Optimal market dealing under constraints *

Etienne Chevalier[†] [‡] M'hamed Gaïgi [§] Vathana Ly Vath [¶] [‡] Mohamed Mnif [‖]

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Abstract

We consider a market dealer acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset in a quote-driven market. The market dealer may benefit from the bid-ask spread but has the obligation to permanently quote both prices while satisfying some liquidity and inventory constraints. The objective is to maximize the expected utility from terminal liquidation value over a finite horizon and subject to the above constraints. We characterize the value function as the unique viscosity solution to the associated HJB equation and further enrich our study with numerical results. The contributions of our study, as compared to previous studies [2], [12], [15], concern both the modelling aspects and the dynamic structure of the control strategies. Important features and constraints characterizing market making problems are no longer ignored. Indeed, along with the obligation to continuously quote bid and ask prices, we do not allow the market maker to stop quoting them when the stock inventory reaches its lower or higher bound. Furthermore, we no longer assume the existence of a reference price.

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†Laboratoire de Mathématiques et Modélisation d'Evry, Université d'Evry, France , etienne.chevalier@univ-evry.fr.

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§ENIT-LAMSIN, University of Tunis El Manar, and Laboratoire de Mathématiques et Modélisation d'Evry, Université d'Evry, mhamed.gaigi@ens.univ-evry.fr

¶Laboratoire de Mathématiques et Modélisation d'Evry, Université d'Evry and ENSIIE, France, ly-vath@ensiie.fr.

‖ENIT-LAMSIN, University of Tunis El Manar, mohamed.mnif@enit.rnu.tn

1 Introduction

We consider a financial market with a single dealer or monopolistic market maker acting as a liquidity provider by continuously setting bid and ask prices for an illiquid asset. In most studies on financial markets, it is assumed that investors are price-takers, i.e. liquidity takers, in the sense that they trade any financial asset at the available prices with a liquidity premium that must be paid for immediacy. It is clear from the structure of financial markets that, in addition to the presence of price-takers, there must necessarily exist market participants who are price-setters or liquidity providers. In limit order book markets or order-driven markets such as the NYSE, traders can post prices and quantities at which they are willing to buy or sell while waiting for a counterparty to engage in that trade. In dealers' markets or quote-driven markets, for instance the Nasdaq or LSE (London Stock Exchange), registered market makers quote bids and offers and serve as the intermediary between public traders. More precisely, registered market makers act as counterparties when an investor wishes to buy or sell the securities.

In this paper, we consider an equity quote-driven market with a single risky equity assets. In the trading of most equity assets in either Nasdaq or LSE, there are several registered market makers in competition. However, in our study, in order to focus on the modelling of the market making strategies, we consider there is only one "representative" registered market maker in the dealing of the equity assets. The presence of several registered market makers dealing in a competitive environment will be studied in future research. In conformity with a dealer market as mentioned above, we assume that the market maker has a contractual obligation to permanently quote bid and ask prices for this security and therefore to satisfy any sell and buy market order from investors. Indeed, in order to obtain the role of a market maker of an assigned security, a firm has to sign an agreement with the stock exchange which contains many obligations that the firm has to satisfy. Continuously quoting binding bid and ask prices inside of the maximum spread is one of those obligations. The role of the market maker is very important in the trading of illiquid assets as she acts as a facilitator of trades between different investors. The market maker may therefore benefit from the bid-ask spread but faces a number of constraints, in particular the liquidity and inventory constraints. Indeed, the obligation imposed upon the market maker to meet investors orders may make the position of the market maker very risky. For instance, when the market maker has to buy stocks successively due to investors' sell market orders, her stock holding position may become very large and positive, which is very risky in the event of a downturn of the market.

The structural constraints imposed upon market makers in dealer markets are proved to be a major challenge. In the study of market making/dealing problems, we may refer to Avellaneda and Stoikov [2], Ho and Stoll [12], and Mildenstein and Schleef [15]. In [2] and [12], the authors consider a market making problem as described above but within a financial market in which the risky assets has a reference price or a fair price S_t which is assumed to follow an arithmetic brownian process. The market maker quotes her ask and bid prices as respectively $S_t + \delta_t^a$ and $S_t - \delta_t^b$, where (δ_t^a, δ_t^b) are both positive and represent the strategy control of the market maker. The price processes, ie. bid, ask or mid prices, are therefore mainly driven by the reference price process. In [15], the authors consider the existence of a constant price at which they liquidate their inventory at terminal date. Within a multi-period setting, they determine the optimal bid and ask prices that maximize the discounted expected cash flow.

In our study, we do not assume the existence of a reference price. The prices are therefore uniquely driven by the equilibrium between buy and sell market orders. An imbalance between buy and sell market orders, for instance, in the case when the arrivals of sell orders largely exceed those of the buy orders, is expected to move down the bid and ask prices quoted by the market maker. Our assumption realistically takes into account the very features of the financial markets. Indeed, the assumption of the existence of a reference price and the possibility to liquidate the inventory at that price may be suitable only in some specific cases. It is the case when the assets security, for instance, some trackers, futures or shares of holding companies with quoted affiliates, have highly liquid underlyings.

In terms of mathematical modelling and resolution, the most difficult challenge to overcome is to take into account the inventory constraints that the market maker is facing. First, we consider, as [15], that the market maker has the obligation to respect the risk constraint imposed upon her by her company's risk department. We may refer to [9] which investigates the impact of the inventory constraints on the market making problem studied in [2]. Indeed, the stock inventory of the market maker is assumed to have upper and lower bounds which could be high enough to allow some trading flexibility to the market maker. However, unlike in [9], once the inventory reaches the lower (upper) bound, we do not allow the market maker to stop submitting limit ask (bid) order since allowing such move violates the agreement that the market maker's firm has agreed with the financial stock exchange to continuously quote bid and ask prices. As such, in our model, the market maker complies with both obligations to quote and to keep her stock inventory within the lower and upper bounds. Should the market maker violate this inventory risk constraint, we may assume that her role as the market maker is terminated by her firm and her inventory position may be liquidated.

A second important difference with the problems studied in [2], [12] and [15], comes from the assumption that the market maker may liquidate her stock inventory on terminal date at the reference price or a constant price independent of the inventory. In other words, the only inventory risk is due to either the randomness of the reference price [2], or to the uncertainty reflected in a diffusion term of the inventory process itself, [12]. However, from different studies on liquidation costs and price impacts, see for instance [6], [11], [14], and [17], it is clear that the degree of ability to liquidate the stock inventory at the reference price or the mid-price should not be ignored. In our paper, we assume that when the market maker has to liquidate her stock inventory, she incurs a liquidity cost and the price per share received (paid) are lower (higher) than the mid-price in the case of a long (short) position. Our assumption on the form of liquidation function is mainly inspired by [11] and [17]. Under this liquidation assumption, there is equally a trade-off between the gains she could get from the bid-ask spread and the potential loss it will occur when she liquidates her position.

Furthermore, as in [15], to take into account the microstructure of the financial markets, we no longer consider continuous price processes. Bid and ask prices quoted by the market maker are realistically assumed to be discrete prices, and more precisely correspond to multiples of a tick value. However, unlike in [15], we do not assume that prices take values in a finite set.

In terms of market orders arrivals, we assume that the arrival of buy (and sell) market orders submitted by investors follow a Cox process with a regime-shifting Markov intensity. This assumption is inspired from recent literature on liquidity problems, see for instance [2] and [7]. One may expect the regime-shifting in intensity to strongly impact the trend of the bid and ask price processes. As in [7], [2], [12] and [15], we assume that the market maker has access to full information on the market and may in particular observe the whole market orders arrivals process, including the Markov intensity process. However, we may refer to Kyle [14] and Glosten and Milgrom [8], where the authors investigate market making problems under the context of asymmetric information. In their studies, the presence of bid-ask spread is purely due to the presence of an insider trader.

The objective of the market maker is to maximize the expected utility of the terminal wealth. However, we consider that the market maker should avoid, as much as possible, violating the inventory risk constraint imposed by her firm, since her firm may terminate her own position as a market maker. We therefore introduce, in the objective function, a penalty cost self-imposed by the market maker herself or her firm, in order to reduce the inventory risk. It is worth noticing that this penalty cost, together with some other features such as the presence of the liquidation costs, largely prevent the market maker from being able to manipulate the stock price.

The contributions of our study, as compared to previous studies [2], [12], [15], concern both the modelling aspects and the dynamic structure of the control strategies. Important features and constraints characterizing market making problems are no longer ignored. Indeed, along with the obligation to continuously quote bid and ask prices, we do not allow the market maker to stop quoting them when the inventory stock reaches its lower or higher bound. Furthermore, we no longer assume the existence of a reference price. We equally stop assuming that the market maker may liquidate her stock inventory at the reference price. Such an assumption is, in our view, in contradiction with the very essence of the studies on market liquidity risk and impact. As a result, these above additional features turn our market making problem into a non-standard control problem under constraints with real modelling and mathematical challenges.

We provide rigorous mathematical characterization and analysis to our control problem by proving that our value functions are the unique viscosity solutions to the associated HJB system. It is always a technical challenge when applying viscosity techniques to non-standard control problems under constraints. In the proof of our comparison theorem, a major problem is to circumvent the difficulty arising from the discontinuity of our HJB operator on some parts of the solvency region boundary. One way to tackle this difficulty is to build specific test functions allowing us to prove the uniqueness by contradiction.

The rest of the paper is organized as follows. We define the model and formulate our optimal market making problem in the following section. In Sections 3 and 4, we obtain some analytical properties and prove the dynamic programming principle related to our control problem. These results enable us to obtain the characterization of the solution of the problem in terms of the unique viscosity solution to the associated HJB system. Finally, in Section 5, we further enrich our study with numerical results.

2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ where T is a finite horizon. We assume that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. We consider a financial market operated as a single dealer market, in which there is a trading risky assets. In this dealer market, there is a market maker who has the obligation to permanently quote bid and ask prices and to act as a counterparty to investors' market orders. We equally assume that investors, considered as price-takers, may only submit either buy or sell market orders.

2.1 Model settings

Trading orders. We denote by $(\theta_i^a)_{i\geq 1}$ (resp. $(\theta_i^b)_{i\geq 1}$) the sequence of non-decreasing $\mathbb{F}$-stopping times corresponding to the arrivals of buy (resp. sell) market orders. From the market maker's point of view, both sequences of stopping times correspond to trading times, i.e. the times when she has to act as a counterparty to investor's market orders. We denote by $(\xi_i)_{i\geq 1}$ the sequence of these trading times. When a buy (resp. sell) market order arrives at time θ_i^a (resp. θ_j^b), the market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by P^a (resp. P^b). As in [9] and [12], we assume here that transactions are of constant size, scaled to 1.

Market making strategies. We define a strategy control as being a $\mathbb{F}$-predictable process $\alpha = (\alpha_t)_{(0\leq t\leq T)} = (\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b)_{0\leq t\leq T}$ where the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ take values in $\{\chi_{min}, .., \chi_{max}\}$, with $-\chi_{min} \in \mathbb{N}$ and $\chi_{max} \in \mathbb{N}^*$.
We assume that when a sell market order arrives at time θ_j^b, the market maker may either keep the bid and ask prices constant or decrease one or both of them by at most χ_{max} ticks or increase one or both of them by at most χ_{min} ticks. Notice the market maker may decide to change the bid/ask prices but transaction prices are assumed to be based on the one quoted before the prices changes. In here, a tick value is denoted by a strictly positive constant δ. On the opposite side, when a buy market order arrives at time θ_k^a, the market maker may either keep the bid and ask prices constant or increase one or both of them by at most χ_{max} ticks or decrease one or both of them by at most χ_{min} ticks .
In order to illustrate our model, in the below Figure 1, we represent a path of the bid and ask prices for a given control process α such that $(\epsilon^a_{\theta_i^b}, \epsilon^b_{\theta_i^b})_{1\leq i\leq 3} = ((1,1);\ (0,0);\ (1,0))$ and $(\eta^a_{\theta_i^a}, \eta^b_{\theta_i^a})_{1\leq i\leq 3} = ((1,0);\ (0,1);\ (1,1))$.

Bid-Ask spread modelling.
We denote by $P^a = (P_t^a)_{0\leq t\leq T}$ (resp. $P^b = (P_t^b)_{0\leq t\leq T}$) the price quoted by the market maker to buyers (resp. sellers). Notice that $P^a \geq P^b$.

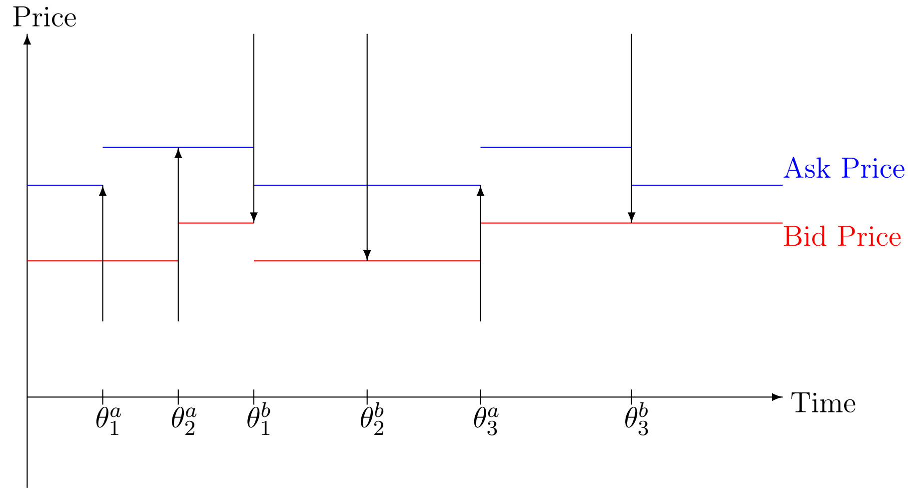

Figure 1: *Representation of bid and ask prices paths*

The dynamics of $P^{a,b}$ evolves according to the following equations

$$\begin{aligned} dP_t^{a,b} &= 0, \ \xi_i < t < \xi_{i+1} \\ P^{a,b}_{\theta^b_{j+1}} &= P^{a,b}_{\theta^{b-}_{j+1}} - \delta\epsilon^{a,b}_{\theta^b_{j+1}} \\ P^{a,b}_{\theta^a_{k+1}} &= P^{a,b}_{\theta^{a-}_{k+1}} + \delta\eta^{a,b}_{\theta^a_{k+1}}. \end{aligned}$$

where i is the number of transactions before time t, j the number of buy transactions before time t for the market maker, k the number of sell transactions before time t, and δ represents one tick.

We denote by P the mid-price and S the bid-ask spread of the stocks. The dynamics of the process (P,S) is given by

$$\begin{aligned} dP_t &= 0, \ \xi_i < t < \xi_{i+1} && (2.1) \\ P_{\theta^b_{j+1}} &= P_{\theta^{b-}_{j+1}} - \frac{\delta}{2}(\epsilon^a_{\theta^b_{j+1}} + \epsilon^b_{\theta^b_{j+1}}) && (2.2) \\ P_{\theta^a_{k+1}} &= P_{\theta^{a-}_{k+1}} + \frac{\delta}{2}(\eta^a_{\theta^a_{k+1}} + \eta^b_{\theta^a_{k+1}}), && (2.3) \end{aligned}$$

$$\begin{aligned} dS_t &= 0, \ \xi_i < t < \xi_{i+1} && (2.4) \\ S_{\theta^b_{j+1}} &= S_{\theta^{b-}_{j+1}} - \delta(\epsilon^a_{\theta^b_{j+1}} - \epsilon^b_{\theta^b_{j+1}}) && (2.5) \\ S_{\theta^a_{k+1}} &= S_{\theta^{a-}_{k+1}} + \delta(\eta^a_{\theta^a_{k+1}} - \eta^b_{\theta^a_{k+1}}). && (2.6) \end{aligned}$$

Regime switching. We first consider the tick time clock associated to a Poisson process $(R_t)_{0\leq t\leq T}$ with deterministic intensity λ defined on $[0,T]$, and representing the random times where the intensity of the orders arrival jumps.

We define a discrete-time stationary Markov chain $(\hat{I}_k)_{k\in\mathbb{N}}$, valued in the finite state space $\{1,...,m\}$, with probability transition matrix $(p_{ij})_{1\leq i,j\leq m}$, i.e. $\mathbb{P}[\hat{I}_{k+1}=j|\hat{I}_k=i]=p_{ij}$ s.t.

$p_{ii} = 0$, independent of R. We define the process

$$I_t = \hat{I}_{R_t},\ t \geq 0 \tag{2.7}$$

$(I_t)_t$ is a continuous time Markov chain with intensity matrix $\Gamma = (\gamma_{ij})_{1 \leq i,j \leq m}$, where $\gamma_{ij} = \lambda p_{ij}$ for $i \neq j$, and $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

We model the arrivals of buy and sell market orders by two Cox processes N^a and N^b. The intensity rate of N_t^a and N_t^b is given respectively by $\lambda^a(t, I_t, P_t, S_t)$ and $\lambda^b(t, I_t, P_t, S_t)$ where λ^a and λ^b are continuous functions valued in $\mathbb{R}$ and defined on $[0,T] \times \{1, ..., m\} \times \frac{\delta}{2}\mathbb{N} \times \delta\mathbb{N}$.

Remark 2.1 *The dependency of intensity rate on the assets prices is inspired by [2] and is used by many previous papers, see for instance [4] and [9], [10].*

We assume that:

$$\bar{\lambda} := \sup_{[0,T] \times \{1,...,m\} \times \frac{\delta}{2}\mathbb{N} \times \delta\mathbb{N}} \Big(\max(\lambda^a, \lambda^b, \lambda) \Big) < +\infty.$$

We now define θ_k^a (resp. θ_k^b) as the k^{th} jump time of N^a (resp. N^b), which corresponds to the k^{th} buy (sell) market order.

We introduce the following stopping times $\rho_j(t) = \inf\{u \geq t, I_u = j\}$ and $\rho(t) = \inf\{u \geq t, R_u > R_t\}$ for $0 \leq t \leq T$ and the notation $Z^{t,i,z,\alpha}$ is the state process associated to the control α such that $(I_t,\ Z_t^{t,i,z,\alpha}) = (i, z)$.

Remark 2.2 Price process under naive strategy.
This remark is inspired by [1]. Consider the so-called naive strategy which consists in increasing (resp. decreasing) both ask and bid prices by one tick when a buy (resp. sell) market order arrives. In that case, the market maker follows the constant strategy $(1,1,1,1)$, the spread is constant and the mid-price has the following dynamics.

$$\begin{cases} dP_t & = & 0,\ \xi_i < t < \xi_{i+1}, \\ P_{\theta_{j+1}^b} & = & P_{\theta_{j+1}^{b-}} - \delta, \\ P_{\theta_{k+1}^a} & = & P_{\theta_{k+1}^{a-}} + \delta. \end{cases}$$

We can show that, under some assumptions on the intensity processes of order arrivals, the mid price weakly converges to the solution of the following stochastic differential equation when the tick, δ goes to 0:

$$dP_t^0 = P_t^0 \left(\mu(t, I_t, P_t^0)dt + \sigma(t, I_t, P_t^0)dW_t \right). \tag{2.8}$$

where μ and σ are defined on $[0,T] \times \{1,..m\} \times \mathbb{R}^+$. Indeed, considering a C^1 function f from $\mathbb{R}$ into $\mathbb{R}$, we have that the infinitesimal generator associated to the mid price process P is given by

$$\begin{aligned} Lf(p) &= \lambda^a(t,i,p)(f(p+\delta) - f(p)) + \lambda^b(t,i,p)(f(p-\delta) - f(p)) \\ &= \delta[\lambda^a - \lambda^b](t,i,p)f'(p) + \frac{\delta^2}{2}[\lambda^a + \lambda^b](t,i,p)f''(p) + \delta^2\varepsilon(\delta). \end{aligned}$$

where $\varepsilon(\delta)$ converges to 0 when δ converges to 0. Under the following assumptions

$$\begin{cases} \delta[\lambda^a - \lambda^b](t,i,p) \longrightarrow \mu(t,i,p) & \text{as } \delta \longrightarrow 0 \\ \delta^2[\lambda^a + \lambda^b](t,i,p) \longrightarrow \sigma^2(t,i,p) & \text{as } \delta \longrightarrow 0, \end{cases}$$

we have

$$Lf(p) \longrightarrow \mu(t,i,p)f^{'}(p) + \frac{\sigma^2(t,i,p)}{2}f^{''}(p), \quad \text{as } \delta \longrightarrow 0.$$

From Jacod and Shiryaev [13], Theorem 4.21. chapter IX, the laws of the process P converges weakly to the law of the diffusion process solution of equation (2.8).

2.2 The control problem

Stock holdings. The number of shares held by the market maker at time $t \in [0,T]$ is denoted by Y_t, and Y satisfies the following equations

$$\begin{aligned} dY_t &= 0, \ \xi_i < t < \xi_{i+1} && (2.9)\\ Y_{\theta^b_{j+1}} &= Y_{\theta^{b-}_{j+1}} + 1 && (2.10)\\ Y_{\theta^a_{k+1}} &= Y_{\theta^{a-}_{k+1}} - 1, && (2.11) \end{aligned}$$

As in [9] and [15], we consider that the market maker has the obligation to respect the risk constraint imposed upon her by her company. Concretely, the stock inventory of the market maker is assumed to have upper and lower bounds which could be high enough to allow some trading flexibility to the market maker. Let $y_{min} < 0 < y_{max}$. We are therefore imposing the following inventory constraint

$$y_{min} \leq Y_t \leq y_{max} \text{ a.s. } 0 \leq t \leq T. \tag{2.12}$$

Cash holdings. We denote by $r > 0$ the instantaneous interest rate. The bank account follows the below equation between two trading times

$$dX_t = rX_t dt, \ \xi_i < t < \xi_{i+1}. \tag{2.13}$$

When a discrete trading occurs at time θ^b_{j+1} (resp. θ^a_{k+1}), the cash amount becomes

$$\begin{aligned} X_{\theta^b_{j+1}} &= X_{\theta^{b-}_{j+1}} - P^b_{\theta^{b-}_{j+1}} && (2.14)\\ X_{\theta^a_{k+1}} &= X_{\theta^{a-}_{k+1}} + P^a_{\theta^{a-}_{k+1}}, && (2.15) \end{aligned}$$

State process. We define the state process as follows:

$$Z = (X,\ Y,\ P := \frac{P^a + P^b}{2},\ S := P^a - P^b). \tag{2.16}$$

Cost of liquidation of the portfolio. If the current mid-price at time $t < T$ is p and the market maker decides to liquidate her portfolio, then we assume that the price she actually gets is

$$Q(t,y,p,s) = (p - \text{sign}(y)\frac{s}{2})f(t,y), \tag{2.17}$$

where f is an impact function defined from $[0, T] \times \mathbb{R}$ into $\mathbb{R}_+$.

We make the following assumption

Assumption (H1) *The impact function f is non-negative, non-increasing in y, and satisfies the following conditions*

$$\begin{array}{rcl} f(t,y) & \leq & f(t,y^{'}) \text{ if } y^{'} \leq y \\ yf(t^{'},y) & \leq & yf(t,y) \text{ if } t^{'} \leq t. \end{array}$$

Remark 2.3 **(H1)** suggests that the further from maturity when the market maker liquidates her block of shares, the more she is penalized. In addition, the bigger the block of shares to liquidate, the more she is penalized. This form of impact function is inspired by [11] and [17].

Liquidation value and Solvency constraints. A key issue for the market maker is to maximize the value of the net wealth at time T. In our framework, we impose a constraint on the spread i.e.

$$0 < S_t \leq K\delta, \ 0 \leq t \leq T,$$

where K is a positive constant. This constraint is consistent with the idea to insure a good level of liquidity in the financial market. The regulatory of the financial market has to find a consensus between the objective of the market maker whose aim is to increase her wealth and the liquidity of the financial market. This bid-ask spread constraint is generally part of the commitments that market maker's firm has taken in its contract with the Stock Exchange. We also impose that the bid price remains positive, therefore the market maker has to use controls such that

$$P_t - S_t/2 > 0.$$

When the market maker has to liquidate her portfolio at time t, her wealth will be $L(t, X_t, Y_t, P_t, S_t)$ where L is the liquidation function defined as follows

$$L(t, x, y, p, s) = x + yQ(t, y, p, s),$$

with Q as defined in 2.17.

Furthermore, we assume that in the case that the cash held by the market maker falls below a negative constant x_{min}, she has to liquidate her position. This constraint on x_{min} is a solvency constraint generally imposed by the market maker's employer since they do not have unlimited financing facilities. From the market maker's point of view, x_{min} is the threshold below which she shall not go.
We may now introduce the following state space

$$\mathbb{S} = (x_{min}, +\infty) \times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, ..., K\}.$$

and then the solvency region

$$\mathcal{S} = \{(t,x,y,p,s) \in [0,T] \times \mathbb{S} : \, p - \frac{s}{2} \geq \delta\}.$$

We denote its boundary and its closure by

$$\partial_x \mathcal{S} \quad = \quad \Big\{(t,x,y,p,s) \in [0,T] \times \bar{\mathbb{S}} : \, x = x_{min}\Big\} \quad \text{and} \quad \bar{\mathcal{S}} \; = \; \mathcal{S} \cup \partial_x \mathcal{S}.$$

Admissible trading strategies. Given $(t,z) := (t,x,y,p,s) \in \mathcal{S}$, we say that the strategy $\alpha = (\epsilon_u^a, \epsilon_u^b, \eta_u^a, \eta_u^b)_{t \leq u \leq T}$ is admissible, if the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{\chi_{min}, ..., \chi_{max}\}$ and for all $u \in [t,T]$, $(u, Z_u^{t,i,z,\alpha}) \in \mathcal{S}$. We denote by $\mathcal{A}(t,z)$ the set of all these admissible policies.

Value functions. The objective of the market maker should be to maximize the expected utility of the terminal wealth, obtained at the expiration of the market making contract. However, we consider that the market maker should avoid violating the inventory risk constraint 2.12 imposed by the risk department of her firm. We therefore introduce, in the objective function, a penalty cost self-imposed by the market maker herself in order to reduce the inventory risk. This penalty cost does not directly impact the wealth of the market maker, but only her control strategy in the optimization problem.

We set g a non-negative penalty function defined on $\{y_{min}, ..., y_{\max}\}$. This penalty may be compared to the holding costs function introduced in [15]. For numerical purposes, in Section 5, we will consider as in [15] a quadratic penalty cost function.

We also consider an exponential utility function U i.e. there exists $\gamma > 0$ such that $U(x) = 1 - e^{-\gamma x}$ for $x \in \mathbb{R}$. We set $U_L = U \text{o} L$.
As such, we consider the following value functions $(v_i)_{i \in \{1,...,m\}}$ which are defined on $\mathcal{S}$ by

$$v_i(t,z) := \sup_{\alpha \in \mathcal{A}(t,z)} J_i^\alpha(t,z) \tag{2.18}$$

where we have set

$$\begin{aligned} J_i^\alpha(t,z) \quad &:= \quad \mathbb{E}^{t,i,z}\left[U_L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T \wedge \tau^{t,i,z,\alpha})^-}) - \int_t^{T \wedge \tau^{t,i,z,\alpha}} g(Y_s^{t,i,y,\alpha}) ds\right], \\ \tau^{t,i,z,\alpha} \quad &:= \quad \inf\{u \geq t : X_u^{t,i,x,\alpha} \leq x_{min} \text{ or } Y_u^{t,i,y,\alpha} \in \{y_{min} - 1, y_{max} + 1\}\}. \end{aligned}$$

3 Analytical properties and dynamic programming principle

We use a dynamic programming approach to derive the system of partial differential equations satisfied by the value functions. First, we state the following Proposition in which we obtain some bounds of our value functions

Proposition 3.1 Bounds of the value functions. *There exist nonnegative constants, C_1, C_2 and C_3, depending on the parameters of our problem, such that*

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t,x,y,p,s) \leq 1, \quad \forall (i,t,x,y,p,s) \in \{1,...,m\} \times \mathcal{S},$$

Proof: Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \mathcal{S}$ and $\alpha \in \mathcal{A}(t, z)$. As U is lower than 1 and g positive, we obviously have $v_i(t, z) \leq 1$. Moreover, if we set $G = \sup_{y \in \{y_{min}, \cdots, y_{max}\}} g(y)$, we get

$$v_i(t, z) \geq 1 - \inf_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\left[\exp\left(-\gamma L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T \wedge \tau^{t,i,z,\alpha})^-})\right)\right] - GT.$$

We conclude the proof by applying the following Lemma. □

Lemma 3.1 *Let $\beta > 0$. For all $(i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times \mathcal{S}$, we have*

$$u_i(t, z) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\exp\left(-\beta L(T \wedge \tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T \wedge \tau^{t,i,z,\alpha})^-})\right)\right] \leq \exp\left(-\beta a + \bar{\lambda} T (e^{\beta b \chi_{max}} - 1)\right) e^{\beta b p},$$

where we have set $a = x_{min} + y_{min} f(0, y_{min}) \frac{K\delta}{2}$ and $b = -y_{min} f(0, y_{min})$.

Proof. See Appendix.

This technical Lemma equally allows us to show the next results on the Hölder continuity of the functions J_i and v_i. We begin with the following Lemma establishing the Hölder continuity with respect to the cash variable for the functions J_i^α.

Lemma 3.2 *For $\xi \in [0, -x_{min} e^{-rT})$, we set*

$$\phi(\xi) = -\frac{1}{r} \ln\left(1 - \frac{\xi e^{rT}}{\mid x_{min} \mid}\right) \quad \text{and} \quad \psi(\xi) = \sqrt{\phi(\xi)} + \phi(\xi) + \xi.$$

Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \bar{\mathcal{S}}$ and $x < x' < x - x_{min} e^{-rT}$. For all $\alpha \in \mathcal{A}(t, z)$, we have

$$\mid J_i^\alpha(t, z') - J_i^\alpha(t, z) \mid \leq K_1(p) \psi(x' - x),$$

where $K_1(p)$ is a positive constant depending only on p and $z' = (x', y, p, s)$.

Proof. See Appendix.

Now we turn to the Hölder continuity of the criterium function with respect to both time and cash variables.

Proposition 3.2
Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \bar{\mathcal{S}}$ and (t', x') in $[0, T] \times (x_{min}, +\infty)$ s.t.

$$x < x' < x - x_{min} e^{-rT}, \text{ and} \tag{3.19}$$

$$\mid t - t' \mid < \min\left(\frac{\mid x_{min} \mid e^{-2rT}}{r \mid x' \mid}, \frac{1}{r} \left| \ln\left(\left|\frac{x'}{x_{min}}\right|\right) \right|\right), \text{ if } x' \neq 0. \tag{3.20}$$

For all $\alpha \in \mathcal{A}(t \wedge t', z)$ such that $\alpha_s = 0$ for all $s \in [t \wedge t', t \vee t']$, we have $\alpha \in \mathcal{A}(t, z) \cap \mathcal{A}(t', z')$ with $z' = (x', y, p, s)$ and

$$\mid J_i^\alpha(t, z) - J_i^\alpha(t', z') \mid \;\leq\; K_2(p) \left(\psi(r e^{rT} \mid x'(t - t') \mid) + \psi(x' - x) + \mid t' - t \mid\right).$$

where $K_2(p)$ is a positive constant depending only on p.

Proof. See Appendix.

Proposition 3.3 ***Uniform Continuity of the value functions***
Let $(i, y, p, s) \in \{1, ..., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N}^* \times \delta\{1, ..K\}$ *such that* $p - \frac{s}{2} > 0$.
The function $(t, x) \to v_i(t, x, y, p, s)$ *is uniformly continuous on* $[0, T] \times [x_{min}, +\infty)$.

Proof: Throughout the proof, we set $V(u, \xi) = v_i(u, \xi, y, p, s)$ on $[0, T] \times [x_{min}, +\infty)$.
Let $(t, z) := (t, x, y, p, s) \in \bar{\mathcal{S}}$ and (t', x') in $[0, T] \times (x_{min}, +\infty)$ s.t. (t, x) and (t', x') satisfy conditions (3.19) and (3.20). We shall prove that

$$| V(t, x) - V(t', x') | \quad \leq \quad K_2(p) \Big(\psi(re^{rT} \mid x'(t - t') \mid) + \psi(x' - x) + \mid t' - t \mid\Big).$$

where $z' = (x', y, p, s)$ and $K_2(p)$ is a positive constant depending only on p.

Let $\varepsilon > 0$ and $\alpha^{*\prime} \in \mathcal{A}(t', z')$ such that $V(t', x') \leq J_i^{\alpha^{*\prime}}(t', x') + \varepsilon$. For $u \in [t \wedge t', T]$, we set $\alpha_u = \alpha_u^{*\prime} 1\!\!1_{\{u \geq t'\}}$. We have $\alpha \in \mathcal{A}(t', z') \cap \mathcal{A}(t, z)$ then it follows from Proposition 3.2 that

$$\begin{aligned} V(t', x') - V(t, x) \quad &\leq \quad J_i^{\alpha}(t', x') - J_i^{\alpha}(t, x) + \varepsilon \\ &\leq \quad K_2(p) \left(\psi(re^{rT} \mid x'(t - t') \mid) + \psi(x' - x) + \mid t' - t \mid\right) + \varepsilon. \end{aligned}$$

Now, we know that there exists $\alpha \in \mathcal{A}(t, z)$ such that $V(t, x) \leq J_i^{\alpha}(t, x) + \varepsilon$. For $u \in [t \wedge t', T]$, we set $\alpha_u = \alpha_u^* 1\!\!1_{\{u \geq t\}}$. We have $\alpha \in \mathcal{A}(t', z') \cap \mathcal{A}(t, z)$ then it follows from Proposition 3.2 that

$$\begin{aligned} V(t', x') - V(t, x) \quad &\geq \quad J_i^{\alpha}(t', x') - J_i^{\alpha}(t, x) - \varepsilon \\ &\geq \quad -K_2(p) \left(\psi(re^{rT} \mid x'(t - t') \mid) + \psi(x' - x) + \mid t' - t \mid\right) - \varepsilon. \end{aligned}$$

Letting ε going to 0, we obtain the result. □

For control problems, dynamic programming principle was frequently used by many authors. In our context, it is formulated as:

Theorem 3.1 ***Dynamic programming principle (DPP)***
Let $(i, t, z) := (i, t, x, y, p, s) \in \{1, ..., m\} \times \mathcal{S}$. *Let* ν *be a stopping time in* $\mathcal{T}_{t,T}$, *we have*

$$v_i(t, z) = \sup_{\alpha \in \mathcal{A}(t,z)} \hat{J}_i^{\alpha,\nu}(t, z), \tag{3.21}$$

where, for $\alpha \in \mathcal{A}(t, z)$, *we have set*

$$\begin{aligned} \hat{J}_i^{\alpha,\nu}(t, z) \quad = \quad &\mathbb{E}\Big[- g(y) \Big(\nu \wedge \hat{\theta} \wedge \hat{\tau}^{\alpha} - t\Big) + v_{I_{\nu \wedge \hat{\theta}}} \Big(\nu \wedge \hat{\theta},\ Z^{t,i,z,\alpha}_{\nu \wedge \hat{\theta}}\Big) 1\!\!1_{\{\nu \wedge \hat{\theta} < \hat{\tau}^{\alpha}\}} \\ &+ U_L \Big(\hat{\tau}^{\alpha}, xe^{r(\hat{\tau}^{\alpha} - t)}, y, p, s\Big) 1\!\!1_{\{\hat{\tau}^{\alpha} \leq \nu \wedge \hat{\theta}\}}\Big], \end{aligned} \tag{3.22}$$

with $\hat{\tau}^{\alpha} = \tau^{t,i,z,\alpha} \wedge T$, $\rho = \inf\{u \geq t : \ R_u > R_{u^-}\}$, $\theta^w = \inf\{u \geq t : \ N_u^{w,i,t,z} > N_{u^-}^{w,i,t,z}\}$, *for* $w \in \{a, b\}$ *and* $\hat{\theta} = \rho \wedge \theta^a \wedge \theta^b$.

Proof: To establish the Dynamic Programming Principle, we may adapt the proof of **Theorem 5.2** in [5]. The proof strongly relies on the continuity of the value functions J_i and v_i established in Propositions 3.2 and 3.3. However, for the sake of completeness, in the Appendix, we will provide the proof of this dynamic programming principle, which is always of interest, especially in the case of such a non-standard control problem.

4 Viscosity characterization of the value function

We first define the following set:

$$A(t,z) := \Big\{ \alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{-\chi_{min}, ..., \chi_{max}\}^4 \text{ s.t. } p - \frac{s}{2} - \delta\varepsilon^b \geq \delta,$$
$$\delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta, \text{ and } \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta \Big\}.$$

For all $(i,t,x,y,p,s) := (i,t,z) \in \{1,...,m\} \times \mathcal{S}$ and $\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in A(t,z)$, we introduce the two following operators:

$$\mathcal{A}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{min},p,s) & \text{if} \quad y = y_{min} \\ v_i(t, x+p+\frac{s}{2}, y-1, p+\frac{\delta}{2}(\eta^a+\eta^b), s+\delta(\eta^a-\eta^b)) & \text{otherwise} \end{cases}.$$

$$\mathcal{B}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{max},p,s) & \text{if} \quad y = y_{max} \\ U_L(t,z) & \text{if} \quad x < x_{min} + p - \frac{s}{2} \\ U_L(t,z) & \text{if} \quad x = x_{min} + p - \frac{s}{2} < 0 \\ v_i(t, x-p+\frac{s}{2}, y+1, p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b), s-\delta(\varepsilon^a-\varepsilon^b)) & \text{otherwise} \end{cases}.$$

On the open set $\{1,...,m\} \times \mathcal{S}$, we have:

$$-\frac{\partial v_i}{\partial t} - \mathcal{H}_i(t,z,v_i,\frac{\partial v_i}{\partial x}) = 0, \ (t,z) \in \mathcal{S}, \tag{4.23}$$

where $\mathcal{H}_i$ is the Hamiltonian associated with state i:

$$\begin{aligned} \mathcal{H}_i(t,z,v_i,\frac{\partial v_i}{\partial x}) &= rx\frac{\partial v_i}{\partial x} + \sum_{j\neq i} \gamma_{ij} \left(v_j(t,x,y,p,s) - v_i(t,x,y,p,s) \right) - g(y) \\ &+ \sup_{\alpha \in A(t,z)} \Big[\lambda_i^a(t,p,s) \left(\mathcal{A}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s) \right) \\ &+ \lambda_i^b(t,p,s) \left(\mathcal{B}v_i(t,x,y,p,s,\alpha) - v_i(t,x,y,p,s) \right) \Big] = 0. \end{aligned}$$

The boundary and terminal conditions are given by :

$$v_i(t,x_{min},y,p,s) = U_L(t,x_{min},y,p,s) \tag{4.24}$$
$$v_i(T,x,y,p,s) = U_L(T,x,y,p,s). \tag{4.25}$$

We now provide a rigorous characterization for the value function by means of viscosity solutions to the HJB equation (4.23) together with the appropriate boundary terminal conditions. The uniqueness property is particularly crucial to numerically solve the associated HJB. Since the value functions v_i is continuous, we shall work with the notion of continuous viscosity solutions.

Definition 4.1 Viscosity properties.

i) Let $(\phi_i)_{1\leq i\leq m}$ a family of functions defined on $\mathcal{S}$. A function ϕ is a viscosity supersolution of the system of variational inequalities (4.23) on $\{1,...,m\} \times \mathcal{S}$ if,

$$-\frac{\partial \psi_{i_0}}{\partial t}(t_0,z_0) - \mathcal{H}_{i_0}\left(t_0, z_0, \psi, \frac{\partial \psi}{\partial x}\right) \geq 0, \tag{4.26}$$

whenever, for all $(j, y, p, s) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(t, x) \rightarrow \psi_j(t, x, y, p, s)$ *is a* $\mathcal{C}^1$ *function on* $\{(t, x) \in [0, T) \times [x_{min}, +\infty) : (t, x, y, p, s) \in \mathcal{S}\}$ *and* $\phi - \psi$ *has a global minimum at* $(i_0, t_0, z_0) \in \{1, ..., m\} \times \mathcal{S}$.

ii) A function ϕ *is a viscosity subsolution of the system of variational inequalities (4.23) on* $\{1, ..., m\} \times \mathcal{S}$ *if,*

$$-\frac{\partial \psi_{i_0}}{\partial t}(t_0, z_0) - \mathcal{H}_{i_0}\left(t_0, z_0, \psi, \frac{\partial \psi}{\partial x}\right) \leq 0, \tag{4.27}$$

whenever, for all $(j, y, p, s) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(t, x) \rightarrow \psi_j(t, x, y, p, s)$ *is a* $\mathcal{C}^1$ *function on* $\{(t, x) \in [0, T) \times [x_{min}, +\infty) : (t, x, y, p, s) \in \mathcal{S}\}$ *and* $\phi - \psi$ *has a global maximum at* $(i_0, t_0, z_0) \in \{1, ..., m\} \times \mathcal{S}$.

iii) A family of functions $(\phi_i)_{1 \leq i \leq m}$ *is a viscosity solution of the system of variational inequalities (4.23) on* $\{1, ..., m\} \times \mathcal{S}$ *if it is both supersolution and subsolution in* $\{1, .., m\} \times \mathcal{S}$.

The following theorem relates the value function v_i to the HJB (4.23) for all $1 \leq i \leq m$.

Theorem 4.2 *The family of value functions* $(v_i)_{1 \leq i \leq m}$ *is the unique family of functions such that*

i) Continuity condition: For all $(i, y, p, s) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(t, x) \rightarrow v_i(t, x, y, p, s)$ *is continuous on* $\{(t, x) \in [0, T) \times [x_{min}, +\infty) : (t, x, y, p, s) \in \mathcal{S}\}$.

ii) Growth condition: There exist C_1, C_2 *and* C_3 *positive constants such that*

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t, x, y, p, s) \leq 1, \quad \text{on } \{1, .., m\} \times \mathcal{S}. \tag{4.28}$$

iii) Boundary and terminal conditions:

$$v_i(t, x_{min}, y, p, s) = U_L(t, x_{min}, y, p, s) \text{ and } v_i(T, x, y, p, s) = U_L(T, x, y, p, s). \tag{4.29}$$

iv) Viscosity solution: $(v_i)_{1 \leq i \leq m}$ *is a viscosity solution of the system of variational inequalities (4.23) on* $\{1, ..., m\} \times \mathcal{S}$.

Assertions *i)*, *ii)* and *iii)* respectively follow from Proposition 3.3, Proposition 3.1 and the value function definition. Therefore, it just remains to establish assertion *iv)* and the uniqueness result. We shall divide our proof in three lemmas: first we prove that $(v_i)_{1 \leq i \leq m}$ is a viscosity subsolution (see Lemma 4.3) then a supersolution (see Lemma 4.4) and finally we prove a comparison theorem (see Lemma 4.6) which will lead to the uniqueness result.

Lemma 4.3 *The family of value functions* $(v_i)_{1 \leq i \leq m}$ *is a subsolution of the system of variational inequalities (4.23) on* $\{1, ..., m\} \times \mathcal{S}$

Proof. See Appendix.

Lemma 4.4 *The family of value functions $(v_i)_{1\leq i\leq m}$ is a supersolution of the system of variational inequalities (4.23) on $\{1,...,m\}\times\mathcal{S}$*

Proof. See Appendix.
We now turn to the uniqueness result. First, we give an equivalent formulation of the viscosity solutions which is useful to prove the comparison result, see [16].

Lemma 4.5
Let $(\phi_i)_{1\leq i\leq m}$ a family of functions defined on $\mathcal{S}$. A function ϕ is a viscosity supersolution (resp. subsolution) of the system of variational inequalities (4.23) on $\{1,...,m\}\times\mathcal{S}$ if,

$$-\frac{\partial\psi_{i_0}}{\partial t}(t_0,z_0)-\mathcal{H}_{i_0}\left(t_0,z_0,\phi,\frac{\partial\psi}{\partial x}\right)\geq 0,\quad (\textit{resp.}\ \leq 0) \tag{4.30}$$

whenever, for all $(j,y,p,s)\in\{1,..,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}\times\delta\{1,..,K\}$, $(t,x)\to\psi_j(t,x,y,p,s)$ is a $\mathcal{C}^1$ function on $\{(t,x)\in[0,T)\times[x_{min},+\infty):(t,x,y,p,s)\in\mathcal{S}\}$ and $\phi-\psi$ has a global minimum (resp. maximum) at $(i_0,t_0,z_0)\in\{1,...,m\}\times\mathcal{S}$.

With this equivalent definition of viscosity solutions, we are now able to establish the comparison result. In the proof of the comparison result, a major problem is to circumvent the difficulty arising from the discontinuity of our HJB operator on some parts of the solvency region boundary. One way to tackle this difficulty is to build specific test functions allowing us to prove the uniqueness by contradiction.

Lemma 4.6 *Let $(u_i)_{1\leq i\leq m}$ (resp. $(w_i))_{1\leq i\leq m}$) be a viscosity subsolution (resp. supersolution) of* (4.23) *on $[0,T]\times\mathcal{S}$ satisfying the growth condition* (4.28) *and such that for all $(j,y,p,s)\in\{1,..,m\}\times\{y_{min},..,y_{max}\}\times\frac{\delta}{2}\mathbb{N}\times\delta\{1,..,K\}$, $(t,x)\to u_j(t,x,y,p,s)$ (resp. $w_j(t,x,y,p,s)$) is a continuous function on $\{(t,x)\in[0,T)\times[x_{min},+\infty):(t,x,y,p,s)\in\mathcal{S}\}$. Assume that for all $(i,t,z)\in\{1,..,m\}\times\mathcal{S}$ we have*

$$\begin{aligned} u_i(T,z) &\leq w_i(T,z), & (4.31)\\ u_i(t,x_{min},y,p,s) &\leq w_i(t,x_{min},y,p,s) & (4.32)\end{aligned}$$

then we have $u_i(t,z)\leq w_i(t,z)$, for all $(i,t,z)\in\{1,..,m\}\times\mathcal{S}$.

Proof: Let β_1, β_2 and β_3 be positive constants such that $\beta_3>C_3$. We set $h(t,z)=e^{\beta_2(T-t)}\left(\beta_1+x^2+e^{\beta_3 p}\right)$, for $(t,z)\in\mathcal{S}$ and $w_i^\gamma=(1-\gamma)w_i+\gamma h$ for $i\in\{1,..,m\}$ and $\gamma\in(0,1)$.
First we show that h is a supersolution of (4.23) on $\mathcal{S}$. For $(i,t,z)\in\{1,..,m\}\times\mathcal{S}$, we have

$$\begin{aligned}-\frac{\partial h}{\partial t}(t,z)-\mathcal{H}_i\left(t,z,h,\frac{\partial h}{\partial x}\right) &= \beta_2 e^{\beta_2(T-t)}\left(\beta_1+x^2+e^{\beta_3 p}\right)-e^{\beta_2(T-t)}2rx^2+g(y)\\ &\quad-\sup_{\alpha\in A(t,z)}\Big[\lambda_i^a(t,p,s)\left(\mathcal{A}h(t,z,\alpha)-h(t,z)\right)\\ &\quad+\lambda_i^b(t,p,s)\left(\mathcal{B}h(t,z,\alpha)-h(t,z)\right)\Big]. \end{aligned}\tag{4.33}$$

For $\alpha \in A(t,z)$ and $\beta_1 > 1$, we have

$$\begin{aligned}
\mathcal{A}h(t,z,\alpha) - h(t,z) &= \Big[U_L(t,z) - h(t,z)\Big]\mathbb{1}_{y=y_{min}} \\
& \quad e^{\beta_2(T-t)}\Big[(x+p+\frac{s}{2})^2 + e^{\beta_3\left(p+\frac{\delta}{2}(\eta^a+\eta^b)\right)} - x^2 - e^{\beta_3 p}\Big]\mathbb{1}_{y\neq y_{min}} \\
&\leq e^{\beta_2(T-t)}\Big[(p+\frac{s}{2})(2x+p+\frac{s}{2}) + e^{\beta_3 p}\left(e^{\beta_3\delta}-1\right)\Big]\mathbb{1}_{y\neq y_{min}} \quad (4.34)
\end{aligned}$$

In the same way, if we set $\mathcal{C} = \{z \in \mathbb{S} :\ y \neq y_{min}$ or $x > x_{min}+p-\frac{s}{2}$ or $x = x_{min}+p-\frac{s}{2} \geq 0\}$ then we have

$$\begin{aligned}
\mathcal{B}h(t,z,\alpha) - h(t,z) &\leq e^{\beta_2(T-t)}\Big[(x-p+\frac{s}{2})^2 + e^{\beta_3\left(p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b)\right)} - x^2 - e^{\beta_3 p}\Big]\mathbb{1}_{z\in\mathcal{C}} \\
&\leq e^{\beta_2(T-t)}\Big[(-p+\frac{s}{2})(2x-p+\frac{s}{2})\Big]\mathbb{1}_{z\in\mathcal{C}} \quad (4.35)
\end{aligned}$$

Now we plug inequalities (4.34) and (4.35) in inequality (4.33), and obtain that for β_1 and β_2 great enough, there exists $\eta > 0$ such that

$$-\frac{\partial h}{\partial t}(t,z) - \mathcal{H}_i\left(t,z,h,\frac{\partial h}{\partial x}\right) > \eta.$$

To prove Lemma 4.6, it suffices to show that for all $\gamma \in (0,1)$, we have

$$\max_{j\in\{1,..,m\}}\sup_{(t,z)\in\mathcal{S}}\left(u_j - w_j^\gamma\right) \leq 0,$$

since the required result is obtained by letting γ going to 0.
We shall argue by contradiction and assume that

$$\zeta := \max_{j\in\{1,..,m\}}\sup_{(t,z)\in\mathcal{S}}\left(u_j - w_j^\gamma\right) > 0.$$

From the growth condition satisfied by u and w, we deduce that for all $(j,t,z) \in \{1,..,m\} \times \mathcal{S}$,

$$\left(u_j - w_j^\gamma\right)(t,z) \leq C_1 + C_2 e^{C_3 p} - \gamma e^{\beta_2(T-t)}\left(\beta_1 + x^2 + e^{\beta_3 p}\right).$$

Hence, we have $\lim_{p+x\to+\infty}\left(u_j - w_j^\gamma\right)(t,z) = -\infty$ and this implies that there exists $(i^*,t^*,z^*) \in \{1,..,m\} \times \mathcal{S}$ such that

$$(u_{i^*} - w_{i^*}^\gamma)(t^*,z^*) = \zeta > 0.$$

If $t^* = T$ or $x^* = x_{min}$, we deduce from the boundary conditions satisfied by u and w that

$$0 < \theta \leq \gamma\Big[u_{i^*}(t^*,z^*) - e^{\beta_2(T-t^*)}\beta_1\Big] \leq 0,$$

since $u_i \leq 1 \leq \beta_1$. Therefore, we have $t^* < T$ and $x^* > x_{min}$.
We now distinguish the following two cases.

First case: Assume that $x^* = x_{min} + p^* - \frac{s^*}{2}$.
We introduce a sequence $(x_n)_{n\geq 0}$ taking values in $(x_{min}, +\infty)\backslash\{x^*\}$ and such that $\lim_{n\to+\infty} x_n = x^*$. We set $z_n = (x_n, y^*, p^*, s^*)$ and, for $n \in \mathbb{N}$,

$$\zeta_n = \max_{j\in\{1,..,m\}} \sup_{(t,z)\in\mathcal{S}} \left[u_j(t,z) - w_j^\gamma(t,z) - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} \right].$$

It follows from the growth conditions satisfied by u and w, that there exists a sequence $(i_n^*, t_n^*, z_n^*)_{n\geq 0}$ taking values in $\{1,..,m\} \times \mathcal{S}$ and such that, for all $n \in \mathbb{N}$,

$$\zeta_n = u_{i_n^*}(t_n^*, z_n^*) - w_{i_n^*}^\gamma(t_n^*, z_n^*) - \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2}.$$

We notice that

$$\zeta \geq \zeta_n \geq \left(u_{i^*} - w_{i^*}^\gamma\right)(t^*, z_n).$$

From the continuity properties of u and w, it follows that $\lim_{n\to+\infty} \zeta_n = \zeta > 0$ and

$$\lim_{n\to+\infty} \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2} \leq \lim_{n\to+\infty} \zeta - \zeta_n = 0.$$

Therefore, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $\zeta_n > 0$, $(y_n^*, p_n^*, s_n^*) = (y^*, p^*, s^*)$ and $x_n^* \in (x_{min}, +\infty) \setminus \{x^*\}$. Let $n \geq N$, if $t_n^* = T$, we would have the following contradiction

$$0 < \zeta_n \leq \gamma \Big[u_{i_n^*}(t_n^*, z_n^*) - \beta_1 \Big] \leq 0.$$

therefore $t_n^* < T$.

For $(t, z, z') \in [0,T] \times \mathbb{S}^2$, we define the function:

$$\Phi_n(t,z,z') \quad = \quad u_{i_n^*}(t,z) - w_{i_n^*}^\gamma(t,z') - \frac{\|z - z'\|^2}{\varepsilon} - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} - \frac{\|z_n - z'\|^2}{\|z_n - z^*\|^2},$$

where $\varepsilon > 0$.

By a classical argument in the theory of viscosity solutions, we can show that there exists $(t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon))_{n\geq 0} \in [0,T] \times \mathbb{S}^2$ such that

$$\Phi_n(t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon)) = \sup_{(t,z,z')\in[0,T]\times\mathbb{S}^2} \Phi_n(t,z,z').$$

Moreover we can prove that

$$\lim_{\varepsilon\to 0}(t_n(\varepsilon), z_n^1(\varepsilon), z_n^2(\varepsilon)) = (t_n^*, z_n^*, z_n^*) \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\|z_n^1(\varepsilon) - z_n^2(\varepsilon)\|^2}{\varepsilon} = 0.$$

To simplify notations, we shall omit to precise the dependency on ε.

Notice that y, p and s are discrete variables thus, for ε small enough, we have

$$(y_n^1, p_n^1, s_n^1) = (y^*, p^*, s^*) = (y_n^2, p_n^2, s_n^2).$$

We introduce two functions defined on $\{1, .., m\} \times \mathcal{S}$ by

$$\begin{aligned}
\psi_i(t, z) &= w^{\gamma}_{i^*_n}(t_n, z^2_n) + \frac{\|z - z^2_n\|^2}{\epsilon} + \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2} + \frac{\|z_n - z^2_n\|^2}{\|z_n - z^*\|^2} \\
\phi_i(t, z) &= u_{i^*_n}(t_n, z^1_n) - \frac{\|z^1_n - z\|^2}{\epsilon} - \frac{\|z_n - z^1_n\|^2}{\|z_n - z^*\|^2} - \frac{\|z_n - z\|^2}{\|z_n - z^*\|^2}.
\end{aligned}$$

From the definition of ψ and ϕ, $u - \psi$ has a local maximum at (i^*_n, t_n, z^1_n) and $w^\gamma - \phi$ has a local minimum at (i^*_n, t_n, z^2_n), which implies from the equivalent formulation of the viscosity solutions that

$$- \frac{\partial \psi_{i^*_n}}{\partial t}(t_n, z^1_n) - \mathcal{H}_{i^*_n}\left(t_n, z^1_n, u, \frac{\partial \psi}{\partial x}\right) \leq 0, \tag{4.36}$$

$$- \frac{\partial \phi_{i^*_n}}{\partial t}(t_n, z^2_n) - \mathcal{H}_{i^*_n}\left(t_n, z^2_n, w^\gamma, \frac{\partial \phi}{\partial x}\right) \geq \gamma\eta > 0. \tag{4.37}$$

From inequalities (4.36) and (4.37), we have

$$\begin{aligned}
-\gamma\eta &\geq -\frac{\partial \psi_{i^*_n}}{\partial t}(t_n, z^1_n) + \frac{\partial \phi_{i^*_n}}{\partial t}(t^2_n, z^2_n) - \mathcal{H}_{i^*_n}\left(t_n, z^1_n, u, \frac{\partial \psi}{\partial x}\right) + \mathcal{H}_{i^*_n}\left(t_n, z^2_n, w^\gamma, \frac{\partial \phi}{\partial x}\right) \\
&\geq \Delta_1 + \Delta_2 + \Delta_3,
\end{aligned} \tag{4.38}$$

where we have set

$$\begin{aligned}
\Delta_1 &= -\frac{\partial \psi_{i^*_n}}{\partial t}(t_n, z^1_n) + \frac{\partial \phi_{i^*_n}}{\partial t}(t_n, z^2_n) - r x^1_n \frac{\partial \psi_{i^*_n}}{\partial x}(t_n, z^1_n) + r x^2_n \frac{\partial \phi_{i^*_n}}{\partial x}(t_n, z^2_n) \\
&= -\frac{2r}{\varepsilon}(x^1_n - x^2_n)^2 + \frac{2r}{(x_n - x^*)^2}\left(x^1_n(x_n - x^1_n) - x^2_n(x_n - x^2_n)\right), \\
\Delta_2 &= \sum_{j \neq i^*_n} \gamma_{i^*_n, j}\left(\left[u_{i^*_n}(t_n, z^1_n) - w^\gamma_{i^*_n}(t_n, z^2_n)\right] - \left[u_j(t_n, z^1_n) - w^\gamma_j(t_n, z^2_n)\right]\right) \geq 0 \\
\Delta_3 &= \sup_{\alpha \in A(t_n, p^*, s^*)}\left[\lambda^{a,*}_n\left(\mathcal{A}w^\gamma_{i^*_n}(t_n, z^2_n, \alpha) - w^\gamma_{i^*_n}(t_n, z^2_n)\right) + \lambda^{b,*}_n\left(\mathcal{B}w^\gamma_{i^*_n}(t_n, z^2_n, \alpha) - w^\gamma_{i^*_n}(t_n, z^2_n)\right)\right] \\
&\quad - \sup_{\alpha \in A(t_n, p^*, s^*)}\left[\lambda^{a,*}_n\left(\mathcal{A}u_{i^*_n}(t_n, z^1_n, \alpha) - u_{i^*_n}(t_n, z^1_n)\right) + \lambda^{b,*}_n\left(\mathcal{B}u_{i^*_n}(t_n, z^1_n, \alpha) - u_{i^*_n}(t_n, z^1_n)\right)\right],
\end{aligned}$$

with $\lambda^{a,*}_n = \lambda^a_{i^*_n}(t_n, p^*, s^*)$ and $\lambda^{b,*}_n = \lambda^b_{i^*_n}(t_n, p^*, s^*)$. We have $\lim_{\varepsilon \to 0} \Delta_1 = 0$ and $\lim_{\varepsilon \to 0} \Delta_2 \geq 0$. Indeed, we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \Delta_2 &= \sum_{j \neq i^*_n} \gamma_{i^*_n, j}\left(\left[u_{i^*_n}(t^*_n, z^*_n) - w^\gamma_{i^*_n}(t^*_n, z^2_n)\right] - \left[u_j(t^*_n, z^*_n) - w^\gamma_j(t^*_n, z^*_n)\right]\right) \\
&= \sum_{j \neq i^*_n} \gamma_{i^*_n, j}\left(\zeta_n - \left[u_j(t^*_n, z^*_n) - w^\gamma_j(t^*_n, z^*_n) - \frac{\|z_n - z^*_n\|^2}{\|z_n - z^*\|^2}\right]\right) \\
&\geq 0.
\end{aligned}$$

We may conclude the proof by proving that $\liminf_{\varepsilon \to 0} \Delta_3 \geq 0$. As $(y^1_n, p^1_n, s^1_n) = (y^*, p^*, s^*) = (y^2_n, p^2_n, s^2_n)$, there exists $\alpha^* \in A(t_n, z^1_n) = A(t_n, z^2_n)$ such that $\Delta_3 \geq \lambda^{a,*}_n \delta_1 + \lambda^{b,*}_n \delta_2$, where

$$\begin{aligned}
\delta_1 &= \left[\mathcal{A}w^\gamma_{i^*_n}(t_n, z^2_n, \alpha^*) - w^\gamma_{i^*_n}(t_n, z^2_n)\right] - \left[\mathcal{A}u_{i^*_n}(t_n, z^1_n, \alpha^*) - u_{i^*_n}(t_n, z^1_n)\right] \\
\delta_2 &= \left[\mathcal{B}w^\gamma_{i^*_n}(t_n, z^2_n, \alpha^*) - w^\gamma_{i^*_n}(t_n, z^2_n)\right] - \left[\mathcal{B}u_{i^*_n}(t_n, z^1_n, \alpha^*) - u_{i^*_n}(t_n, z^1_n)\right].
\end{aligned}$$

We set $\alpha^* = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b)$ and for $z = (x, y, p, s) \in \mathbb{S}$, we shall use the following notations:

$$\begin{aligned}
\bar{z} &= \left(x + p + \frac{s}{2}, y - 1, p + \frac{\delta}{2}(\eta^a + \eta^b), s + \delta(\eta^a - \eta^b)\right) \\
\hat{z} &= \left(x - p + \frac{s}{2}, y + 1, p - \frac{\delta}{2}(\varepsilon^a + \varepsilon^b), s + \delta(\varepsilon^a - \varepsilon^b)\right).
\end{aligned}$$

If $y^* = y_{min}$, we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \delta_1 &= \lim_{\varepsilon \to 0} \Big[U_L(t_n, z_n^2) - w^\gamma_{i_n^*}(t_n, z_n^2)\Big] - \Big[U_L(t_n, z_n^1) - u_{i_n^*}(t_n, z_n^1)\Big] \\
&= \lim_{\varepsilon \to 0} \Big[u_{i_n^*}(t_n, z_n^1) - w^\gamma_{i_n^*}(t_n, z_n^2)\Big] \\
&= u_{i_n^*}(t_n^*, z_n^*) - w^\gamma_{i_n^*}(t_n^*, z_n^*) \\
&= \zeta_n + \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2} > 0.
\end{aligned}$$

If $y^* > y_{min}$, we then have

$$\begin{aligned}
\lim_{\varepsilon \to 0} \delta_1 &= \lim_{\varepsilon \to 0} \Big[w^\gamma_{i_n^*}(t_n, \bar{z}_n^2) - w^\gamma_{i_n^*}(t_n, z_n^2)\Big] - \Big[u_{i_n^*}(t_n, \bar{z}_n^1) - u_{i_n^*}(t_n, z_n^1)\Big] \\
&= \zeta_n - \Big[u_{i_n^*}(t^*, \bar{z}_n^*) - w^\gamma_{i_n^*}(t_n^*, \bar{z}_n^*)\Big] + \frac{\|z_n - z_n^*\|^2}{\|z_n - z^*\|^2} \\
&\geq \zeta_n - \zeta.
\end{aligned}$$

If $y^* = y_{max}$ or $x_n^* < x_{min} + p^* - \frac{s^*}{2}$, then, for n large enough, we get

$$\lim_{\varepsilon \to 0} \delta_2 = \lim_{\varepsilon \to 0} \Big[U_L(t_n, z_n^2) - w^\gamma_{i_n^*}(t_n, z_n^2)\Big] - \Big[U_L(t_n, z_n^1) - u_{i_n^*}(t_n, z_n^1)\Big] \geq \zeta_n > 0.$$

Moreover, if $y^* < y_{max}$ and $x_n^* > x_{min} + p^* - \frac{s^*}{2}$, we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} \delta_2 &= \lim_{\varepsilon \to 0} \Big[w^\gamma_{i_n^*}(t_n, \hat{z}_n^2) - w^\gamma_{i_n^*}(t_n, z_n^2)\Big] - \Big[u_{i_n^*}(t_n, \hat{z}_n^1) - u_{i^*}(t_n, z_n^1)\Big] \\
&\geq \zeta_n - \Big[u_{i_n^*}(t^*, \hat{z}_n^*) - w^\gamma_{i_n^*}(t_n^*, \hat{z}_n^*)\Big] \\
&\geq \zeta_n - \zeta.
\end{aligned}$$

Hence, if we let ε going to 0 in inequality (4.38), we get $-\gamma\eta \geq 2(\zeta_n - \zeta)$. We obtain a contradiction by letting n going to $+\infty$.

<u>Second case:</u> We assume that $x^* \neq x_{min} + p^* - \frac{s^*}{2}$. As we shall work far from the set of discontinuity of the operator $\mathcal{B}$, this case is more simple and we just give the sketch of the proof. For $(t, z, z') \in [0, T] \times \mathbb{S}^2$, we define the function:

$$\Phi(t, z, z') \quad = \quad u_{i^*}(t, z) - w^\gamma_{i^*}(t, z') - \frac{\|z - z'\|^2}{\varepsilon},$$

where $\varepsilon > 0$.

By a classical argument in the theory of viscosity solutions, we can show that there exists $(t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) \in [0, T] \times \mathbb{S}^2$ such that

$$\Phi(t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) = \sup_{(t,z,z') \in [0,T] \times \mathbb{S}^2} \Phi(t, z, z').$$

Moreover we can prove that

$$\lim_{\varepsilon\to 0}(t(\varepsilon), z^1(\varepsilon), z^2(\varepsilon)) = (t^*, z^*, z^*) \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{\|z^1(\varepsilon)-z^2(\varepsilon)\|^2}{\epsilon} = 0.$$

The function $\hat{\phi}$ and $\hat{\psi}$ defined by

$$\begin{aligned}
\hat{\psi}_i(t,z) &= w^\gamma_{i^*}(t(\epsilon), z^2(\varepsilon)) + \frac{\|z-z^2(\varepsilon)\|^2}{\epsilon}\\
\hat{\phi}_i(t,z) &= u_{i^*}(t(\epsilon), z^1(\epsilon)) - \frac{\|z^1(\epsilon)-z\|^2}{\epsilon}
\end{aligned}$$

are respectively super and sub solution of equation (4.23) then we get

$$-\gamma\eta \geq -\frac{\partial\hat{\psi}_{i^*}}{\partial t}(t(\varepsilon), z^1(\varepsilon)) + \frac{\partial\hat{\phi}_{i^*}}{\partial t}(t(\varepsilon), z^2(\varepsilon)) - \mathcal{H}_{i^*}\left(t(\varepsilon), z^1(\varepsilon), u, \frac{\partial\hat{\psi}}{\partial x}\right) + \mathcal{H}_{i^*}\left(t(\varepsilon), z^2(\varepsilon), w^\gamma, \frac{\partial\hat{\phi}}{\partial x}\right).$$

We conclude the proof by letting ε going to and get the following contradiction $0 > -\gamma\eta \geq 0$ $\square$

5 Numerical Results

In this paragraph, we present the results of the numerical method we used to approximate the solution of the system of equations (4.23).

5.1 Numerical scheme

To solve the HJB equation (4.23) arising from the stochastic control problem (3.21), one can use either probabilistic or deterministic numerical method. We choose to use a deterministic method based on a finite difference scheme, which is well known to have the monotonicity, consistency and stability properties. These properties ensure the convergence of this scheme, see [3].

To compute numerically the value function, we used the following iterative scheme allowing us to obtain the HJB (4.23) as a limit of HJB equations :

$\forall i \in \{1, ..., m\}$

$$\begin{aligned}
v_i^0(t,z) &= U_L(t,z), \qquad (t,z) \in \bar{\mathcal{S}}\\
-\frac{\partial v_i^n}{\partial t} - \hat{\mathcal{H}}_i(t,z,v_i^n, v_i^{n-1}, \frac{\partial v_i^n}{\partial x}) &= 0, \qquad (t,z) \in \mathcal{S},
\end{aligned}$$

where $\hat{\mathcal{H}}_i$ is the Hamiltonian associated with state i and defined as follows:

$$\begin{aligned}
\hat{\mathcal{H}}_i(t,z,v_i^n, v_i^{n-1}, \frac{\partial v_i^n}{\partial x}) &= rx\frac{\partial v_i^n}{\partial x} + \sum_{j\neq i}\gamma_{ij}\left(v_j^n(t,x,y,p,s) - v_i^n(t,x,y,p,s)\right) - g(y)\\
&+ \sup_{\alpha\in A(t,z)}\Big[\lambda_i^a(p,s)\left(\mathcal{A}v_i^{n-1}(t,x,y,p,s,\alpha) - v_i^n(t,x,y,p,s)\right)\\
&+ \lambda_i^b(p,s)\left(\mathcal{B}v_i^{n-1}(t,x,y,p,s,\alpha) - v_i^n(t,x,y,p,s)\right)\Big].
\end{aligned}$$

The boundary and terminal conditions are given by :

$$\begin{aligned} v_i^n(t, x_{min}, y, p, s) &= U_L(t, x_{min}, y, p, s) \\ v_i^n(T, p, s, x, y) &= U_L(T, x, y, p, s). \end{aligned}$$

After localizing the problem on the discretized grid $[0, T) \times (x_{min}, x_{max}] \times \{y_{min}, ..., y_{max}\} \times [p_{min}, p_{max}] \times \delta\{1, ..., K\}$, where p_{min} and p_{max} are nonnegative constants, and $x_{max} > x_{min}$, each HJB equation is approximated by a finite difference scheme assuming a Dirichlet boundary condition on the localized boundary

$$v_i^n(t, x_{max}, y, p, s) = U_L(t, x_{max}, y, p, s).$$

Let h and d_x be, respectively, the time discretization step and the space disretization step in the direction x. For (t, z) in the time-space grid described above, we consider approximations of the following form :

$$\begin{aligned} \frac{\partial v_i^n}{\partial t}(t, x, y, p, s) &\sim \frac{v_i^n(t+h, x, y, p, s) - v_i^n(t, x, y, p, s)}{h} \\ \frac{\partial v_i^n}{\partial x}(t, x, y, p, s) &\sim \pm \frac{v_i^n(t, x \pm d_x, y, p, s) - v_i^n(t, x, y, p, s)}{d_x}. \end{aligned}$$

Considering a two-regime case, the HJB relative to the regime 1, can be rewritten as follows:

$$\begin{aligned} \frac{v_1^n(t+h, x, y, p, s) - v_1^n(t, x, y, p, s)}{h} &+ rx\left(\frac{v_1^n(t, x, y, p, s) - v_1^n(t, x - d_x, y, p, s)}{d_x}\right)\mathbb{1}_{\{x<0\}} \\ &+ rx\left(\frac{v_1^n(t, x + d_x, y, p, s) - v_1^n(t, x, y, p, s)}{d_x}\right)\mathbb{1}_{\{x\geq 0\}} \\ &+ \gamma_{12}\left(v_2^n(t, x, y, p, s) - v_1^n(t, x, y, p, s)\right) - g(y) \\ &+ \sup_{\alpha \in A(t,z)}\Big[\lambda_1^a(p, s)\left(\mathcal{A}v_1^{n-1}(t, x, y, p, s, \alpha) - v_1^n(t, x, y, p, s)\right) \\ &+ \lambda_1^b(p, s)\left(\mathcal{B}v_1^{n-1}(t, x, y, p, s, \alpha) - v_1^n(t, x, y, p, s)\right)\Big] = 0. \end{aligned}$$

Which leads to

$$\begin{aligned} v_1^n(t, z) &= \frac{C_1^1(t, z)C_2^2(t, z) + C_2^1(t, z)}{1 - C_1^1(t, z)C_1^2(t, z)} \\ v_2^n(t, z) &= C_1^2(t, z)v_1^n(t, z) + C_2^2(t, z) \end{aligned}$$

where:

$$\begin{aligned} C_1^i(t, z) &= C(i, t, z)\gamma_{12}d_x h, \\ C_2^i(t, z) &= C(i, t, z)\Big\{d_x v_i^n(t+h, x, y, p, s) \pm rxhv_i^n(t, x \pm dx, y, p, s) - d_x h g(y) \\ &+ d_x h \sup_{\alpha \in A(t,z)}[\lambda_i^a(p, s)\mathcal{A}v_i^{n-1}(t, x, y, p, s, \alpha) + \lambda_i^b(p, s)\mathcal{B}v_i^{n-1}(t, x, y, p, s, \alpha)]\Big\}, \\ C(i, t, z) &= \left(d_x \pm rxh + d_x h\left(\gamma_{12} + \lambda_i^a(p, s) + \lambda_i^b(p, s)\right)\right)^{-1}, \end{aligned}$$

such that $i \in \{1, 2\}$.

5.2 Numerical computation and results

The approximated value function is now explicit and can be computed numerically. Numerical tests are performed for a two-regime case with the following numerical data:

- Market values:

  $\rightarrow$ Initial conditions: $x = 4, \quad y = 2, \quad p = 1, \quad s = 0.02.$

  $\rightarrow r = 0.05, \quad \delta = 0.018, \quad \lambda = 20.$

  $\rightarrow$ Impact function: $f(t, y) = \exp(-0.09y)$.

  $\rightarrow$ Intensity functions:

$$
\begin{aligned}
\lambda_i^a(p, s) &= \frac{\psi_i^a}{p} \exp\big(- s - 0.01(p - 1)\big) \\
\lambda_i^b(p, s) &= \psi_i^b p \exp\big(- s + 0.01(p - 1)\big),
\end{aligned}
$$

$$
\text{where } \psi_1^a = 120, \quad \psi_2^a = 80, \quad \psi_1^b = 80, \quad \psi_2^b = 120.
$$

- Constraints:

  $\rightarrow x_{min} = -2, \quad y_{min} = -10, \quad y_{max} = 10, \quad K = 5, \quad T = 1.$
  $\rightarrow$ Penalty function: $g(y) = y^2 \times 10^{-3}$.
  $\rightarrow$ Utility function: $U(l) = 1 - e^{-0.01l}$ i.e. $\gamma = 0.01$.

- Numerical values:

  $\rightarrow$ Localisation: $x_{max} = 18, \quad p_{min} = 1 - 20 \times \frac{\delta}{2}, \quad p_{max} = 1 + 20 \times \frac{\delta}{2}$.
  $\rightarrow$ Discretization: $n_x = 100$ and $n_t = 20$.
  $\rightarrow$ Transition probabilities: $p_{12} = p_{21} = 0.5$.

Remark 5.4 1.) The impact function f and the intensity functions λ_i^a and λ_i^b are respectively inspired from the models studied in [11] and [2].
2.) With these choices of λ^a and λ^b, we suggest that the intensity of the buy market orders is non-increasing with respect to the mid price while the intensity of the sell market orders is non-decreasing with respect to the mid price. Both intensities are non-increasing with respect to the spread.
3.) These choices have the following financial assumption: when prices get higher, there are likely to have many more investors willing to sell and fewer willing to buy. In other way around, when the spread gets higher, fewer trading orders are expected.

Shape of the value function

We represent in Figure 2 (resp. Figure 3) the shape of the value function associated to the regime 1 for fixed (t, x, y) such that y is positive (resp. y is negative).

Remark 5.5 *(On the shape of the value function)*
- The value function is non-decreasing (non-increasing) in P, the mid-price of the assets,

when the stockholding, y, is positive (negative).

- The value function is non-decreasing in S, the bid-ask spread, but only up to a threshold, beyond which the value function would start to decrease. This is a clear evidence that the optimal strategy for the market maker is not necessarily to increase the bid-ask spread. Indeed, a high spread will negatively impact on the frequency of the trades, in order words, investors may turn away from illiquid assets. This is precisely the reason why the Stock Exchange sets a maximum level of bid-ask spread beyond which the market maker may not allow to go.

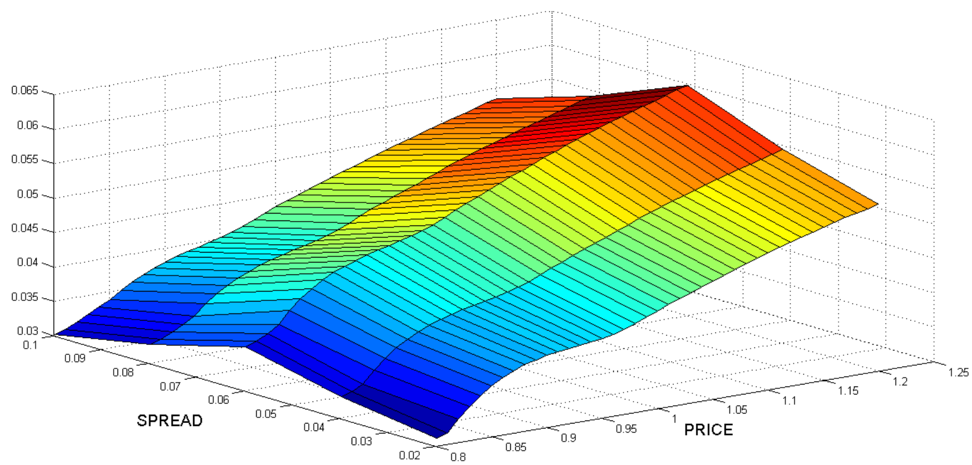

Figure 2: *Value function for $y \geq 0$*

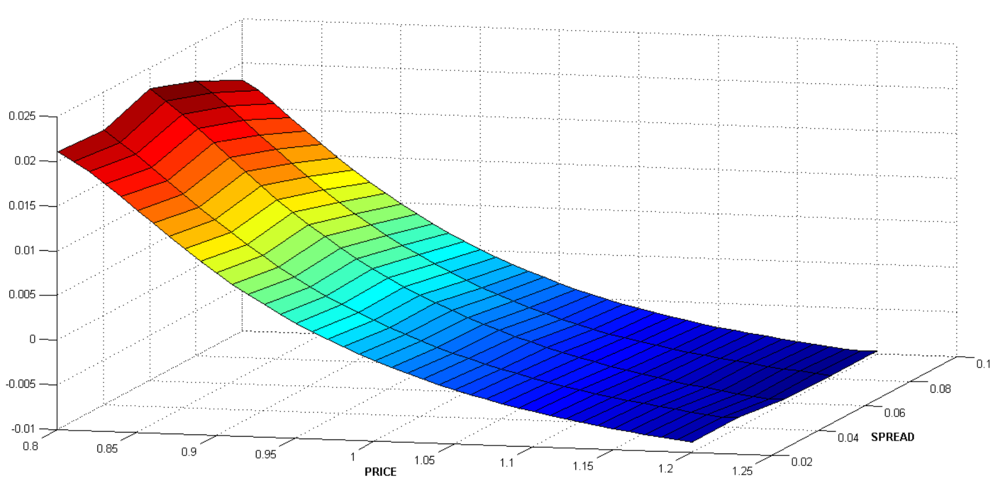

Figure 3: *Value function for $y \leq 0$*

Optimal market making strategies

Figure 4 describes the optimal control strategies for the market marker when a sell market order arrives and when the market maker's inventory is around zero. In our model, the market maker has to adjust optimally the bid and ask prices in order to maximize its objective function. To achieve that aim, the idea for the market maker is to incite trades in both directions. The best scenario is to get alternately buy and sell market orders in order earn the bid-ask spread while maintaining her inventory close to zero.

In the case showed in Figure 4, after the arrival of sell market order, the market maker should act in a way to encourage the arrival of a buy market order, i.e., increasing the buy market order arrival intensity relatively to the intensity of the sell market order. Given the properties of the intensity functions, one needs to decrease the mid-price.

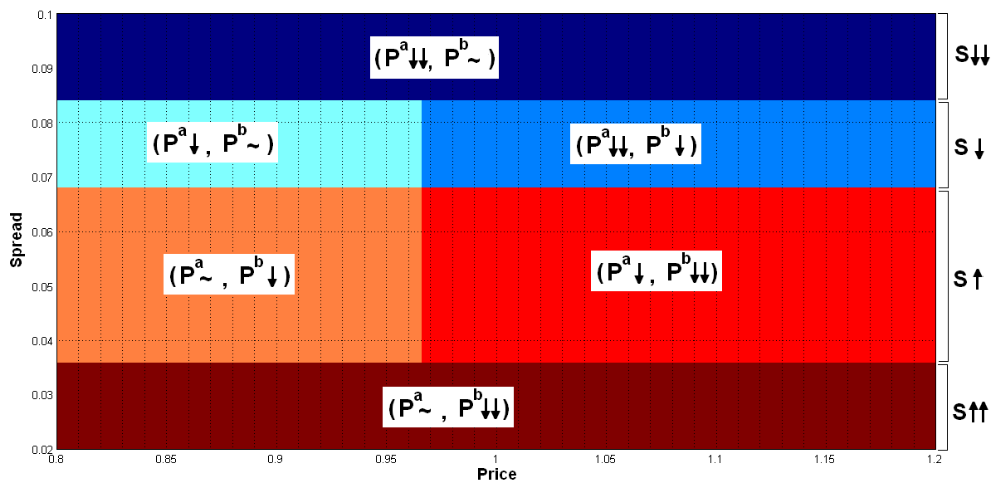

Figure 4: *Optimal strategy when a sell market order arrives*

From Figure 4, we may make the following observations:

- when the spread is very low, the market maker has to decrease the bid price more than the ask price, see region where the spread value is below 0.07.
- when the spread is high and close to the maximum spread allowed, the market maker should decrease the ask price. She should decrease the spread in order to encourage trades.

Notice that the market maker may make a profit of 3 ticks in the favorable case, i.e., the next market order is a buy order.

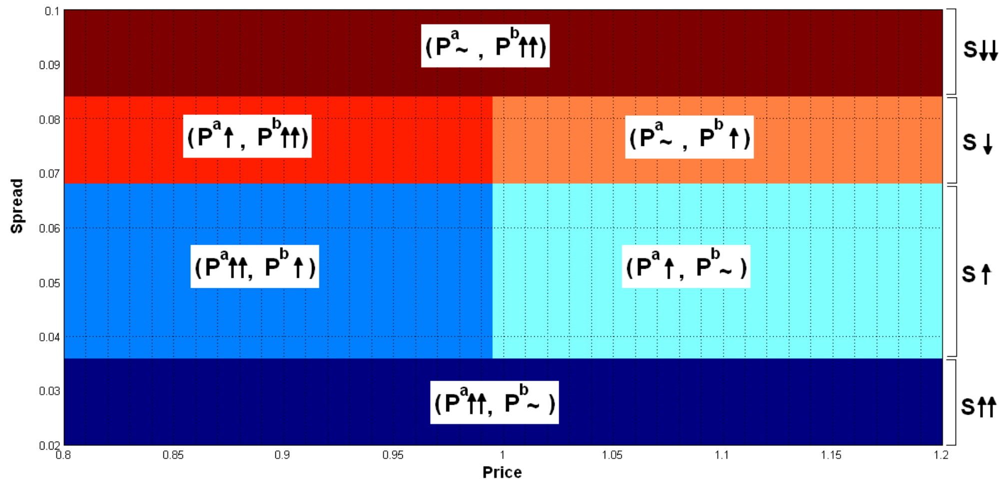

Figure 5: *Optimal strategy when a buy market order arrives*

Figure 5 shows the symmetric case when a buy market order arrives.

Some simulated paths

To obtain these below simulated paths, we compute the optimal strategy of the market maker via the numerical procedure described above. Then, we simulate the regime switching (the Poisson process R) and the order arrivals (the Cox processes N^a and N^b with intensities $\lambda^a(t, I_t, P_t, S_t)$ and $\lambda^b(t, I_t, P_t, S_t)$ respectively). We adjust the mid-price and the spread of the market maker according to her optimal strategy and take into account her new cash and stock inventory positions after each order arrival.
We represent in Figure 6 and 7 simulated trajectories of the bid-ask prices and her stock inventory.

Remark 5.6

- *Figure 7 shows that between $t = 0.20$ and $t = 0.32$, there is clearly an imbalance between buy and sell market orders (with sell market orders largely exceeding buy orders). Therefore, the market maker has to buy the stock in order to satisfy those sell market orders, resulting in a decrease of the bid and ask prices (see Figure 6).*

- *Between $t = 0.32$ and $t = 0.45$, there is a reversal of the situation with an imbalance of orders in favor of the buy market orders. The market maker takes that opportunity to sell back the shares and controls her inventory risk by keeping her stockholder position near to zero.*

- *Between $t = 0.45$ and $t = 1$, there seems to be a fair balance between buy and sell markets.*

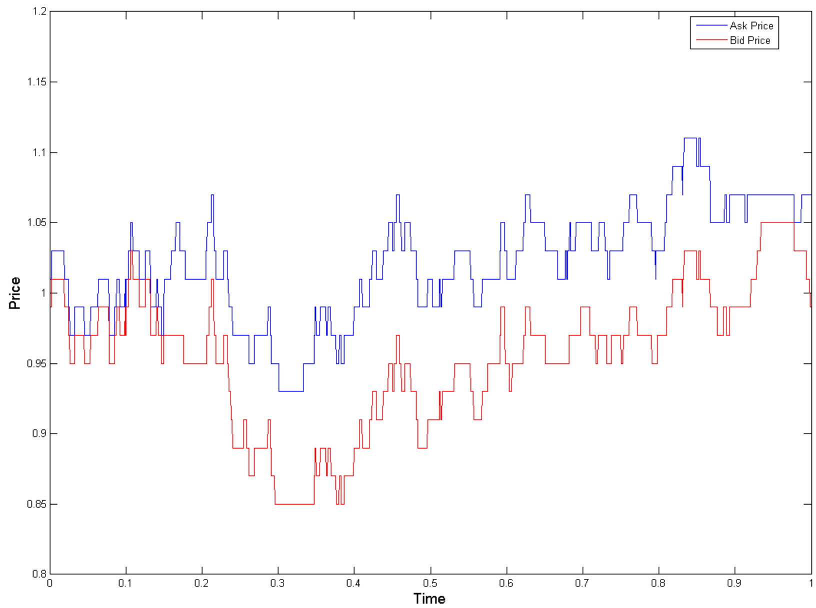

Figure 6: *Bid and Ask Price Paths*

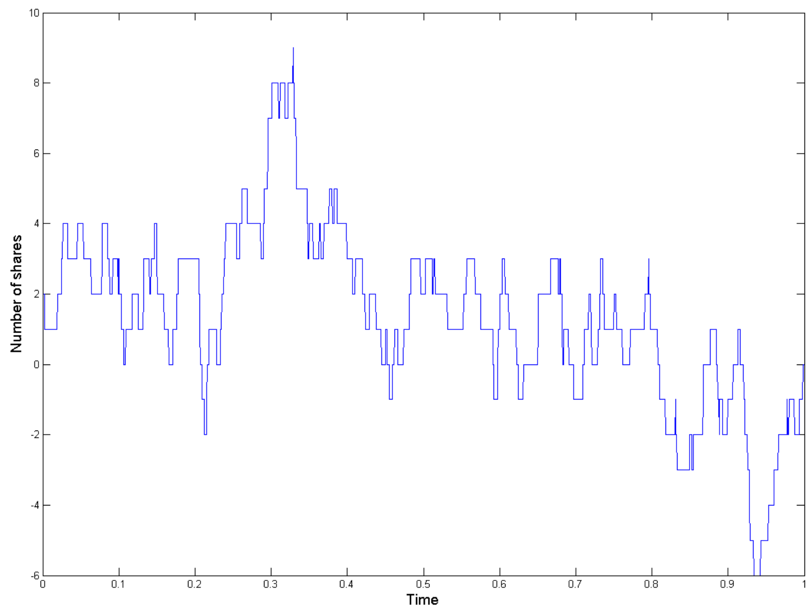

Figure 7: *A simulated trajectory of the market maker's stock inventory position*

Market maker's net wealth and comparison with naive strategy

In Figure 8, we represent the mean, over 1000 trajectories, of the market maker's net wealth obtained via the optimal strategies and the naive strategy.

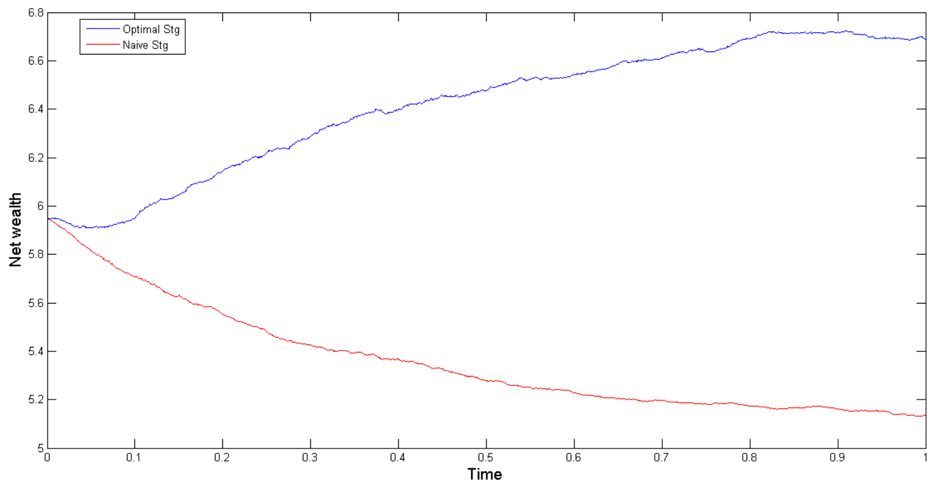

Figure 8: *Optimal strategy v.s. Naive strategy*

In the case of the naive strategy, we assume that the market maker behaves in a trivial way by replacing her optimal strategy $\alpha = (\alpha_t)_{(0\leq t\leq T)} = (\epsilon^a_t, \epsilon^b_t, \eta^a_t, \eta^b_t)_{0\leq t\leq T}$ with the strategy denoted $\hat{\alpha} = (\hat{\alpha}_t)_{(0\leq t\leq T)} = (1,1,1,1)$, as mentioned in Remark 2.2.

Remark 5.7 *As expected, Figure 8 shows that*

- *the trend of the market maker's net wealth using her optimal strategy is positive which means that she is able to increase her initial wealth smoothly throughout maturity while satisfying all the constraints.*
- *the optimal strategy we computed via our numerical procedure is by far a better strategy for the market maker than the naive one.*

6 Conclusion

We have formulated a market making model in a dealer market. We have studied our problem by addressing the following three main aspects. First, the modelling aspect which includes important features and constraints characterizing market making problems. Then, we have rigourously characterized the value function as the unique constrained viscosity solution to a system of variational Hamilton-Jacobi-Bellman inequalities. In this problem, a major challenge to overcome is to take into account the inventory constraints that the market maker is facing. Finally, we completed our studies with numerical results by solving the

HJB equation (4.23) arising from the stochastic control problem (3.21) using a deterministic method based on a finite difference scheme.

This paper is our first contribution in the study of market making problems. We will further investigate several natural theoretical questions in future research. An interesting extension is to relax the assumption regarding full information access on the market and in particular the observability of the Markov intensity process. One way to tackle the problem with an unobservable Markov process I_t is to use filtering theory and draw information for the intensity process given the observation of the market orders. Another natural extension of our study is to consider a competitive market making problem under inventory constraints, with the presence of several market makers in the market.

Appendix

Proof of Lemma 3.1.
Let $i \in \{1, ..., m\}$, $(t,z) := (t,x,y,p,s) \in \mathcal{S}$ and $\alpha \in \mathcal{A}(t,z)$. We have

$$0 \leq S^{t,i,s,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq K\delta, \quad y_{min} \leq Y^{t,i,y,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq y_{max} \quad \text{and} \quad x_{min} \leq X^{t,i,x,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} .$$

Hence, we get

$$\begin{aligned} L\left(T\wedge\tau^{t,i,z,\alpha}, Z^{t,i,x,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}\right) &\geq x_{min} + y_{min} f(0,y_{min})(P^{t,i,p,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} + \frac{K\delta}{2}) \\ &\geq a - bP^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} , \end{aligned}$$

where $a = x_{min} + y_{min} f(0,y_{min})\frac{K\delta}{2}$ and $b = -y_{min} f(0,y_{min}) > 0$. Moreover, it follows from the definition of P that $P^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-} \leq p + \chi_{max} N^{a,t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-}$. Therefore, we obtain

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\beta L(T\wedge\tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})^-})\right)\right] &\leq e^{-\beta(a-bp)}\mathbb{E}\left[\exp\left(\beta b\chi_{max} N^{a,t,i,z,\alpha}_{(T\wedge\tau^{t,z,\alpha})^-})\right)\right] \\ &\leq e^{-\beta(a-bp)}\mathbb{E}\left[\exp\left(\beta b\chi_{max}\bar{N}_T)\right)\right] , \end{aligned}$$

where $\bar{N}$ is a Poisson process with intensity $\bar{\lambda}$. we conclude the proof by observing that $\mathbb{E}\left[\exp\left(\beta b\chi_{max}\bar{N}_T)\right)\right] = \exp\left((\bar{\lambda}T(e^{\beta b\chi_{max}} - 1)\right)$. □

Proof of Lemma 3.2.
Let $\alpha \in \mathcal{A}(t,z) = \mathcal{A}(t,z')$. To simplify notations, we set $\tau = \tau^{t,i,z,\alpha}$ and $\tau' = \tau^{t,i,z',\alpha}$. We first notice that $\tau \leq \tau'$, then $(Y^{t,i,z,\alpha}_\tau, P^{t,i,z,\alpha}_\tau, S^{t,i,z,\alpha}_\tau) = (Y^{t,i,z',\alpha}_\tau, P^{t,i,z',\alpha}_\tau, S^{t,i,z',\alpha}_\tau)$. Therefore we get:

$$\begin{aligned} \Delta J_i^\alpha &:= J_i^\alpha(t,z') - J_i^\alpha(t,z) \\ &= \mathbb{E}\left[U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right) - \int_{T\wedge\tau}^{T\wedge\tau'} g(Y^{t,i,z,\alpha}_u)\,du\right] \\ &\geq \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right) - \int_{\tau}^{T\wedge\tau'} g(Y^{t,i,z,\alpha}_u)\,du\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\right] . \end{aligned}$$

Indeed, on $\{\tau \wedge T = \tau' \wedge T\}$, we have

$$U(L(T\wedge\tau', Z^{t,i,z',\alpha}_{(T\wedge\tau')^-})) = U((x'-x)e^{r(T\wedge\tau-t)} + L(T\wedge\tau, Z^{t,i,z,\alpha}_{(T\wedge\tau)^-})) \geq U(L(T\wedge\tau, Z^{t,i,z,\alpha}_{(T\wedge\tau)^-})).$$

Notice that on $\{\tau < \tau'\}$, we have $y_{min} \leq Y^{t,i,z,\alpha}_{\tau} \leq y_{max}$ and therefore $X^{t,i,z,\alpha}_{\tau} \leq x_{min}$.
We introduce θ the first order arrival time after τ:

$$\theta := \inf\{u > \tau :\ N^a_u > N^a_{u^-} \text{ or } N^b_u > N^b_{u^-}\}.$$

As $x_{min}e^{-r\tau\wedge T} + (x'-x)e^{-rt} \leq e^{-rt}(x_{min}e^{-r(T-t)} + x' - x) < 0$, we can also define the following stopping time, greater than τ:

$$\nu := \tau - \frac{1}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{\mid x_{min}\mid}\right).$$

As g is bounded by $G > 0$ and $U \leq 1$, we get $\Delta J^{\alpha}_i \geq \delta_1 - \delta_2 + \delta_3 - \delta_4$ where

$$\begin{aligned}
\delta_1 &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right)\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}\right]\\
\delta_2 &:= \mathbb{E}\left[\int_{\tau}^{T\wedge\tau'} g(Y^{t,i,z,\alpha}_u)\,du\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}\right]\\
\delta_3 &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(\tau,\ Z^{t,i,z,\alpha}_{\tau^-}\right)\right)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right]\\
\delta_4 &:= \mathbb{E}\left[\int_{\tau}^{T\wedge\tau'} g(Y^{t,i,z,\alpha}_u)\,du\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right].
\end{aligned}$$

We first find a lower bound for δ_1 and an upper bound for δ_2. On $\{\nu < \theta\} \cap \{\tau < \tau' \wedge T\}$, we have

$$\begin{aligned}
X^{t,i,z',\alpha}_{\nu} &= (x'-x)e^{r(\nu-t)} + X^{t,i,z,\alpha}_{\tau}e^{r(\nu-\tau)}\\
&\leq e^{r\nu}\left((x'-x)e^{-rt} + x_{min}e^{-r\tau}\right)\\
&\leq x_{min}\frac{(x'-x)e^{r(\tau-t)} + x_{min}}{(x'-x)e^{r(T-t)} + x_{min}}\\
&\leq x_{min},
\end{aligned}$$

where the second inequality is deduced from the definition of ν.

Hence, on $\{\nu < \theta\}\cap\{\tau < \tau'\wedge T\}$, we have $\tau' \leq \nu < \theta$ and it follows from the monotonicity of the function: $t \to yf(t,y)$, see **Assumption (H1)**.

$$\begin{aligned}
L(T\wedge\tau', Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}) &\geq x_{min} + y(p - \text{sign}(y)\frac{s}{2})f(T\wedge\tau', y)\\
&\geq x_{min} + y(p - \text{sign}(y)\frac{s}{2})f(\tau, y)\\
&\geq L(\tau, Z^{t,i,z,\alpha}_{\tau}).
\end{aligned}$$

Since U is non-decreasing, we have $\delta_1 \geq 0$. Moreover g is non-negative, as such, we get

$$\delta_2 \leq G\mathbb{E}[(\nu-\tau)\mathbb{1}_{\{\tau<T\wedge\tau'\}}\mathbb{1}_{\{\nu<\theta\}}] \leq -\frac{G}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{\mid x_{min}\mid}\right).$$

Now we find a lower bound for $\delta_3 - \delta_4$. As g is bounded by $G > 0$ and $U \leq 1$, we get

$$\delta_3 - \delta_4 \geq \mathbb{E}\left[U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\mathbb{1}_{\{\tau < T\wedge\tau'\}}\mathbb{1}_{\{\theta\leq\nu\}}\right] - (1+TG)\mathbb{P}(\theta \leq \nu).$$

Therefore, we deduce from Cauchy-Schwarz inequality that

$$\delta_3 - \delta_4 \geq -\mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)^2\right]^{\frac{1}{2}}(\mathbb{P}(\theta\leq\nu))^{\frac{1}{2}} - (1+TG)\mathbb{P}(\theta \leq \nu).$$

Applying Lemma 3.1 with $\beta = 2\gamma$, we show that there exists $C(p) > 0$ such that.

$$\begin{aligned}
\mathbb{E}\left[\left(U_L\left(T \wedge \tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)^2\right] &\leq 1 + \mathbb{E}[\exp\left(-2\gamma L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right)\right)] \\
&\leq 1 + C(p).
\end{aligned}$$

Hence, we get $\delta_3 - \delta_4 \geq -(1+C(p))^{\frac{1}{2}}(\mathbb{P}(\theta\leq\nu))^{\frac{1}{2}} - (1+TG)\mathbb{P}(\theta\leq\nu)$. Moreover, we have

$$\begin{aligned}
\mathbb{P}(\theta\leq\nu) &\leq \mathbb{P}(N^a_\nu - N^a_\tau > 0) + \mathbb{P}(N^b_\nu - N^b_\tau > 0) \\
&\leq 2\mathbb{P}(\bar{N}_\nu - \bar{N}_\tau > 0) \\
&\leq -\frac{2\bar{\lambda}}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{\mid x_{min}\mid}\right),
\end{aligned}$$

where $\bar{N}$ is a Poisson process with intensity $\bar{\lambda}$. To conclude, as $\phi(x'-x) = -\frac{1}{r}\ln\left(1 - \frac{(x'-x)e^{r(T-t)}}{|x_{min}|}\right)$, we have shown that

$$\Delta J^\alpha_i \geq -G\phi(x'-x) - (2\bar{\lambda}(1+C(p)))^{\frac{1}{2}}\left(\phi(x'-x)\right)^{\frac{1}{2}} - 2\bar{\lambda}(1+TG)\phi(x'-x).$$

It remains to find an upper bound for ΔJ^α_i. As g is positive, we have

$$\begin{aligned}
\Delta J^\alpha_i &\leq \mathbb{E}\left[U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right] \\
&\leq \hat{\delta}_1(\alpha) + \hat{\delta}_2(\alpha),
\end{aligned}$$

where we have set

$$\begin{aligned}
\hat{\delta}_1(\alpha) &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right)\mathbb{1}_{\{\nu<\theta\}}\right] \\
\hat{\delta}_2(\alpha) &:= \mathbb{E}\left[\left(U_L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) - U_L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right)\right)\mathbb{1}_{\{\theta\leq\nu\}}\right]
\end{aligned}$$

On $\{\nu < \theta\}$, we have seen that

$$L\left(T\wedge\tau',\ Z^{t,i,z',\alpha}_{(T\wedge\tau')^-}\right) = (x'-x)e^{r\tau'} + L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right).$$

Hence, it follows from the concavity of U and its monotony that

$$\begin{aligned}
\hat{\delta}_1(\alpha) &\leq (x'-x)e^{rT}\mathbb{E}\left[U'\left(L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\mathbb{1}_{\{\nu<\theta\}}\right] \\
&\leq (x'-x)e^{rT}\mathbb{E}\left[U'\left(L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\right] \\
&= \gamma(x'-x)e^{rT}\mathbb{E}\left[\exp\left(-\gamma L\left(T\wedge\tau,\ Z^{t,i,z,\alpha}_{T\wedge\tau}\right)\right)\right].
\end{aligned}$$

From Lemma 3.1, it follows that there exists $C(p) > 0$ such that $\hat{\delta}_1(\alpha) \leq C(p)(x' - x)$. Finally, we deduce from Cauchy-Schwarz inequality and Lemma 3.1 that there exists $C(p) > 0$ such that

$$\begin{aligned}\hat{\delta}_2(\alpha) &\leq \mathbb{P}(\theta \leq \nu) + \left(\mathbb{E}\left[(U_L\left(T \wedge \tau,\ Z^{t,i,z,\alpha}_{(T\wedge\tau)^-}\right))^2\right]\right)^{\frac{1}{2}} (\mathbb{P}(\theta \leq \nu))^{\frac{1}{2}} \\ &\leq 2\bar{\lambda}\phi(x' - x) + C(p)(2\bar{\lambda}\phi(x' - x))^{\frac{1}{2}}.\end{aligned}$$

□

Proof of Proposition 3.2.

Let $\alpha \in \mathcal{A}(t \wedge t', z)$ s. t. $\alpha_{|[t\wedge t', t\vee t']} = 0$. As y, p and s are fixed, we will write $J_i^\alpha(t, \zeta)$ instead of $J_i^\alpha(t, \zeta, y, p, s)$ for $\zeta \in [x_{min}, +\infty)$.

We set $\hat{x}' := x' e^{r(t-t')}$. We have $\mid x' - \hat{x}' \mid \leq \mid x' \mid re^{rT} \mid t - t' \mid$ since $\mid e^\zeta - 1 \mid \leq \mid \zeta \mid e^{|\zeta|}$. As such, from the condition on $|t - t'|$ in (3.20), we may apply Lemma 3.2 and obtain

$$\begin{aligned}\mid J_i^\alpha(t', x') - J_i^\alpha(t, x) \mid &\leq \mid J_i^\alpha(t', x') - J_i^\alpha(t, \hat{x}') \mid \\ &\quad + \mid J_i^\alpha(t, \hat{x}') - J_i^\alpha(t, x') \mid + \mid J_i^\alpha(t, x') - J_i^\alpha(t, x) \mid \\ &\leq \mid J_i^\alpha(t', x') - J_i^\alpha(t, \hat{x}') \mid \\ &\quad + K_1(p)\left(\psi(\mid x' - \hat{x}' \mid) + \psi(x' - x)\right).\end{aligned}$$

As ψ is increasing, we have

$$\psi(\mid x' - \hat{x}' \mid) \leq \psi(re^{rT} \mid x'(t - t') \mid).$$

Therefore, we just have to prove that there exists $C(p) > 0$ such that:

$$\mid J_i^\alpha(t', x') - J_i^\alpha(t, \hat{x}') \mid \leq C(p) \mid t' - t \mid. \tag{6.39}$$

We first set $t_0 = \min(t,\ t')$, $\quad t_1 = max(t,\ t')$ and

$$x_k = x' \exp\left(\frac{r}{2}(t - t' - (-1)^k(t_1 - t_0))\right) \text{ for } k \in \{0, 1\}.$$

With these notations, if $t_0 = t'$ then $x_0 = x'$ and $x_1 = \hat{x}'$ else $t_1 = t'$, $x_1 = x'$ and $x_0 = \hat{x}'$. Hence, we aim at proving that

$$\mid J_i^\alpha(t_1, z_1) - J_i^\alpha(t_0, z_0) \mid \leq C(p)(t_1 - t_0) \text{ with } x_1 = x_0 e^{r(t_1 - t_0)},\ z_k = (x_k, y, p, s)\ \forall k \in \{0, 1\}.$$

To simplify notations, we set $\tau^1 = \tau^{t_1,i,z_1,\alpha}$ and $\tau^0 = \tau^{t_0,i,z_0,\alpha}$. As $\alpha \in \mathcal{A}(t_0, z_0)$, we have

$$J_i^\alpha(t_0, z_0) = \mathbb{E}\left[U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right].$$

Now, if we set $\hat{\theta} := \inf\{u \geq t_0 :\ N_u^a > N_{u^-}^a \text{ or } N_u^b > N_{u^-}^b \text{ or } R_u > R_{u^-}\}$. We have

$$\begin{aligned}J_i^\alpha(t_0, z_0) &= \mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right)\mathbb{1}_{\{\hat{\theta} \leq t_1\}}\right] \\ &\quad + \mathbb{E}\left[\mathbb{E}\left[\left(U_L\left(T \wedge \tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_1}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\, du\right) |\mathcal{F}_{t_1}\right]\mathbb{1}_{\{t_1 < \hat{\theta}\}}\right] \\ &\quad - \mathbb{E}\left[\left(\int_{t_0}^{t_1} g(Y_u^{t_0,i,z_0,\alpha})\, du\right)\mathbb{1}_{\{t_1 < \hat{\theta}\}}\right].\end{aligned}$$

We can notice that on $\{t_1 < \hat{\theta}\}$, we have

$$I_{t_1} = i, \quad \tau^1 = \tau^0 \quad \text{and} \quad \forall u \in [t_1, T\wedge\tau^1],\ Z_u^{t_0,i,z_0,\alpha} = Z_u^{t_1,i,z_1,\alpha}.$$

Therefore, we deduce from the Markov property that

$$\begin{aligned} J_i^\alpha(t_0,z_0) &= \mathbb{E}\left[\left(U_L\left(T\wedge\tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right) - \int_{t_0}^{T\wedge\tau^0} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{\hat{\theta}\le t_1\}}\right] \\ &\quad + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1<\hat{\theta}) - \mathbb{E}\left[\left(\int_{t_0}^{t_1} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{t_1<\hat{\theta}\}}\right]. \end{aligned}$$

Then, it follows from Cauchy-Schwarz inequality and Lemma 3.1 that there exists $C(p)>0$ such that

$$\begin{aligned} J_i^\alpha(t_0,z_0) &\ge -\mathbb{E}\left[\left(U_L\left(T\wedge\tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}\right)\right)^2\right]^{\frac12}\left(\mathbb{P}(\hat{\theta}\le t_1)\right)^{\frac12} \\ &\quad -TG\mathbb{P}(\hat{\theta}\le t_1) + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1<\hat{\theta}) - G(t_1-t_0) \\ &\ge -C(p)\left(\mathbb{P}(\hat{\theta}\le t_1)\right)^{\frac12} - TG\mathbb{P}(\hat{\theta}\le t_1) + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1<\hat{\theta}) - G(t_1-t_0). \end{aligned}$$

Recalling that $J_i^\alpha(t_1,z_1)\le 1$, we get

$$\begin{aligned} J_i^\alpha(t_1,z_1) - J_i^\alpha(t_0,z_0) &\le C(p)\left(\mathbb{P}(\hat{\theta}\le t_1)\right)^{\frac12} + (J_i^\alpha(t_1,z_1)+TG)\mathbb{P}(\hat{\theta}\le t_1) + G(t_1-t_0) \\ &\le C(p)(3\bar{\lambda})^{\frac12}\mid t_1-t_0\mid^{\frac12} + (3(1+TG)\bar{\lambda}+G)\mid t_1-t_0\mid. \end{aligned}$$

The last inequality follows from the following one:

$$\begin{aligned} \mathbb{P}(\hat{\theta}\le t_1) &\le \mathbb{P}(N^a_{t_1} > N^a_{t_0}) + \mathbb{P}(N^b_{t_1} > N^b_{t_0}) + \mathbb{P}(R_{t_1} > R_{t_0}) \\ &\le 3\mathbb{P}(\bar{N}_{t_1} > \bar{N}_{t_0}) \\ &\le 3\bar{\lambda}(t_1-t_0). \end{aligned}$$

Now we follow the same idea to find a lower bound for $J_i^\alpha(t_1,z_1) - J_i^\alpha(t_0,z_0)$. We have

$$J_i^\alpha(t_0,z_0) = E\left[U_L(T\wedge\tau^0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau^0)^-}) - \int_{t_0}^{T\wedge\tau^0} g(Y_s^{t_0,i,z_0,\alpha})ds\right].$$

Once again, we notice that on $\{t_1 < \hat{\theta}\}$, we have

$$I_{t_1} = i, \quad \tau_1 = \tau_0 \quad \text{and} \quad \forall u \in [t_1, T\wedge\tau_1],\ Z_u^{t_0,i,z_0,\alpha} = Z_u^{t_1,i,z_1,\alpha}.$$

Therefore, it follows from Markov property again that

$$\begin{aligned} J_i^\alpha(t_0,z_0) &\le \mathbb{E}\left[\left(U_L\left(T\wedge\tau_0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau_0)^-}\right) - \int_{t_0}^{T\wedge\tau_0} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{\hat{\theta}\le t_1\}}\right] \\ &\quad + \mathbb{E}\left[U_L(T\wedge\tau_1,\ Z^{t_1,i,z_1,\alpha}_{(T\wedge\tau_1)^-}) - \int_{t_1}^{T\wedge\tau_1} g(Y_s^{t_1,i,z_1,\alpha})ds\right]\mathbb{P}(t_1<\hat{\theta}) \\ &\le \mathbb{E}\left[\left(U_L\left(T\wedge\tau_0,\ Z^{t_0,i,z_0,\alpha}_{(T\wedge\tau_0)^-}\right) - \int_{t_0}^{T\wedge\tau_0} g(Y_u^{t_0,i,z_0,\alpha})\,du\right)\mathbb{1}_{\{\hat{\theta}\le t_1\}}\right] \\ &\quad + J_i^\alpha(t_1,z_1)\mathbb{P}(t_1<\hat{\theta}). \end{aligned}$$

As $U \leq 1$ and $g \geq 0$, we obtain

$$J_i^\alpha(t_0, z_0) \leq \mathbb{P}(\hat{\theta} \leq t_1) + J_i^\alpha(t_1, z_1)\mathbb{P}(t_1 < \hat{\theta}).$$

Hence, from the proof of Proposition 3.1, we know that there exists $C(p) > 0$ such that

$$\begin{aligned} J_i^\alpha(t_1, z_1) - J_i^\alpha(t_0, z_0) &\geq (J_i^\alpha(t_1, z_1) - 1)\mathbb{P}(\hat{\theta} \leq t_1) \\ &\geq -(C(p) + 1)\mathbb{P}(\hat{\theta} \leq t_1) \\ &\geq -3(C(p) + 1)\bar{\lambda}(t_1 - t_0). \end{aligned}$$

□

Proof of Lemma 4.3.

Proof: Let $(i, t, z) \in \{1, .., m\} \times \mathcal{S}$ and $(\psi_j)_{1 \leq j \leq m}$ a family of functions such that for all $(j, \zeta, \pi, \sigma) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(u, \xi) \to \psi_j(u, \xi, \zeta, \pi, \sigma)$ is a $\mathcal{C}^1$ function on $\{(u, \xi) \in [0, T) \times [x_{min}, +\infty) : (u, \xi, \zeta, \pi, \sigma) \in \mathcal{S}\}$ and $v - \psi$ has a global maximum at $(i, t, z) \in \{1, ..., m\} \times \mathcal{S}$. Without loss of generality we assume that $0 = (v - \psi)(i, t, z)$.

Let $0 < h < T - t$ such that

$$\begin{aligned} &\text{If} \quad x < x_{min} + p - \tfrac{s}{2} \quad \text{then } xe^{ru} < x_{min} + p - \frac{s}{2} \text{ for all } u \in [0, h] \qquad (6.40) \\ &\text{If} \quad x > x_{min} + p - \tfrac{s}{2} \quad \text{then } xe^{ru} > x_{min} + p - \frac{s}{2} \text{ for all } u \in [0, h]. \end{aligned}$$

Notice that if $x = x_{min} + p - \frac{s}{2} < 0$ then $xe^{ru} < x_{min} + p - \frac{s}{2}$ for all $u \in (0, h]$ and if $x = x_{min} + p - \frac{s}{2} \geq 0$ then $xe^{ru} \geq x_{min} + p - \frac{s}{2}$ for all $u \in [0, h]$.

We choose an admissible strategy $\alpha \in \mathcal{A}(t, z)$ and set $\widehat{\tau}^\alpha := \tau^{i,t,z,\alpha} \wedge T$ such that the dynamic programming principle (3.21) implies

$$\begin{aligned} \psi_i(t, z) &= v_i(t, z) \\ &\leq \mathbb{E}\Big[- g(y)\,(\nu - t) + v_{I_{(t+h)\wedge\hat{\theta}}}\Big((t + h) \wedge \hat{\theta},\ Z^{t,i,z,\alpha}_{(t+h)\wedge\hat{\theta}}\Big)\, \mathbb{1}_{\{(t+h)\wedge\hat{\theta} < \hat{\tau}^\alpha\}} \\ &\quad + U_L\Big(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s\Big)\, \mathbb{1}_{\{\hat{\tau}^\alpha \leq (t+h)\wedge\hat{\theta}\}}\Big] + h^2, \\ &\leq \mathbb{E}\Big[- g(y)\Big((t + h) \wedge \hat{\theta} \wedge \hat{\tau}^\alpha - t\Big) + \psi_{I_\nu}(\nu, Z_\nu) \qquad (6.41) \\ &\quad + \Big(U_L\Big(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s\Big) - \psi_i(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s)\Big)\, \mathbb{1}_{\{\hat{\tau}^\alpha \leq (t+h)\wedge\hat{\theta}\}}\Big] + h^2, \end{aligned}$$

where we have set $\nu := (t + h) \wedge \hat{\theta} \wedge \hat{\tau}^{\alpha -}$. Applying Itô's formula to $\psi_{I_u}(u, Z_u)$ between t and ν , we have

$$\begin{aligned} \psi_{I_\nu}(\nu, Z_\nu) &= \psi_i(t, z) + \int_t^\nu \frac{\partial \psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial \psi_i}{\partial x}(u, Z_u) r X_u du \\ &+ \sum_{t \leq u \leq \nu} \Big(\psi_{I_u}(u, Z_u) - \psi_{I_{u^-}}(u, Z_{u-})\Big) \\ &= \psi_i(t, z) + \int_t^\nu \frac{\partial \psi_i}{\partial t}(u, Z_u)du + \int_t^\nu \frac{\partial \psi_i}{\partial x}(u, Z_u) r X_u du \end{aligned}$$

$$
\begin{aligned}
&+ \quad \int_t^\nu \Big(\psi_{I_u}(u,Z_u)-\psi_i(u,Z_{u-})\Big)dN_u^a+\int_t^\nu \Big(\psi_{I_u}(u,Z_u)-\psi_i(u,Z_{u-})\Big)dN_u^b \\
&+ \quad \int_t^\nu \Big(\psi_{I_u}(u,Z_u)-\psi_i(u,Z_{u^-})\Big)dR_u.
\end{aligned}
$$

By taking expectation, we obtain

$$
\begin{aligned}
\mathbb{E}[\psi_{I_\nu}(\nu,Z_\nu)] &= \psi_i(t,z)+\mathbb{E}\Big[\int_t^\nu \frac{\partial\psi_i}{\partial t}(u,Z_u)du+\int_t^\nu \frac{\partial\psi_i}{\partial x}(u,Z_u)rX_udu\Big] \qquad (6.42)\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^a(t,p,s)\Big(\psi_i(u,X_{u-}+p+\frac{s}{2},y-1,p+\frac{\delta}{2}(\eta_u^a+\eta_u^b),s+\delta(\eta_u^a-\eta_u^b))\\
&\qquad -\psi(u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y>y_{min}\}}du\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^b(t,p,s)\Big(\psi(u,X_{u-}-p+\frac{s}{2},y+1,p-\frac{\delta}{2}(\eta_u^a+\eta_u^b),s-\delta(\eta_u^a-\eta_u^b))\\
&\qquad -\psi(u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y<y_{max}\text{ and }X_{u-}-p+\frac{s}{2}\geq x_{min}\}}du\Big]\\
&+ \sum_{j\neq i}\mathbb{E}\Big[\int_t^\nu \gamma_{i,j}\Big(\psi_j(u,Z_u)-\psi_i(u,Z_u)\Big)\,du\Big].
\end{aligned}
$$

Plugging (6.42) into (6.41), we obtain

$$
\begin{aligned}
\psi_i(t,z) &\leq \mathbb{E}[-g(y)\,(\nu-t)]+\psi_i(t,z) \qquad (6.43)\\
&+ \mathbb{E}\Big[\int_t^\nu \frac{\partial\psi_i}{\partial t}(u,Z_u)du+\int_t^\nu \frac{\partial\psi_i}{\partial x}(u,Z_u)rX_udu\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^a(t,p,s)\Big(\psi_i(u,X_{u-}+p+\frac{s}{2},y-1,p+\frac{\delta}{2}(\eta_u^a+\eta_u^b),s+\delta(\eta_u^a-\eta_u^b))\\
&\qquad -\psi_i(u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y>y_{min}\}}du\Big]\\
&+ \mathbb{E}\Big[\int_t^\nu \lambda_i^b(t,p,s)\Big(\psi_i(u,X_{u-}-p+\frac{s}{2},y+1,p-\frac{\delta}{2}(\eta_u^a+\eta_u^b),s-\delta(\eta_u^a-\eta_u^b))\\
&\qquad -\psi_i(u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y<y_{max}\text{ and }X_{u-}-p+\frac{s}{2}\geq x_{min}\}}du\Big]\\
&+ \sum_{j\neq i}\mathbb{E}\Big[\int_t^\nu \gamma_{i,j}\Big(\psi_j(u,Z_u)-\psi_i(u,Z_u)\Big)\,du\Big]\\
&+ \mathcal{R}_i(t,z)+h^2,
\end{aligned}
$$

where we have set

$$
\mathcal{R}_i(t,z)=\mathbb{E}\Big[\,\Big(U_L\left(\hat\tau^\alpha,xe^{r(\hat\tau^\alpha-t)},y,p,s\right)-\psi_i(\hat\tau^\alpha,xe^{r(\hat\tau^\alpha-t)},y,p,s)\Big)\,\mathbb{1}_{\{\hat\tau^\alpha\leq(t+h)\wedge\hat\theta\}}\Big].
$$

As $x>x_{min}$, we have $xe^{r(\hat\tau^\alpha-t)}>x_{min}$ on $\{\hat\tau^\alpha\leq t+h\}$ for h small enough. Hence, we have:

$$
\begin{aligned}
\mathcal{R}_i(t,z) &= \mathbb{E}\Big[[U_L-\psi_i]\left(\theta^a,xe^{r(\theta^a-t)},y_{min},p,s\right)\mathbb{1}_{\{\hat\tau^\alpha=\theta^a\leq(t+h)\wedge\hat\theta;\ y=y_{\min}\}}\Big]\\
&\quad +\mathbb{E}\Big[[U_L-\psi_i]\left(\theta^b,xe^{r(\theta^b-t)},y,p,s\right)\mathbb{1}_{\{\hat\tau^\alpha=\theta^b\leq(t+h)\wedge\hat\theta;\ y=y_{max}\text{ or }xe^{r(\hat\tau^\alpha-t)}<x_{min}+p-\frac{s}{2}\}}\Big]\\
&= \mathbb{E}\Big[\int_t^\nu \lambda_i^a(t,p,s)\Big[U_L-\psi_i\Big](u,X_{u-},y,p,s)\mathbb{1}_{\{y=y_{min}\}}du\Big] \qquad (6.44)\\
&\quad +\mathbb{E}\Big[\int_t^\nu \lambda_i^b(t,p,s)\Big[U_L-\psi_i\Big](u,X_{u-},y,p,s)\Big)\mathbb{1}_{\{y=y_{max}\text{ or }xe^{r(u-t)}<x_{min}+p-\frac{s}{2}\}}du\Big].
\end{aligned}
$$

At this point, we remark that, for $u \in (t, \nu]$,

$$\begin{array}{rcl}\mathbb{1}_{\{xe^{r(u-t)} \geq x_{min}+p-\frac{s}{2}\}} & = & 1_{\{x>x_{min}+p-\frac{s}{2} \text{ or } x=x_{min}+p-\frac{s}{2}\geq 0\}} \\ \mathbb{1}_{\{xe^{r(u-t)} < x_{min}+p-\frac{s}{2}\}} & = & 1_{\{x<x_{min}+p-\frac{s}{2} \text{ or } x=x_{min}+p-\frac{s}{2}<0\}}.\end{array}$$

Therefore, plugging (6.44) into (6.43), it follows from the definition of the Hamiltonian that

$$0 \leq \mathbb{E}[\int_t^\nu -g(y) + \mathcal{H}_i(u, Z_{u-}, \psi, \frac{\partial \psi}{\partial x})\, du].$$

From the right continuity of the processes $(R_t)_t$, $(N_t^a)_t$ and $(N_t^b)_t$, we get

$$\lim_{h \to 0^+} \frac{1}{h}[\int_t^\nu -g(y) + \mathcal{H}_i(u, Z_{u-}, \psi, \frac{\partial \psi}{\partial x})\, du] = -g(y) + \mathcal{H}_i(t, z, \psi, \frac{\partial \psi}{\partial x}) \quad \text{a.s.}$$

Since ψ is smooth with respect to the variables t and x and the process $(u, Z_u)_{t \leq u \leq t+h}$ is bounded on $\{u \leq \nu\}$, we deduce from the dominated convergence theorem that :

$$0 \leq \lim_{h \to 0^+} \frac{1}{h}\mathbb{E}[\int_t^\nu -g(y) + \mathcal{H}_i(u, Z_{u-}, \psi, \frac{\partial \psi}{\partial x})\, du] = -g(y) + \mathcal{H}_i(t, z, \psi, \frac{\partial \psi}{\partial x}).$$

Therefore $(v_i)_{1 \leq i \leq m}$ is a subsolution of the system of variational inequalities (4.23) on $\{1, ..., m\} \times \mathcal{S}$.

□

Proof of Lemma 4.4.
Proof: The proof is very similar to the one of the previous Lemma. Indeed, let $(i, t, z) \in \{1, .., m\} \times \mathcal{S}$ and $(\psi_j)_{1 \leq j \leq m}$ a family of functions such that for all $(j, \zeta, \pi, \sigma) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(u, \xi) \to \psi_j(u, \xi, \zeta, \pi, \sigma)$ is a $\mathcal{C}^1$ function on $\{(u, \xi) \in [0, T) \times [x_{min}, +\infty) : (u, \xi, \zeta, \pi, \sigma) \in \mathcal{S}\}$ and $v - \psi$ has a global minimum at $(i, t, z) \in \{1, ..., m\} \times \mathcal{S}$. Without loss of generality we assume that $0 = (v - \psi)(i, t, z)$.
Let $0 < h < T - t$, $\alpha \in \mathcal{A}(t, z)$ an admissible strategy and set $\widehat{\tau}^\alpha := \tau^{i,t,z,\alpha} \wedge T$. The dynamic programming principle (3.21) implies the opposite inequality of (6.41) without the term h^2. Then, we may apply Itô's formula to $\psi_{I_u}(u, Z_u)$ between t and ν and by taking expectation, we obtain equation (6.42). Finally, we obtain the opposite inequality of (6.43) for any admissible strategy. Therefore, we get

$$0 \geq \mathbb{E}[\int_t^\nu -g(y) + \mathcal{H}_i(u, Z_{u-}, \psi, \frac{\partial \psi}{\partial x})\, du]$$

and we conclude by dividing by h and letting h going to 0.

□

Proof of Theorem 3.1.
First step: We prove that $v_i(t, z) \leq \sup_{\alpha \in \mathcal{A}(t,z)} \hat{J}_i^{\alpha,\nu}(t, z)$.

Let $\alpha \in \mathcal{A}(t, z)$. We have

$$\begin{array}{rcl} J_i^\alpha(t, z) & = & \mathbb{E}\left[U_L(\hat{\tau}^\alpha, Z_{\hat{\tau}^\alpha -}^{t,i,z,\alpha}) - \int_t^{\hat{\tau}^\alpha} g(Y_s^{t,i,y,\alpha}) ds\right] \\ & = & \mathbb{E}\left[\mathbb{E}\left[U_L(\hat{\tau}^\alpha, Z_{\hat{\tau}^\alpha -}^{t,i,z,\alpha}) - \int_t^{\hat{\tau}^\alpha} g(Y_s^{t,i,y,\alpha}) ds | \mathcal{F}_{\nu \wedge \hat{\theta}}\right] \mathbb{1}_{\{\nu \wedge \hat{\theta} < \hat{\tau}^\alpha\}}\right] \\ & & + \mathbb{E}\left[\left(U_L(\hat{\tau}^\alpha, xe^{r(\hat{\tau}^\alpha - t)}, y, p, s) - g(y)(\hat{\tau}^\alpha - t)\right) \mathbb{1}_{\{\hat{\tau}^\alpha \leq \nu \wedge \hat{\theta}\}}\right]. \end{array}$$

Now, we shall work on $\{\nu\wedge\hat\theta<\hat\tau^\alpha\}$ and we have:

$$\mathbb{E}\left[U_L(\hat\tau^\alpha, Z^{t,i,z,\alpha}_{\hat\tau^\alpha-})-\int_t^{\hat\tau^\alpha} g(Y^{t,i,y,\alpha}_s)ds|\mathcal{F}_{\nu\wedge\hat\theta}\right] \;\;\leq\;\; v_{I_{\nu\wedge\hat\theta}}\left(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\right)-(\nu\wedge\hat\theta-t)g(y).$$

Second step: We prove that $\hat v_\nu(i,t,z):=\sup_{\alpha\in\mathcal{A}(t,z)}\hat J^{\alpha,\nu}_i(t,z)\leq v_i(t,z)$.
Let $\varepsilon>0$. We first notice that

$$X^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\leq x^+e^{rT}+p+\frac{K}{2}\delta \text{ and } P^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\leq p+\delta.$$

Therefore $(\nu\wedge\hat\theta, Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})$ takes values in the bounded set $B(t,x,p)$ where

$$B(t,x,p)=\left\{(u,\xi,\zeta,\pi,\sigma)\in[t,T]\times\mathcal{S}:\;\xi\leq x^+e^{rT}+p+\frac{K}{2}\text{ and }\pi\leq p+\delta\right\}.$$

We now define a countable partition of $B(t,x,p)$ with Borel subsets $\mathcal{B}_k$ such that for all $k\in\mathbb{N}$, $\mathcal{B}_k=I\times J\times\{a\}\times\{b\}\times\{c\}$ where $I\times J\subset[t,T]\times(x_{min},+\infty)$, $a\in\{y_{min},...,y_{max}\}$, $b\in\frac{\delta}{2}\mathbb{N}$ and $c\in\delta,...,K\delta\}$. For $k\in\mathbb{N}$, we choose $((t_k,z_k):=(t_k,x_k,y_k,p_k,s_k)\in\bar{\mathcal{B}}_k$ such that t_k is the largest time in the trace of $\bar{\mathcal{B}}_k$ in $[t,T]$.
From Propositions 3.2 and 3.3, we can choose $\mathcal{B}=(\mathcal{B}_k)_k$ such that for all $k\in\mathbb{N}$, all $i\in\{1,...,m\}$, all (u,ξ,ζ,π,σ) in $\mathcal{B}_k$ and all $\alpha\in\mathcal{A}(u,z_k)$ s.t. $\alpha_{|[u,t_k]}=0$, we have

$$|v_i(u,\xi,\zeta,\pi,\sigma)-v_i(t_k,z_k)|+|J^\alpha_i(u,\xi,\zeta,\pi,\sigma)-J^\alpha_i(t_k,z_k)|\leq\varepsilon. \tag{6.45}$$

Let $\alpha\in\mathcal{A}(t,z)$ such that

$$\hat v_\nu(i,t,z)\leq\varepsilon+\hat J^{\alpha,\nu}_i(t,z).$$

As $(\mathcal{B}_k)_{k\in\mathbb{N}}$ is a partition of $[t,T]\times\mathcal{S}$, we get

$$\begin{aligned}
\hat v_\nu(i,t,z) &\leq \varepsilon+\sum_{k=0}^{+\infty}\mathbb{E}\Big[v_{I_{\nu\wedge\hat\theta}}\left(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\right)\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_k\}}\Big]\\
&\quad+\mathbb{E}\Big[-g(y)\left(\nu\wedge\hat\theta\wedge\hat\tau^\alpha-t\right)+U_L\left(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha-t)},y,p,s\right)\mathbb{1}_{\{\hat\tau^\alpha\leq\nu\wedge\hat\theta\}}\Big]\\
&\leq 2\varepsilon+\sum_{k=0}^{+\infty}\mathbb{E}\Big[v_{I_{\nu\wedge\hat\theta}}\left(t_k,z_k\right)\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_k\}}\Big]\\
&\quad+\mathbb{E}\Big[-g(y)\left(\nu\wedge\hat\theta\wedge\hat\tau^\alpha-t\right)+U_L\left(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha-t)},y,p,s\right)\mathbb{1}_{\{\hat\tau^\alpha\leq\nu\wedge\hat\theta\}}\Big],
\end{aligned}$$

where the latter inequality derives from (6.45).
Now, for $j\in\{1,...,m\}$ and $k\in\mathbb{N}$, we introduce $\alpha^{j,k}\in\mathcal{A}(t_k,j,z_k)$ such that

$$v_j(t_k,z_k)\leq\varepsilon+J^{\alpha^{j,k}}_j(t_k,z_k).$$

Let $k\in\mathbb{N}$, we get

$$\begin{aligned}
V_k &:= \mathbb{E}\Big[v_{I_{\nu\wedge\hat\theta}}\left(t_k,z_k\right)\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_k\}}\Big]\\
&\leq \sum_{j=1}^{m}\mathbb{E}\Big[\left(\varepsilon+J^{\alpha^{j,k}}_j\left(t_k,z_k\right)\right)\mathbb{1}_{\{I_{\nu\wedge\hat\theta}=j\}}\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\; Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_k\}}\Big].
\end{aligned}$$

Now, we define the random variable κ with values in $\mathbb{N}$, null on $\{\nu\wedge\hat\theta \geq \hat\tau^\alpha\}$ and such that for $\omega\in\{\nu\wedge\hat\theta<\hat\tau^\alpha\}$, $(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})(\omega)\in\mathcal{B}_{\kappa(\omega)}$. Notice that κ is $\mathcal{F}_{\nu\wedge\hat\theta}$-measurable.
For $\omega\in\{\nu\wedge\hat\theta<\hat\tau^\alpha\}$, we set:

$$\hat\alpha_s(\omega) := \begin{cases} \alpha_s(\omega) & \text{if} \quad t\leq s\leq \nu\wedge\hat\theta(\omega) \\ 0 & \text{if} \quad \nu\wedge\hat\theta(\omega)<s<t_{\kappa(\omega)} \\ \alpha_s^{j,\kappa(\omega)}(\omega) & \text{if} \quad t_{\kappa(\omega)}\leq s \text{ and } I_{\nu\wedge\hat\theta}(\omega)=j. \end{cases} \tag{6.46}$$

On $\{\nu\wedge\hat\theta\geq\hat\tau^\alpha\}$, we set $\hat\alpha=\alpha$. To simplify notations, we shall write $\hat\alpha$ for $\hat\alpha_{|[t_\kappa,T]}$ and $\hat\alpha_{|[\nu\wedge\hat\theta,T]}$. We get

$$V_k \leq \mathbb{E}\Big[\Big(\varepsilon + J^{\hat\alpha}_{I_{\nu\wedge\hat\theta}}(t_\kappa,z_\kappa)\Big)\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_\kappa\}}\Big].$$

From the definition of $\hat\alpha$, we have $\hat\alpha\in\mathcal{A}\left(\nu\wedge\hat\theta, I_{\nu\wedge\hat\theta}, Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\right)$, therefore we deduce from the last inequality that

$$\begin{aligned}
\sum_{k=0}^{+\infty}V_k &\leq \varepsilon+\sum_{k=0}^{+\infty}\mathbb{E}\Big[J^{\hat\alpha}_{I_{\nu\wedge\hat\theta}}(t_\kappa,z_\kappa)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_\kappa\}}\Big] \\
&\leq 2\varepsilon+\sum_{k=0}^{+\infty}\mathbb{E}\Big[J^{\hat\alpha}_{I_{\nu\wedge\hat\theta}}\Big(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\Big)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\mathbb{1}_{\{(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta})\in\mathcal{B}_\kappa\}}\Big] \\
&= 2\varepsilon+\mathbb{E}\Big[J^{\hat\alpha}_{I_{\nu\wedge\hat\theta}}\Big(\nu\wedge\hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\Big)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\Big] \\
&= 2\varepsilon+\mathbb{E}\Big[\Big(U_L(\hat\tau^{\hat\alpha}, Z^{\nu\wedge\hat\theta, I_{\nu\wedge\hat\theta}, Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta},\hat\alpha}_{\hat\tau^{\hat\alpha}-}) - \int_{\nu\wedge\hat\theta}^{\hat\tau^{\hat\alpha}} g(Y_s^{\nu\wedge\hat\theta, I_{\nu\wedge\hat\theta}, Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta},\hat\alpha})ds\Big)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\Big] \\
&= 2\varepsilon+\mathbb{E}\Big[\Big(U_L(\hat\tau^{\hat\alpha}, Z^{t,i,z,\hat\alpha}_{\hat\tau^{\hat\alpha}-}) - \int_{\nu\wedge\hat\theta}^{\hat\tau^{\hat\alpha}} g(Y_s^{t,i,z,\hat\alpha})ds\Big)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\Big],
\end{aligned}$$

where the last equality is obtained from the definition in (6.46) To conclude we notice that on $\{\hat\tau^\alpha\leq\nu\wedge\hat\theta\}$, we have $\hat\tau^\alpha=\hat\tau^{\hat\alpha}$ and then

$$\begin{aligned}
\hat v_\nu(i,t,z) &\leq 4\varepsilon+\mathbb{E}\Big[-g(y)\Big(\nu\wedge\hat\theta\wedge\hat\tau^\alpha-t\Big)+U_L\Big(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha-t)},y,p,s\Big)\,\mathbb{1}_{\{\hat\tau^\alpha\leq\nu\wedge\hat\theta\}}\Big] \\
&\quad +\mathbb{E}\Big[\Big(U_L(\hat\tau^{\hat\alpha}, Z^{t,i,z,\hat\alpha}_{\hat\tau^{\hat\alpha}-}) - \int_{\nu\wedge\hat\theta}^{\hat\tau^{\hat\alpha}} g(Y_s^{t,i,z,\hat\alpha})ds\Big)\,\mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}}\Big] \\
&= 4\varepsilon+\mathbb{E}\Big[U_L(\hat\tau^{\hat\alpha}, Z^{t,i,z,\hat\alpha}_{\hat\tau^{\hat\alpha}-}) - \int_t^{\hat\tau^{\hat\alpha}} g(Y_s^{t,i,z,\hat\alpha})ds\Big] \\
&= 4\varepsilon+J_i^{\hat\alpha}(t,z) \\
&\leq 4\varepsilon+v_i(t,z).
\end{aligned}$$

Sending ε to zero, we may conclude the proof. □

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