Q_i &= 12\left(\frac{d(\hat{z}_i')}{d(\hat{z}_i')} - 1\right)^2 Dd(\hat{z}_i')Dd(\hat{z}_i')^{\intercal} + 4\left(\frac{d(\hat{z}_i')}{d(\hat{z}_i')} - 1\right)^3 D^2 d(\hat{z}_i'). \end{aligned}$$

Here $^{\intercal}$ denotes the transpose operator. After some straightforward calculation, we then get :

$$\mathcal{E}_i = 3\frac{(\hat{p}_i' - \hat{p}_i)^2}{\varepsilon_i} + A_i \left(\begin{pmatrix} 3P_i & -2Q_i \\ -2P_i & 3Q_i \end{pmatrix} + \varepsilon_i \begin{pmatrix} P_i^2 & 0 \\ 0 & Q_i^2 \end{pmatrix} \right) A_i^{\intercal},$$

which converges also to zero from (5.25) and (5.27). Therefore, by sending i to infinity into (5.33), we see that the lim sup of its l.h.s. is nonnegative, which gives the required contradiction : $0 \leq -\delta/m$.

• *Case 2.* : $z_0 \in \mathcal{S} \cap \mathcal{K}$.

This case is dealt similarly as in Case 1. and its proof is omitted. It suffices e.g. to consider the function

$$\begin{aligned} \Psi_i(t, z, z') &= u(t, z) - w_m(t, z') - \psi_i(t, z, z') \\ \psi_i(t, z, z') &= |t - t_0|^2 + |z - z_0|^4 + \frac{i}{2}|z - z'|^2, \end{aligned}$$

for $i \geq 1$, and to take a maximum $(\tilde{t}_i, \tilde{z}_i, \tilde{z}'_i)$ of Ψ_i. We then show that the sequence $(\tilde{t}_i, \tilde{z}_i, \tilde{z}'_i)_{i\geq 1}$ converges to (t_0, z_0, z_0) as i goes to infinity and we apply Ishii's lemma to get the required contradiction.

By combining previous results, we then finally obtain the following PDE characterization of the value function.

Corollary 5.1 *The value function v is continuous on $[0,T) \times \mathcal{S}$ and is the unique (in $[0,T) \times \mathcal{S}$) constrained viscosity solution to (5.1) lying in $\mathcal{G}_\gamma([0,T] \times \bar{\mathcal{S}})$ and satisfying the boundary condition :*

$$\lim_{(t',z')\to(t,z)} v(t', z') \quad = \quad 0, \quad \forall (t, z) \in [0, T) \times D_0,$$

and the terminal condition

$$v(T^-, z) := \lim_{\substack{(t,\, z') \,\to\, (T,\, z) \\ t \,<\, T,\, z' \,\in\, \mathcal{S}}} v(t, z') \quad = \quad \bar{U}(z), \quad \forall z \in \bar{\mathcal{S}}.$$

Proof. From Theorem 5.1, v^* is an usc viscosity subsolution to (5.1) in $[0,T) \times \bar{\mathcal{S}}$ and v_* is a lsc viscosity supersolution to (5.1) in $[0,T) \times \mathcal{S}$. Moreover, by Corollary 4.6 and Proposition 4.3, we have $v^*(t,z) = v_*(t,z) = 0$ for all $(t,z) \in [0,T) \times D_0$, and $v^*(T,z) = v_*(T,z) = \bar{U}(z)$ for all $z \in \bar{\mathcal{S}}$. Then by Theorem 5.2, we deduce $v^* \leq v_*$ on $[0,T] \times \mathcal{S}$, which proves the continuity of v on $[0,T) \times \mathcal{S}$. On the other hand, suppose that $\tilde{v}$ is another constrained viscosity solution to (5.1) with $\lim_{(t',z')\to(t,z)} v(t',z') = 0$ for $(t,z) \in [0,T) \times D_0$ and $\tilde{v}(T^-, z) = \bar{U}(z)$ for $z \in \bar{\mathcal{S}}$. Then, $\tilde{v}^*(t,z) = v_*(t,z) = v^*(t,z) = \tilde{v}_*(t,z)$ for $(t,z) \in [0,T) \times D_0$ and $\tilde{v}^*(T,z) = v_*(T,z) = v^*(T,z) = \tilde{v}_*(T,z)$ for z in $\bar{\mathcal{S}}$. We then deduce by Theorem 5.2 that $v^* \leq \tilde{v}_* \leq \tilde{v}^* \leq v_*$ on $[0,T] \times \mathcal{S}$. This proves $v = \tilde{v}$ on $[0,T) \times \mathcal{S}$. □

6 Conclusion

We formulated a model for optimal portfolio selection under liquidity risk and price impact. Our main result is a characterization of the value function as the unique constrained viscosity solution to the quasi-variational Hamilton-Jacobi-Bellman inequality associated to this impulse control problem under solvency constraint. The main technical difficulties come from the nonlinearity due to price impact, and the state constraint. They are overcomed with the specific exponential form of the price impact function : a natural theoretical question is to extend our results for general price impact functions. Once we have provided a complete PDE characterization of the value function, the next step, from an applied view point, is to numerically solve this quasi-variational inequality. This can be realized for instance by iterated optimal stopping problems as done in [10]. Moreover, from an economic viewpoint, it would be of course interesting to analyse the effects of liquidity risk and price impact in our model on the optimal portfolio in a classical market without frictions, e.g. the Merton model. These numerical and economical studies are in current investigation.

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Appendix : Proof of Lemma 4.1

We first prove the following elementary lemma.

Lemma A.1 *For any $y \in \mathbb{R}$, there exists an unique $\bar{\zeta}(y) \in \mathbb{R}$ s.t.*

$$\bar{g}(y) \;:=\; \max_{\zeta \in \mathbb{R}} g(y,\zeta) \;=\; \bar{\zeta}(y)(e^{-\lambda y} - e^{\lambda \bar{\zeta}(y)}). \tag{A.1}$$

The function $\bar{g}$ is differentiable, decreasing on $(-\infty, 0)$, increasing on $(0,\infty)$, with $\bar{g}(0) = 0$, $\lim_{y\to-\infty} \bar{g}(y) = \infty$, $\lim_{y\to\infty} \bar{g}(y) = e^{-1}/\lambda$, and for all $p > 0$,

$$\ell(y,p) + p\bar{g}(y) < 0 \;\; if \;\; y < 0 \quad\; and \quad -\ell(y,p) + p\bar{g}(y) < 0 \;\; if \;\; 0 < y \le \frac{1}{\lambda}.$$

Proof. (i) For fixed y, a straightforward study of the differentiable function $\zeta \to g_y(\zeta) := g(y,\zeta)$ shows that there exists an unique $\bar{\zeta}(y) \in \mathbb{R}$ such that :

$$\begin{aligned} G(y,\bar{\zeta}(y)) \;=\; g'_y(\bar{\zeta}(y)) \quad &= \quad e^{-\lambda y} - e^{\lambda\bar{\zeta}(y)}(1+\lambda\bar{\zeta}(y)) \;=\; 0, \\ g'_y(\zeta) > (\text{ resp. } <)\ 0 \quad &\Longleftrightarrow \quad \zeta < (\text{ resp. } >)\ \bar{\zeta}(y) \end{aligned}$$

This proves that g_y is increasing on $(-\infty,\bar{\zeta}(y))$ and decreasing on $(\bar{\zeta}(y),\infty)$ with

$$\max_{\zeta\in\mathbb{R}} g_y(\zeta) \;=\; g_y(\bar{\zeta}(y)) \;:=\; \bar{g}(y),$$

i.e. (A.1). Since $g'_y(-1/\lambda) = e^{-\lambda y} > 0$, we notice that $\bar{\zeta}(y)$ is valued in $(-1/\lambda,\infty)$ for all $y \in \mathbb{R}$. Moreover, since the differentiable function $(y,\zeta) \to G(y,\zeta) := g'_y(\zeta)$ is decreasing in y on $\mathbb{R}$: $\dfrac{\partial G}{\partial y} < 0$ and decreasing in ζ on $(-1/\lambda,\infty)$: $\dfrac{\partial G}{\partial \zeta} < 0$, we derive by the implicit functions theorem that $\bar{\zeta}(y)$ is a differentiable decreasing function on $\mathbb{R}$. Since $G(y,-1/\lambda) = e^{-\lambda y}$ goes to zero as y goes to infinity, we also obtain that $\bar{\zeta}(y)$ goes to $-1/\lambda$ as y goes to infinity. By noting that for all ζ, $G(y,\zeta)$ goes to ∞ when y goes to $-\infty$, we deduce that $\bar{\zeta}(y)$ goes to ∞ as y goes to $-\infty$. Since $G(0,0) = 0$, we also have $\bar{\zeta}(0) = 0$. Notice also that $G(y,-y) = \lambda y e^{-\lambda y}$: hence, when $y < 0$, $G(y,-y) < 0 = G(y,\bar{\zeta}(y))$ so that $\bar{\zeta}(y) < -y$, and when $y > 0$, $G(y,-y) > 0 = G(y,\bar{\zeta}(y))$ so that $\bar{\zeta}(y) > -y$.

(ii) By the envelope theorem, the function $\bar{g}$ defined by $\bar{g}(y) = \max_{\zeta\in\mathbb{R}} g(y,\zeta) = g(y,\bar{\zeta}(y))$ is differentiable on $\mathbb{R}$ with

$$\bar{g}'(y) \;=\; \frac{\partial g}{\partial y}(y,\bar{\zeta}(y)) \;=\; -\lambda\bar{\zeta}(y)e^{-\lambda\bar{\zeta}(y)}, \quad y \in \mathbb{R}.$$

Since $\bar{\zeta}(y) >$ (resp. $<$) 0 iff $y <$ (resp. $>$) 0 with $\bar{\zeta}(0) = 0$, we deduce the decreasing (resp. increasing) property of $\bar{g}$ on $(-\infty,0)$ (resp. $(0,\infty)$) with $\bar{g}(0) = 0$. Since $\bar{\zeta}(y)$ converges to $-1/\lambda$ as y goes to infinity, we immediately see from expression (A.1) of $\bar{g}$ that $\bar{g}(y)$ converges to e^{-1}/λ as y goes to infinity. For $y < 0$ and by taking $\zeta = -y/2$ in the maximum in (A.1), we have $\bar{g}(y) \geq -y(e^{-\lambda y} - e^{-\lambda y/2})/2$, which shows that $\bar{g}(y)$ goes to infinity as y goes to $-\infty$. When $y < 0$, we have $0 < \bar{\zeta}(y) < -y$, and thus by (A.1), we get :

$$\bar{g}(y) \;<\; -y\left(e^{-\lambda y} - e^{\lambda\bar{\zeta}(y)}\right),$$

and so $\ell(y,p) + p\bar{g}(y) < pye^{\lambda\bar{\zeta}(y)} < 0$ for all $p > 0$. When $y > 0$, we have $\bar{\zeta}(y) < 0$ and thus by (A.1), we get : $\bar{g}(y) < -\bar{\zeta}(y)e^{\lambda\bar{\zeta}(y)}$. Now, since the function $\zeta \mapsto -\zeta e^{\lambda\zeta}$ is decreasing on $[-1/\lambda, 0]$, we have for all $0 < y \leq 1/\lambda$, $-1/\lambda \leq -y < \bar{\zeta}(y)$ and so

$$\bar{g}(y) \;<\; ye^{-\lambda y}.$$

This proves $p\bar{g}(y) \leq \ell(y,p)$ for all $0 < y \leq 1/\lambda$ and $p > 0$. □

Proof of Lemma 4.1. For any $z \in \bar{\mathcal{S}}$, we write $\mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$ where $\mathcal{C}_0(z) = \{\zeta \in \mathbb{R} : L_0(\Gamma(z,\zeta)) \geq 0\}$ and $\mathcal{C}_1(z) = \{\zeta \in \mathbb{R} : L_1(\Gamma(z,\zeta)) \geq 0, y + \zeta \geq 0\}$. From (4.1) and by noting that the function $\zeta \mapsto pg(y,\zeta)$ goes to $-\infty$ as $|\zeta|$ goes to infinity, we see that $\mathcal{C}_0(z)$ is bounded. Since the function $\zeta \mapsto p\theta(\zeta,p)$ goes to infinity as ζ goes to infinity, we also see that $\mathcal{C}_1(z)$ is bounded. Hence, $\mathcal{C}(z)$ is bounded. Moreover, for any $z = (x,y,p) \in \bar{\mathcal{S}}$, the function $\zeta \mapsto L(\Gamma(z,\zeta))$ is uppersemicontinuous : it is indeed continuous on $\mathbb{R} \setminus \{-y\}$ and uppersemicontinuous on $-y$. This implies the closure property and then the compactness of $\mathcal{C}(z)$.

$\star$ Fix some arbitrary $z \in \partial^y \mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = x - k + pg(0,\zeta) - k$. Since $g(0,\zeta) \leq 0$ for all $\zeta \in \mathbb{R}$ and $x \leq k$, we see that $L(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = x - \theta(\zeta,p) - k$. Since $\theta(\zeta,p) \geq 0$ for all $\zeta \geq 0$, and recalling that $x < k$, we also see that $L_1(\Gamma(z,\zeta)) = x - \theta(\zeta,p) - k < 0$ for all $\zeta \geq 0$. Therefore $L(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$ and so $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_0^x \mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k$. Now, we recall from Remark 2.2 that $\ell(y,p) \leq p/(\lambda e) < k$. Moreover, by Lemma A.1, we have $pg(y,\zeta) \leq p\bar{g}(y) \leq p/(\lambda e) < k$. This implies $L_0(\Gamma(z,\zeta)) < 0$ for any $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = -\theta(\zeta,p) - k$. Since $\theta(\zeta,p) \geq -p/(\lambda e)$ for all $\zeta \in \mathbb{R}$, we get $L_1(\Gamma(z,\zeta)) \leq p/(\lambda e) - k < 0$. Therefore $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_1^x \mathcal{S}$. Then, for any $\zeta \in \mathbb{R}$, we have $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k$. Now, we recall from Remark 2.2 that $\ell(y,p) < k$. Moreover, since $0 < y \leq 1/\lambda$, we get from Lemma A.1 : $pg(y,\zeta) \leq p\bar{g}(y) < \ell(y,p) < k$ for all $\zeta \in \mathbb{R}$. This implies $L_0(\Gamma(z,\zeta)) < 0$ for any $\zeta \in \mathbb{R}$. On the other hand, we have $L_1(\Gamma(z,\zeta)) = -\theta(\zeta,p) - k$. Since the function $\zeta \mapsto \theta(\zeta,p)$ is increasing on $[-1/\lambda, \infty)$ and $y < 1/\lambda$, we have for all $\zeta \geq -y$, $\theta(\zeta,p) \geq \theta(-y,p) = -\ell(y,p)$, and so $-\theta(\zeta,p) - k \leq \ell(y,p) - k < 0$. This implies $L_1(\Gamma(z,\zeta)) < 0$ for all $\zeta \in \mathbb{R}$ and thus $\mathcal{C}(z)$ is empty.

$\star$ Fix some arbitrary $z \in \partial_2^x \mathcal{S}$. Then for $\zeta = -1/\lambda$, we have $\theta(\zeta,p) = -p/(\lambda e)$ and $y + \zeta > 0$ (see Remark 2.2). Hence, $L(\Gamma(z,-1/\lambda)) \geq L_1(\Gamma(z,-1/\lambda)) \geq 0$ and so $-1/\lambda \in \mathcal{C}(z)$. Moreover, take some arbitrary $\zeta \in \mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$. In the case where $\zeta \in \mathcal{C}_0(z)$, i.e. $L_0(\Gamma(z,\zeta)) = \ell(y,p) - k + pg(y,\zeta) - k \geq 0$, and recalling that $\ell(y,p) < k$, we must have necessarily $g(y,\zeta) > 0$. This implies $-y < \zeta < 0$. Similarly, when $\zeta \in \mathcal{C}_1(z)$, i.e. $-\theta(\zeta,p) - k \geq 0$ and $y + \zeta \geq 0$, we must have $-y < \zeta < 0$. Therefore, $\mathcal{C}(z) \subset (-y, 0)$.

$\star$ Fix some arbitrary $z \in \partial_\ell^- \mathcal{S} \cup \partial_\ell^+ \mathcal{S}$. Then we have $L(\Gamma(z,-y)) = L_1(\Gamma(z,-y)) = 0$, which shows that $\zeta = -y \in \mathcal{C}(z)$. Consider now the case where $z \in \partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}$. We claim that $\mathcal{C}_1(z) = \{-y\}$. Indeed, take some $\zeta \in \mathcal{C}_1(z)$, i.e. $x - \theta(\zeta,p) - k \geq 0$ and $y + \zeta \geq 0$. Then, $\theta(\zeta,p) \geq \theta(-y,p) = -\ell(y,p)$ (since $\zeta \mapsto \theta(\zeta,p)$ is increasing on $[-1/\lambda, \infty)$ and $-y \geq -1/\lambda$) and so $0 \leq x - \theta(\zeta,p) - k \leq x + \ell(y,p) - k = 0$. Hence, we must have $\zeta = -y$. Take now some arbitrary $\zeta \in \mathcal{C}_0(z)$. Hence, $L_0(\Gamma(z,\zeta)) = pg(y,\zeta) - k \geq 0$, and we must have necessarily $g(y,\zeta) \geq 0$. Since $y \leq 0$, this implies $0 \leq \zeta \leq -y$. We have then showed that $\mathcal{C}(z) \subset [-y, 0]$. Consider now the case where $z \in \partial_\ell^+ \mathcal{S}$ and take some arbitrary $\zeta \in \mathcal{C}(z) = \mathcal{C}_0(z) \cup \mathcal{C}_1(z)$. If $\zeta \in \mathcal{C}_0(z)$, then similarly as above, we must have $pg(y,\zeta) - k \geq 0$. Since $y > 0$, this implies $-y \leq \zeta < 0$. If $\zeta \in \mathcal{C}_1(z)$, i.e. $x - \theta(\zeta,p) - k \geq 0$ and $y + \zeta \geq 0$, and recalling that $x < k$, we must have also $-y \leq \zeta < 0$. We have then showed that $\mathcal{C}(z) \subset [-y, 0)$.

Notice that for $z \in (\partial_\ell^- \mathcal{S} \cup \partial_\ell^+ \mathcal{S}) \cap \mathcal{N}_\ell$, we have $L_0(\Gamma(z,\zeta)) \leq p\bar{g}(y) - k < 0$ for all $\zeta \in \mathbb{R}$. Hence, $\mathcal{C}_0(z) = \emptyset$. We have already seen that $\mathcal{C}_1(z) = \{-y\}$ when $z \in \partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}$ and so $\mathcal{C}(z) = \{-y\}$ when $z \in (\partial_\ell^- \mathcal{S} \cup \partial_\ell^{+,\lambda} \mathcal{S}) \cap \mathcal{N}_\ell$.