Optimal exit strategies for investment projects

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Abstract

We study the problem of an optimal exit strategy for an investment project which is unprofitable and for which the liquidation costs evolve stochastically. The firm has the option to keep the project going while waiting for a buyer, or liquidating the assets at immediate liquidity and termination costs. The liquidity and termination costs are governed by a mean-reverting stochastic process whereas the rate of arrival of buyers is governed by a regime-shifting Markov process. We formulate this problem as a multidimensional optimal stopping time problem with random maturity. We characterize the objective function as the unique viscosity solution of the associated system of variational Hamilton–Jacobi–Bellman inequalities. We derive explicit solutions and numerical examples in the case of power and logarithmic utility functions when the liquidity premium factor follows a mean-reverting CIR process.

Keywords: real options, stochastic control, liquidity discount, regime shifting, viscosity solutions, system of variational inequalities.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Introduction

There is often a time when a firm is engaged in a project that does not produce to its full potential and faces the difficult dilemma of shutting it down or keeping it alive in the hope that it will become profitable once again. When an investment in not totally irreversible, assets can be sold at their scrap value minus some liquidation and project termination costs, which may include for example termination pay to workers, legal fees and a liquidity premium in the case of fire sale of the assets. Since these closing costs may be substantial, it may be worthwhile to wait for the project to be profitable again or to wait for an interested buyer that will pay the fair value of the assets and put them to better use. In this study, we give an analytical solution to this problem when the liquidation costs and the value of the assets are diffusion processes and the arrival time of a buyer is modeled by means of an intensity function depending on the current state of a Markov chain.

There is a vast literature on firm's investment decisions in stochastic environments, see for instance [2], [4] [6], [15], [18], [19] and [23]. In relation to our study, Dixit and Pindyck [9] consider various firm's decisions problems with entry, exit, suspension and/or abandonment scenarios in the case of an asset given by a geometric Brownian motion. The firm's strategy can then be described in terms of stopping times given by the time when the value of the assets hit certain threshold levels characterized as free boundaries of a variational problem. Duckworth and Zervos [10], and Lumley and Zervos [16] solve an optimal investment decision problem with switching costs in which the firm controls the production rate and must decide at which time it exits and re-enters production.

The firm, we consider, in this paper, must decide between liquidating the assets of an underperforming project and waiting for the project to become once again profitable, in a setting where the liquidation costs and the value of the assets are given by general diffusion processes. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting.

Amongst the large literature on optimal stopping problems, we may refer to some related works including Bouchard, El Karoui and Touzi [1], Carr [3], Dayanik and Egami [7], Dayanik and Karatzas [8], Guo and Zhang [12], Lamberton and Zervos [13]. In [8] and [13], the authors study optimal stopping problems with general 1-dimensional processes. Random maturity in optimal stopping problem was introduced in [3] and [1]. It allowed to reduce the dimension of their problems as well as addressing the numerical issues. We may refer to Dayanik and Egami [7] for a recent paper on optimal stopping time and random maturity. For optimal stopping problem with regime shifting, we may refer to Guo and Zhang [12], where an explicit optimal stopping rule and the corresponding value function in a closed form are obtained.

In this paper, our optimal stopping problem combines all the above features, i.e., random maturity and regime shifting, in the bi-dimensional framework. We are able to characterize the value function of our problem and provide explicit solution in some particular cases where we manage to reduce the dimension of our control problem.

In the general bi-dimensional framework, the main difficulty is related to the proof of the continuity property and the PDE characterization of the value function. Since it is not possible to get the smooth-fit property, the PDE characterization may be obtained only via the viscosity approach. To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.

In the particular cases where it is possible to reduce our problem to a one-dimensional problem, we are able to provide explicit solution. Our reduced one-dimensional problem is highly related to previous studies in the literature, see for instance Zervos, Johnson and Alezemi [24] and Leung, Li and Wang [14].

The rest of the paper is organized as follows. We define the model and formulate our optimal stopping problem in the following section. In Section 3, we characterize the solution of the problem in terms of the unique viscosity solution to the associated HJB system and obtain some qualitative description of these functions. In Sections 4 and 5, we derive explicit solutions in the case of power and logarithmic utility functions when the liquidity discount factor follows a mean-reverting CIR process, and provide numerical examples.

2 The Investment Project

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times. Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation ρ, i.e. $d[W, B]_t = \rho dt$ for all t.

We consider a firm which owns assets that are currently locked up in an investment project which currently produces no output per unit of time. Because the firm is currently not using the assets at its full potential, it considers two possibilities. The first is to liquidate the assets in a fire sale and recover any remaining value. The cash flow obtained in the latter case is the fair value of the assets minus liquidation and project termination costs. We denote by θ the moment at which the firm decides to take this option. However, liquidation and project termination costs may be high. As a result, the second option is to wait for the project to become profitable once again, or equivalently, to wait for an investor or another firm who will purchase the assets as a whole at their fair value S_τ and put them to good use in a profitable investment project. We denote by τ the moment when this happens, and for simplicity we refer to this moment as the recovery time. Intuitively, the firm will choose the first option if the probability that the assets can be sold at their fair value is too low.

The fair value of the assets are given by $S = \exp(X)$, in which

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0 \\ X_0 &= x. \end{aligned} \tag{2.1}$$

Assume that μ and σ are Lipschitz functions on $\mathbb{R}$ satisfying the following growth condition

$$\lim_{|x|\to\infty} \frac{|\mu(x)| + |\sigma(x)|}{|x|} = 0. \tag{2.2}$$

Liquidation and Termination Costs. We model the liquidation cost of the assets and terminal costs of the project as a given process $(f(Y_t))_{t\geq 0}$, where f is strictly decreasing C^2 function defined on $\mathbb{R}^+ \to [0,1]$, and satisfies the following conditions:

$$\begin{aligned} f(0)=1 \text{ and } \lim_{y\to\infty} f(y) &= 0 \\ \exists\, c>0, \text{ such that } \lim_{y\to\infty} f(y)\exp(y^c) &= 0. \end{aligned} \tag{2.3}$$

Unlike the value of financial assets, it is natural to model liquidation costs with mean-reverting properties. As such, the costs, given by $f(Y_t)$ at time t, is defined in terms of the mean-reverting non-negative process Y which is governed by the following SDE:

$$\begin{aligned} dY_t &= \alpha(Y_t)dt + \gamma(Y_t)dW_t, \quad (2.4) \\ Y_0 &= y, \end{aligned}$$

where α is a Lipschitz function on $\mathbb{R}^+$ and, for any $\varepsilon > 0$, γ is a Lipschitz function on $[\varepsilon, \infty)$. We assume that α and γ satisfy the following growth condition

$$\limsup_{|y|\to\infty} \frac{|\alpha(y)|+|\gamma(y)|}{|y|} < +\infty. \tag{2.5}$$

Furthermore, to insure the mean-reverting property, we assume that there exists $\beta > 0$ such that $(\beta - y)\alpha(y)$ is positive for all $y \geq 0$. Should the firm decide to terminate the project operations and liquidate the assets, the resulting cash flow is $S_t f(Y_t)$.

The simplest example is $f(y) = \exp(-y)$ with the process Y given as a CIR process:

$$\begin{aligned} dY_t &= \kappa\,(\beta - Y_t)\,dt + \gamma\sqrt{Y_t}dW_t, \quad (2.6) \\ Y_0 &= y, \end{aligned}$$

with κ, β and γ positive constants.

The recovery time. We model the arrival time of a buyer, denoted by τ, or equivalently the time when the project becomes profitable again, by means of an intensity function λ_i depending on the current state i of a continuous-time, time-homogenous, irreducible Markov chain L, independent of W and B, with $m+1$ states. The states of the chain represent liquidity states of the assets. The generator of the chain L under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=0,\ldots m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain L from state i to state j $(0 \leq i,j \leq m)$. Without loss of generality we assume

$$\lambda_0 > \lambda_1 > \ldots > \lambda_m > 0. \tag{2.7}$$

Utility function. We let U denote the utility function of the firm. We assume that U satisfies the following assumptions.

Assumption 2.1 $U : \mathbb{R}^+ \to \mathbb{R}$ *is non-decreasing, concave and twice continuously differentiable, and satisfies*

$$\lim_{x\to 0} x\, U'(x) \;<\; +\infty. \tag{2.8}$$

Assumption 2.2 *U is supermeanvalued w.r.t. S, i.e.*

$$U(S_t) \geq \mathbb{E}[U(S_\theta)|\mathcal{F}_t] \tag{2.9}$$

for any stopping time $\theta \in \mathcal{T}$.

The financial interpretation of the supermeanvalued property of U w.r.t. S is as follows: it is always better to accept right away an offer to buy the assets at their fair value then to wait for a later one. Indeed, if an offer at the fair value S_t arrives at time t, then the utility of the obtained value S_t, is greater that the expected utility obtained at any fair value in the future. For more details on the supermeanvalued property, which is closely related to the concept of superharmonicity, we may refer to Dynkin [11] and Oksendal [20].

Objective function. The objective of the firm is to maximize the expected profit obtained from the sale of the illiquid asset, either through liquidation or at its fair value at the exogenously given stopping time τ. As such, we consider the following value function:

$$v(i,x,y) := \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\,\mathbb{1}_{\theta \leq \tau} + U(e^{X_\tau})\,\mathbb{1}_{\theta > \tau}\right], \quad x \in \mathbb{R},\, y \in \mathbb{R}^+,\, i \in \{0,\dots,m\} \tag{2.10}$$

where $\mathbb{E}^{i,x,y}$ stands for the expectation with initial conditions $X_0 = x$, $Y_0 = y$ and $L_0 = i$, and $h(x,y) = U(e^x f(y))$. Recall that τ is defined through the Markov chain L.

For the rest of the paper, we sometimes write $v_i(x,y)$ instead of $v(i,x,y)$ depending on the context.

In the next section, we characterize the value function v in terms of the unique viscosity solution to the associated HJB system, and describe qualitative properties of the liquidation and continuation regions. In Section 4 and 5, we consider specific cases for μ, σ, and the utility function U, and derive explicit formulas.

3 Characterization of the value function

We first obtain some descriptive properties of these functions including the monotonicity and continuity of the functions v_i.

We denote by $\mathcal{L}$ the second order differential operator associated to the state processes (X,Y):

$$\mathcal{L}\phi(x,y) = \mu(x)\frac{\partial \phi}{\partial x} + \alpha(y)\frac{\partial \phi}{\partial y} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2} + \rho\gamma(y)\sigma(x)\frac{\partial^2 \phi}{\partial x \partial y} + \frac{1}{2}\gamma^2(y)\frac{\partial^2 \phi}{\partial y^2}. \tag{3.11}$$

The main result of this section is that the value function v is the unique viscosity solution of the following variational inequality:

$$\min\Big[-\mathcal{L}v(i,x,y) - \mathcal{G}_i v(.,x,y) - \mathcal{J}_i v(i,x,y) - \eta(x)\,,\, v(i,x,y) - h(x,y)\Big] = 0, \tag{3.12}$$

in which the operators $\mathcal{G}_i$ and $\mathcal{J}_i$ are defined as

$$\begin{aligned} \mathcal{G}_i\varphi(.,x,y) &= \sum_{j \neq i} \vartheta_{i,j}\left(\varphi(j,x,y) - \varphi(i,x,y)\right) \\ \mathcal{J}_i\varphi(i,x,y) &= \lambda_i\left(e^x - \varphi(i,x,y)\right). \end{aligned}$$

We can also say that the family of value functions $v_i (i = 0, \ldots, m)$ is the unique solution of the above system of variational inequalities, meaning that each v_i satisfies the variational inequality (3.12) for $i = 0, \ldots, m$. In the same reasoning, we sometimes write $\mathcal{G}_i v_.(x, y)$ instead of $\mathcal{G}_i v(., x, y)$ when referring to the v_i's as a family of functions. Before stating the main result, we derive a number of analytical properties of the value functions.

Proposition 3.1 *The value functions v_i are non-decreasing in x and non-increasing in y and verify the following inequalities*

$$Max\big(h(x,y), \mathbb{E}^x[U(e^{X_\tau})]\big) \leq v_i(x,y) \leq U(e^x) \text{ on } \mathbb{R}\times\mathbb{R}^+ . \tag{3.13}$$

Proof. From the definition of the value function, by considering $\theta = 0$, it is obvious that $v_i(x, y) \geq h(x, y)$.
For any $t > 0$, we also have

$$v_i(x,y) \geq \mathbb{E}^{i,x,y}\left[h(X_t, Y_t)\, \mathbb{1}_{t\leq\tau}\mathbb{1}_{Y_t>0} + U(e^{X_\tau})\, \mathbb{1}_{t>\tau}\right].$$

As τ is almost surely finite, letting t going to $+\infty$, we find that $v_i(x, y) \geq \mathbb{E}^x[U(e^{X_\tau})]$.
Since U is non-decreasing and $0 \leq f(Y) \leq 1$, we also have the following inequalities:

$$\begin{aligned} v_i(x,y) &\leq \sup_{\theta\in\mathcal{T}} \mathbb{E}\left[U(e^{X_\theta})\mathbb{1}_{\theta\leq\tau} + U(e^{X_\tau})\mathbb{1}_{\theta>\tau}\right] \\ &\leq \sup_{\theta\in\mathcal{T}} \mathbb{E}\left[U(e^{X_{\theta\wedge\tau}})\right] \\ &\leq \sup_{\theta\in\mathcal{T}} \mathbb{E}\left[U(e^{X_\theta})\right]. \end{aligned}$$

Using the supermeanvalued property of U w.r.t S, we obtain $v_i(x, y) \leq U(e^x)$.

From the uniqueness of the solution of the stochastic differential equation (2.1) combined with the non-decreasing property of U, we obtain that v_i is non-decreasing in x. We may apply the same argument to obtain that v_i is non-increasing in y, but one should be careful when using the uniqueness of the trajectory of Y which only holds up to $\xi_y := \inf\{t > 0, Y_t^y = 0\}$. See Remark 3.1 below in this regard. Since $f(0) = \sup_{y\in\mathbb{R}^+} f(y) = 1$, the non-increasing property of v_i in y is verified. □

From Proposition 3.1, we obviously obtain that $v_i(x, 0) = U(e^x)$, which states that when the liquidation value matches the fair value of the asset, it is optimal to immediately sell the asset. Furthermore, if $\mathbb{E}^x[U(e^{X_\tau})] = U(e^x)$, we find that $v_i(x, y) = U(e^x)$ so that the optimal policy is to wait until τ, i.e. the arrival of an interested buyer willing to pay the fair price.

We now deals with the analysis of the continuity of the value function. We highlight two main difficulties that need a no-standard treatment. The first one comes from the SDE satisfied by Y (2.4) since we do not assume the standard hypothesis of Lipschitz coefficients see Remark 3.1. We overcome this drawback showing that the local Lipschitz property is satisfied until the smallest optimal exit time from the investment, see Lemma 3.1. The second difficulty is related to the bi-dimensional setting where the classical arguments used to show the regularity of value function are not longer available. We then need to show the continuity in term of limits of sequences and to distinguish different sub-sequences with ad-hoc proofs, see Proposition 3.2.

The complexity of the proof of Proposition 3.2 suggests that a direct proof of differentiability, i.e. smooth-fit property, of the value function is probably out of reach in our setting. We will then turn to the viscosity characterization approach to overcome the impossibility to use a verification approach.

Remark 3.1 Noticing that the coefficients of the SDE governing X are Lipschitz continuous, we have the continuity of $X(t,x) := X_t^x$ in variables (t,x), for almost all ω. In particular, for any given $t > 0$, the mapping which associates x to the trajectory of X:

$$\begin{array}{rcl} \mathbb{R} & \to & C([0,t],\mathbb{R}) \\ x & \mapsto & X(.,\omega,x) \end{array}$$

is continuous. In here, $C([0,t],\mathbb{R})$ denotes the space of continuous real functions defined on $[0,t]$. On the other hand, since the coefficients of the SDE of Y are only locally Lipschitz, the mapping which associates y to the trajectory of Y:

$$\begin{array}{rcl} \mathbb{R} & \to & C([0,t],\mathbb{R}) \\ y & \mapsto & Y(.,\omega,y) \end{array}$$

is continuous only on the open set $A_y := \{y : \ \xi_y > t\}$, where as above $\xi_y = \inf\{t > 0, Y_t^y = 0\}$.

Before turning to the continuity of the value functions, we show the existence of an optimal stopping time.

Lemma 3.1 *There exists a stopping time $\theta^*_{i,x,y}$ such that*

$$v(i,x,y) = \mathbb{E}^{i,x,y}\left[h(X_{\theta^*_{i,x,y}}, Y_{\theta^*_{i,x,y}})\, \mathbb{1}_{\theta^*_{i,x,y} \le \tau \wedge \xi_y} + U(e^{X_\tau})\, \mathbb{1}_{\theta^*_{i,x,y} > \tau \wedge \xi_y}\right]. \tag{3.14}$$

Moreover, on $\{\xi_y \le \tau\}$, we have $\theta^*_{i,x,y} \le \xi_y$.

Proof. We have

$$v(i,x,y) = \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\, \mathbb{1}_{\theta \le \tau} + U(e^{X_\tau})\, \mathbb{1}_{\theta > \tau}\right].$$

We consider the process Z defined as

$$Z_t = h(X_t, Y_t)\, \mathbb{1}_{t \le \tau} + U(e^{X_\tau})\, \mathbb{1}_{t > \tau}.$$

The process $(v(L_t, X_t, S_t))_{t \ge 0}$ is the Snell envelope of Z. As such,

$$v(i,x,y) = \mathbb{E}^{i,x,y}\left[Z_{\theta^*_{i,x,y}}\right]$$

where

$$\theta^*_{i,x,y} = \inf\{t \ge 0; v(L_t^i, X_t^x, Y_t^y) \le Z_t\}.$$

From the definition of the stopping time ξ_y and since $v(i,x,0) = U(e^x)$, we have

$$
\begin{aligned}
v(L^i_{\xi_y}, X^x_{\xi_y}, Y^y_{\xi_y}) &= v(L^i_{\xi_y}, X^x_{\xi_y}, 0) \\
&= U(e^{X^x_{\xi_y}})\mathbb{1}_{\xi_y \le \tau} + U(e^{X^x_{\xi_y}})\mathbb{1}_{\tau < \xi_y} \\
&= Z_{\xi_y} + \left(U(e^{X^x_{\xi_y}}) - U(e^{X^x_{\tau}})\right)\mathbb{1}_{\tau < \xi_y}.
\end{aligned}
$$

Therefore, on the set $\{\xi_y \le \tau\}$, we have $v(L^i_{\xi_y}, X^x_{\xi_y}, Y^y_{\xi_y}) = Z_{\xi_y}$ so that $\theta^*_{i,x,y} \le \xi_y$. Moreover, we find

$$
\{\theta^*_{i,x,y} \le \tau\} = \{\theta^*_{i,x,y} \le \tau \wedge \xi_y\} \qquad \text{and} \qquad \{\theta^*_{i,x,y} > \tau\} = \{\theta^*_{i,x,y} > \tau \wedge \xi_y\}.
$$

It allows us to conclude the proof, by observing that

$$
\begin{aligned}
v(i,x,y) &= \mathbb{E}^{i,x,y}\left[Z_{\theta^*_{i,x,y}}\right] \\
&= \mathbb{E}^{i,x,y}\left[h(X_{\theta^*_{i,x,y}}, Y_{\theta^*_{i,x,y}})\,\mathbb{1}_{\theta^*_{i,x,y} \le \tau\wedge\xi_y} + U(e^{X_\tau})\,\mathbb{1}_{\theta^*_{i,x,y} > \tau\wedge\xi_y}\right].
\end{aligned}
$$

□

We now prove the continuity of the value functions.

Proposition 3.2 *The value functions v_i are continuous on $\mathbb{R} \times \mathbb{R}^+$ and satisfy:*

$$
\lim_{(u,y)\to(x,0^+)} v_i(u,y) = v_i(x,0) = U(e^x).
$$

Proof. Since both $h(x,y)$ and $U(e^x)$ are continuous, using relation (3.13), we obtain

$$
\lim_{(u,y)\to(x,0^+)} v_i(u,y) = v_i(x,0) = U(e^x),
$$

leading to the continuity of v_i on $\mathbb{R} \times \{0\}$.

We now examine the continuity of v_i at a given $(x,y) \in \mathbb{R} \times (0,\infty)$ and $i \in \{0,...,m\}$. We consider a sequence $(x_n,y_n)_{n\ge 0}$ which converges to (x,y). Without loss of generality, we may consider $(x_n,y_n) \in (x-1,x+1) \times ((y-1)_+, y+1)$. We need to show that $\lim_{n\to\infty} v_i(x_n,y_n) = v_i(x,y)$, which we will show in two steps.

<u>**Step 1.**</u> We first show that for a given $\varepsilon > 0$, there exists an $N > 0$, such that $\forall n \ge N$, we have

$$
v_i(x_n,y_n) - v_i(x,y) \le \varepsilon.
$$

We separate the sequence (x_n,y_n) into two subsequences,

- $(\tilde{x}_n, \tilde{y}_n)$, the subsequence containing only $y_n \ge y$ and
- $(\bar{x}_n, \bar{y}_n)$ the subsequence containing only $y_n < y$.

(a) Sequence $(\tilde{x}_n, \tilde{y}_n)$. Since v_i is non-increasing in y, we have

$$v_i(\tilde{x}_n, \tilde{y}_n) - v_i(x,y) \le v_i(\tilde{x}_n, y) - v_i(x,y).$$

For n, such that $\tilde{x}_n \le x$, we have

$$v_i(\tilde{x}_n, y) - v_i(x,y) \le 0. \tag{3.15}$$

For all n such that $\tilde{x}_n > x$, from Lemma 3.1, there exists θ_n, such that

$$v_i(\tilde{x}_n, y) = \mathbb{E}^{i,\tilde{x}_n,y}\left[h(X_{\theta_n}, Y_{\theta_n})\mathbb{1}_{\theta_n \le \xi_y \wedge \tau} + U(e^{X_\tau})\mathbb{1}_{\tau\wedge\xi_y<\theta_n}\right].$$

As such, we have

$$\begin{aligned} 0 \le v_i(\tilde{x}_n, y) - v_i(x,y) \quad &\le \quad \mathbb{E}^i\Big[\Big(h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n})\Big)\mathbb{1}_{\theta_n\le\tau\wedge\xi_y} \\ &\qquad + \Big(U(e^{X^{\tilde{x}_n}_\tau}) - U(e^{X^x_\tau})\Big)\mathbb{1}_{\tau\wedge\xi_y<\theta_n}\Big]. \end{aligned}$$

We let

$$\begin{aligned} A_n &:= h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \ \ge\ 0, \\ B_n &:= \Big(U(e^{X^{\tilde{x}_n}_\tau}) - U(e^{X^x_\tau})\Big) \ \ge\ 0. \end{aligned}$$

We first notice h is continuous in both variables and, in particular, continuous in the first variable, uniformly on any compact set of the second variable. Using Remark 3.1 and noticing that the function f is valued in the compact $[0,1]$, we obtain

$$\lim_{n\to\infty} A_n = 0,\ a.s.$$

Furthermore, by well-known comparison theorems for SDEs, for n big enough, we have

$$\begin{aligned} |A_n| &= h(X^{\tilde{x}_n}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \\ &\le h(X^{x+1}_{\theta_n}, Y^y_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n}) \\ &\le \sup_{t\le\tau}\left[h(X^{x+1}_t, Y^y_t) - h(X^x_t, Y^y_t)\right]. \end{aligned}$$

Using the properties of the utility function U, which is non-decreasing and concave, there exists $s_0 > 0$ and $M > 0$ such that $\forall\ 0 < s < s_0$, we have $sU'(s) < M$. As such, we have

$$\begin{aligned} h(X^{x+1}_t, Y^y_t) - h(X^x_t, Y^y_t) \quad &< \quad f(Y^y_t)\left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(f(Y^y_t)e^{X^x_t})\mathbb{1}_{f(Y^y_t)e^{X^x_t}<s_0} \\ &+ \quad f(Y^y_t)\left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0)\mathbb{1}_{f(Y^y_t)e^{X^x_t}\ge s_0} \\ &\le \quad M\left(e^{X^{x+1}_t - X^x_t} - 1\right) + \left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0). \end{aligned}$$

Then, we obtain

$$|A_n| \ \le\ \sup_{t\le\tau}\left[M\left(e^{X^{x+1}_t - X^x_t} - 1\right) + \left(e^{X^{x+1}_t} - e^{X^x_t}\right)U'(s_0)\right],$$

which is integrable.

It is equally clear that $\lim_{n\to\infty} B_n = 0$, $a.s.$ and $|B_n| \le \left| U(e^{X_\tau^{x+1}}) - U(e^{X_\tau^x}) \right|$, which is integrable.

Applying the dominated convergence theorem, we obtain that for all $\varepsilon > 0$, there exists $N > 0$, such that $\forall n \ge N$,

$$v_i(\tilde{x}_n, \tilde{y}_n) - v_i(x, y) \le \varepsilon. \tag{3.16}$$

(b) Sequence $(\bar{x}_n, \bar{y}_n)$, i.e. the sequence for which $\bar{y}_n \le y$. We set $\xi_n := \inf\{t \ge 0; Y_t^{\bar{y}_n} = 0\}$. We have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, y) \le [v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n)] + [v_i(x, \bar{y}_n) - v_i(x, y)] \,.$$

We first consider the term $v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n)$. For n such that $\bar{x}_n \le x$, we have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \le 0.$$

For all n such that $\bar{x}_n > x$, from Lemma 3.1, there exists θ_n such that

$$\begin{aligned} v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \quad \le \quad & \mathbb{E}^i \Big[\big(h(X_{\theta_n}^{\bar{x}_n}, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^{x}, Y_{\theta_n}^{\bar{y}_n}) \big) \mathbb{1}_{\theta_n \le \tau \wedge \xi_n} \\ & + \Big(U(e^{X_\tau^{\bar{x}_n}}) - U(e^{X_\tau^x}) \Big) \mathbb{1}_{\tau < \theta_n \wedge \xi_n} \Big]. \end{aligned}$$

We let

$$\begin{aligned} C_n \quad &:= \quad h(X_{\theta_n}^{\bar{x}_n}, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^{x}, Y_{\theta_n}^{\bar{y}_n}) > 0, \\ D_n \quad &:= \quad U(e^{X_\tau^{\bar{x}_n}}) - U(e^{X_\tau^x}) > 0. \end{aligned}$$

Using the same argument as in (a), we have $\lim_{n\to\infty} C_n(\omega) = 0$, $\lim_{n\to\infty} D_n(\omega) = 0$, and C_n and D_n are dominated by integrable random variables. Applying the dominated convergence theorem, we obtain for a given ε, there exists an $N > 0$, such that $\forall n \ge N$, we have

$$v_i(\bar{x}_n, \bar{y}_n) - v_i(x, \bar{y}_n) \le \frac{\varepsilon}{2}. \tag{3.17}$$

We now consider the term $v_i(x, \bar{y}_n) - v_i(x, y)$. From Lemma 3.1, for all $n \ge 1$ there exists θ_n such that

$$v_i(x, \bar{y}_n) - v_i(x, y) \le \mathbb{E}^i \left[\big(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y) \big) \mathbb{1}_{\theta_n \le \tau \wedge \xi_n} \right].$$

Since $\bar{y}_n < y$, we have $\xi_n \le \xi_y$, and thanks to the monotonicity of h, we may write

$$E_n := \big(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y) \big) \mathbb{1}_{\theta_n \le \tau \wedge \xi_n} \le \big(h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y) \big) \mathbb{1}_{\theta_n \le \tau \wedge \xi_y}.$$

Moreover, it follows from Lemma 3.1 that $\{\theta_n = \xi_y \le \tau\} = \{\xi_n = \theta_n = \xi_y \le \tau\}$. Therefore, we find

$$\begin{aligned} E_n \quad \le \quad & \big[h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y) \big] \mathbb{1}_{\theta_n < \tau \wedge \xi_y} \\ & + \big[h(X_{\theta_n}^x, 0) - h(X_{\theta_n}^x, 0) \big] \mathbb{1}_{\theta_n = \xi_n = \xi_y \le \tau} + \big[h(X_\tau^x, Y_\tau^{\bar{y}_n}) - h(X_\tau^x, Y_\tau^y) \big] \mathbb{1}_{\tau < \xi_y} \\ = \quad & \big[h(X_{\theta_n}^x, Y_{\theta_n}^{\bar{y}_n}) - h(X_{\theta_n}^x, Y_{\theta_n}^y) \big] \mathbb{1}_{\theta_n < \tau \wedge \xi_y} + \big[h(X_\tau^x, Y_\tau^{\bar{y}_n}) - h(X_\tau^x, Y_\tau^y) \big] \mathbb{1}_{\tau < \xi_y}. \end{aligned}$$

Using Remark 3.1 and the same convergence argument as above, we obtain the pointwise convergence of E_n. Noticing that, for n big enough,

$$\begin{aligned} \left[h(X^x_{\theta_n}, Y^{\bar{y}_n}_{\theta_n}) - h(X^x_{\theta_n}, Y^y_{\theta_n})\right] \mathbb{1}_{\theta_n<\tau\wedge\xi_y} &\leq \sup_{t<\tau\wedge\xi_y} \left[h(X^x_t, Y^{\frac{y}{2}}_t) - h(X^x_t, Y^y_t)\right] \mathbb{1}_{t<\tau\wedge\xi_y}, \\ \text{and } \left[h(X^x_\tau, Y^{\bar{y}_n}_\tau) - h(X^x_\tau, Y^y_\tau)\right] \mathbb{1}_{\tau<\xi_y} &\leq \left[h(X^x_\tau, Y^{\frac{y}{2}}_\tau) - h(X^x_\tau, Y^y_\tau)\right] \mathbb{1}_{\tau<\xi_y}, \end{aligned}$$

we obtain an integrable upper bound for $|E_n|$, leading therefore to the desired results, i.e. there exists an $N > 0$, such that $\forall n \geq N$, we have

$$v_i(x, \bar{y}_n) - v_i(x, y) \leq \frac{\varepsilon}{2}. \tag{3.18}$$

Combining inequalities (3.16), (3.17) and (3.18), we obtain that there exists $N > 0$, such that $\forall n \geq N$, we have

$$v_i(x_n, y_n) - v_i(x, y) \leq \varepsilon. \tag{3.19}$$

Step 2. We use the same arguments as in Step 1 to show that for a given $\varepsilon > 0$, there exists an $N > 0$, such that $\forall n \geq N$, we have $v_i(x, y) - v_i(x_n, y_n) \leq \varepsilon$. This part of the proof is easier as the optimal stopping time from Lemma 3.1 does not depend on n in some cases.

Combining the two steps, we obtain the continuity of v_i on $\mathbb{R} \times \mathbb{R}^+$. □

3.1 Viscosity Characterization of the value function

This section is dedicated to the proof of the existence and uniqueness of a solution of the HJB system. As underlined before, we turn to the viscosity characterization approach in order to overcome the impossibility to use a verification approach since a direct proof of the smooth-fit property is out of reach in a multidimensional case.

In order to prove that the value function v is the unique solution of the HJB system, we shall assume that the following dynamic programming principle holds: for any $(i, x, y) \in \{0, \ldots, m\} \times \mathbb{R} \times (0, \infty)$, for all $\nu \in \mathcal{T}$, we have

$$(\mathbf{DP}) \quad v(i, x, y) = \sup_{\theta\in\mathcal{T}} \mathbb{E}^{i,x,y} \left[h(X_\theta, Y_\theta)\mathbb{1}_{\theta\leq\tau\wedge\nu} + U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\nu} + v(L_\nu, X_\nu, Y_\nu)\mathbb{1}_{\nu<\theta}\mathbb{1}_{\nu\leq\tau}\right].$$

We then have the PDE characterization of the value functions.

Theorem 3.1 *The value functions v_i, $i \in \{0, \ldots, m\}$, are continuous on $\mathbb{R} \times \mathbb{R}^+$, and constitute the unique viscosity solution on $\mathbb{R} \times \mathbb{R}^+$ with growth condition*

$$|v_i(x, y)| \leq |U(e^x)| + |U(e^x) f(y)|,$$

and boundary condition

$$\lim_{y\downarrow 0} v_i(x, y) = U(e^x),$$

to the system of variational inequalities :

$$\begin{aligned} \min\Big[-\mathcal{L}v_i(x, y) - \mathcal{G}_i v_\cdot(x, y) - \mathcal{J}_i v_i(x, y)\,,\ v_i(x, y) - U(e^x f(y))\Big] = 0, \\ \forall\,(x, y) \in \times\mathbb{R} \times \mathbb{R}^+_*, \text{ and } i \in \{0, \ldots, n\}. \end{aligned} \tag{3.20}$$

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.2 *The value functions $(v_i)_{0\leq i\leq m}$ constitute a subsolution to the system of variational inequalities (3.20).*

Proof of lemma 3.2: We prove the subsolution property by contradiction. Suppose that the claim is not true. Then there exists $(\bar{i},\bar{x},\bar{y}) \in \{0,1,\dots,m\}\times\mathbb{R}\times\mathbb{R}^+$, a neighborhood of $B_{(\bar{x},\bar{y})}(\delta) := \{(x,y)\in\mathbb{R}\times\mathbb{R}^+; |x-\bar{x}|\leq\delta; |y-\bar{y}|\leq\delta\}$ where $\delta>0$, C^2 functions φ_i $(0\leq i\leq m)$ with $(\varphi_{\bar{i}}-v_{\bar{i}})(\bar{x},\bar{y})=0$, and $\varphi_i\geq v_i$ on $B_{(\bar{x},\bar{y})}(\delta)$ $(i\in\{0,1,\dots,m\})$, and $\eta>0$, such that for all $(x,y)\in\bar{B}_{(\bar{x},\bar{y})}(\delta)$, we have

$$\begin{aligned} -\mathcal{L}\varphi_{\bar{i}}(x,y)-\mathcal{G}_{\bar{i}}\varphi_{.}(x,y)-\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(x,y) &> \eta, & (3.21)\\ \varphi_{\bar{i}}(x,y)-U(e^{x}f(y)) &> \eta. & (3.22)\end{aligned}$$

We consider the exit time

$$\tau_\delta = \inf\{t\geq 0\ ;\ (L_t^{\bar{i}}, X_t^{\bar{x}}, Y_t^{\bar{y}})\notin\{\bar{i}\}\times B_{(\bar{x},\bar{y})}(\delta)\}.$$

Let $\theta\in\mathcal{T}$, and apply Itô's Formula to φ between 0 and $\gamma_\delta := \tau_\delta\wedge\theta\wedge\tau$. Taking an expectation, we obtain

$$\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})\right] = \varphi_{\bar{i}}(\bar{x},\bar{y})+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\int_0^{\gamma_\delta}\left(\mathcal{L}\varphi_{\bar{i}}(X_t,Y_t)+\mathcal{G}_{\bar{i}}\varphi_{.}(X_t,Y_t)\right)dt\right].$$

From relation (3.21), the above inequality becomes

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})+\int_0^{\gamma_\delta}\left(\eta+\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(X_t,Y_t)\right)dt\right]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta},Y_{\gamma_\delta})+\left(U(e^{X_\tau})-\varphi_{\bar{i}}(X_\tau,Y_\tau)\right)\mathbb{1}_{\tau<\theta\wedge\tau_\delta}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\end{aligned}$$

since $\left(U(e^{X_\tau})-\varphi_{\bar{i}}(X_\tau,Y_\tau)\right)\mathbb{1}_{\tau\leq t}-\int_0^t\mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(X_s,Y_s)ds$ is a martingale on $[0,\gamma_\delta]$. Hence,

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\varphi_{L_{\gamma_\delta}}(X_{\theta\wedge\tau_\delta},Y_{\theta\wedge\tau_\delta})\mathbb{1}_{\tau\geq\theta\wedge\tau_\delta}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\varphi_{\bar{i}}(X_\theta,Y_\theta)\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}+\varphi_{L_{\gamma_\delta}}(X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]\\ &\quad+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta].\end{aligned}$$

Using (3.22) and the fact that $\varphi_i\geq v_i$ on $B_{(\bar{x},\bar{y})}(\delta)$ for all $i\leq m$, we obtain for any $\theta\in\mathcal{T}$

$$\begin{aligned}\varphi_{\bar{i}}(\bar{x},\bar{y}) &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+\left(U(e^{X_\theta}f(Y_\theta))+\eta\right)\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}\right]\\ &\quad+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[v(L_{\tau_\delta},X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[\gamma_\delta]\\ &\geq \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta}+U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\leq\tau\wedge\tau_\delta}\right]\\ &\quad+\mathbb{E}^{\bar{i},\bar{x},\bar{y}}\left[v(L_{\tau_\delta},X_{\tau_\delta},Y_{\tau_\delta})\mathbb{1}_{\tau_\delta<\theta}\mathbb{1}_{\tau_\delta\leq\tau}\right]+\eta\mathbb{E}^{\bar{i},\bar{x},\bar{y}}[1\wedge\tau_\delta\wedge\tau].\end{aligned}$$

Using the Dynamic Programming Principle, we obtain

$$\varphi_{\bar{i}}(\bar{x},\bar{y}) \geq v_{\bar{i}}(\bar{x},\bar{y})+\eta\mathbb{E}[1\wedge\tau_\delta\wedge\tau].$$

Noticing that $\eta\mathbb{E}[1\wedge\tau_\delta\wedge\tau]>0$, we obtain the contradiction and therefore leading us to the subsolution property.

Lemma 3.3 *The value functions v_i, $i \in \{0,1,\dots,m\}$ constitute a supersolution to the system of variational inequalities (3.20).*

Proof of lemma 3.3: We consider the C^2 test functions φ_i ($i \in \{0,1,\dots,m\}$), such that $v_{\bar{i}}(\bar{x},\bar{y}) = \varphi_{\bar{i}}(\bar{x},\bar{y})$ and $\varphi_i \le v_i$ ($i \in \{0,1,\dots,m\}$). We can also assume w.l.o.g. that $\varphi_i(x,y) < v_i(x,y)$ on $\{0,1,\dots,m\} \times \mathbb{R} \times \mathbb{R}^+ \setminus (\bar{i},\bar{x},\bar{y})$. We have to prove that

$$\min\Big[-\mathcal{L}\varphi_{\bar{i}}(\bar{x},\bar{y}) - \mathcal{G}_{\bar{i}}\varphi_.(\bar{x},\bar{y}) - \mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(\bar{x},\bar{y})\,,\, \varphi_{\bar{i}}(\bar{x},\bar{y}) - U(e^{\bar{x}}f(\bar{y}))\Big] \ge 0.$$

We first note that $\varphi_{\bar{i}}(\bar{x},\bar{y}) = v_{\bar{i}}(\bar{x},\bar{y}) \ge U(e^{\bar{x}}f(\bar{y}))$, so we just have to show that

$$-\mathcal{L}\varphi_{\bar{i}}(\bar{x},\bar{y}) - \mathcal{G}_{\bar{i}}\varphi_.(\bar{x},\bar{y}) - \mathcal{J}_{\bar{i}}\varphi_{\bar{i}}(\bar{x},\bar{y}) \ge 0.$$

For the state variables starting initially from $(\bar{i},\bar{x},\bar{y})$ and a stopping time $\theta \in \mathcal{T}$, we consider the exit time

$$\tau_\delta = \inf\{t \ge 0\;;\; (L_t^{\bar{i}}, X_t^{\bar{x}}, Y_t^{\bar{y}}) \notin \{\bar{i}\} \times B_{(\bar{x},\bar{y})}(\delta)\},$$

where, as before, $B_{(\bar{x},\bar{y})}(\delta) := \{(x,y) \in \mathbb{R}\times\mathbb{R}^+; |x-\bar{x}| \le \delta; |y-\bar{y}| \le \delta\}$.
Using the dynamic programming principle for v applied to the stopping time $\tau_\delta \wedge t$, with $t > 0$, we find

$$\begin{aligned}
\varphi_{\bar{i}}(\bar{x},\bar{y}) &= v_{\bar{i}}(\bar{x},\bar{y}) \\
&\ge \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\Big[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta\wedge t} + U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\le\tau\wedge\tau_\delta\wedge t} \\
&\quad + v(L_{\tau_\delta\wedge t}, X_{\tau_\delta\wedge t}, Y_{\tau_\delta\wedge t})\mathbb{1}_{\tau_\delta\wedge t<\theta}\mathbb{1}_{\tau_\delta\wedge t\le\tau}\Big] \\
&\ge \mathbb{E}^{\bar{i},\bar{x},\bar{y}}\Big[U(e^{X_\tau})\mathbb{1}_{\tau<\theta\wedge\tau_\delta\wedge t} + U(e^{X_\theta}f(Y_\theta))\mathbb{1}_{\theta\le\tau\wedge\tau_\delta\wedge t} \\
&\quad + \varphi_{L_{\tau_\delta\wedge t}}(X_{\tau_\delta\wedge t}, Y_{\tau_\delta\wedge t})\mathbb{1}_{\tau_\delta\wedge t<\theta}\mathbb{1}_{\tau_\delta\wedge t\le\tau}\Big],
\end{aligned} \tag{3.23}$$

for any $\theta \in \mathcal{T}$.

Now applying Itô's formula to φ between 0 and $\gamma_\delta := \tau_\delta \wedge \tau \wedge t$, we obtain by taking an expectation

$$\mathbb{E}\Big[\varphi_{L_{\gamma_\delta}}(X_{\gamma_\delta}, Y_{\gamma_\delta})\Big] = \varphi_{\bar{i}}(\bar{x},\bar{y}) + \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_.)(X_t,Y_t)dt\right],$$

and, with inequality (3.23) with $\theta > \tau_\delta \wedge t$, we obtain

$$\begin{aligned}
0 &\ge \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_.)(X_t,Y_t)dt\right] + \mathbb{E}\left[\Big(U(e^{X_\tau}) - \varphi_{\bar{i}}(X_\tau,Y_\tau)\Big)\mathbb{1}_{\tau<\tau_\delta\wedge t}\right] \\
&\ge \mathbb{E}\left[\int_0^{\gamma_\delta} (\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_. + \mathcal{J}_{\bar{i}}\varphi_{\bar{i}})(X_t,Y_t)dt\right].
\end{aligned}$$

From the definition of τ_δ, we readily see that the integrand part of (3.24) is bounded. Dividing the previous inequality by t and taking t to 0, we may apply the dominated convergence theorem and obtain

$$-(\mathcal{L}\varphi_{\bar{i}} + \mathcal{G}_{\bar{i}}\varphi_. + \mathcal{J}_{\bar{i}}\varphi_{\bar{i}})(\bar{x},\bar{y}) \ge 0,$$

leading us to the supersolution property.

□

The following lemma is the key to show the uniqueness of the solution.

Lemma 3.4 *Let $(w_i)_{0\leq i\leq m}$ be a continuous viscosity supersolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, and consider the following C^2 function:*

$$g(x,y) = \begin{cases} ax^4 + by^n + k + U(1)\theta(0) + A_1 x + \frac{1}{2}A_2 x^2 & x \leq 0 \\ ax^4 + by^n + k + U(e^x)\theta(x) & x > 0. \end{cases} \tag{3.24}$$

with $\theta(x) = ln(4+x)$, $A_1 = U'(1)\theta(0) + U(1)\theta'(0)$, $A_2 = U''(1)\theta(0) + 2U'(1)\theta'(0) + U(1)\theta''(0)$, a, b and k strictly positive constants and $n \geq c$, with c as defined in (2.3)*.*

Let $w_i^\gamma := (1-\gamma)w_i + \gamma g$, $0 \leq i \leq m$. Then, $(w_i^\gamma)_{0\leq i\leq m}$ is strict supersolution to the HJB system, i.e., there exists some $\delta > 0$ such that $(w_i^\gamma)_{0\leq i\leq m}$ is a supersolution of

$$\min\Big[-\mathcal{L}w_i^\gamma(x,y) - \mathcal{G}_i w_\cdot^\gamma(x,y) - \mathcal{J}_i w_i^\gamma(x,y)\,,\, w_i^\gamma(x,y) - U(e^x f(y))\Big] \geq \delta, \tag{3.25}$$

$(i,x,y) \in \{0,\dots,m\}\times\mathbb{R}\times\mathbb{R}^+$.

Proof The proof of this lemma is quite straightforward and is therefore omitted. □

Remark 3.2 We notice that the function g dominates the upper bound $U(e^x)$ and the lower bound $U(f(y)e^x)$ of the value functions when $|x|$ and y go to ∞, i.e.,

$$\lim_{|x|,y\to\infty} \frac{|U(f(y)e^x)| + |U(e^x)|}{g(x,y)} = 0.$$

Indeed, g has been precisely constructed to satisfy the above property as well as the strict supersolution property defined in Lemma 3.4.

We can now show a comparison theorem from which the uniqueness of (3.20) follows.

Lemma 3.5 *Let $(u_i)_{0\leq i\leq m}$ a continuous viscosity subsolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, and $(w_i)_{0\leq i\leq m}$ a continuous viscosity supersolution to the system of variational inequalities (3.20) on $\mathbb{R}\times\mathbb{R}^+$, satisfying the boundary conditions $\lim_{y\downarrow 0} u_i(x,y) \leq \lim_{y\downarrow 0} w_i(x,y)$, $i \in \{0,\dots,m\}, x\in\mathbb{R}$, and the following growth condition*

$$|u_i(x,y)| + |w_i(x,y)| \leq |U(e^x)| + |U(e^x f(y)|.$$

Then,

$$u_i(x,y) \leq w_i(x,y), \text{ on } \mathbb{R}\times\mathbb{R}^+,\ i\in\{0,\dots,m\}.$$

The proof of Lemma 3.5 is postponed in Appendix A.

3.2 Liquidation and continuation regions

We now prove useful qualitative properties of the liquidation regions of the optimal stopping problem. We introduce the following liquidation and continuation regions:

$$\begin{aligned}\mathcal{LR} &= \left\{(i,x,y) \in \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+ \,|\, v(i,x,y) = h(x,y)\right\} \\ \mathcal{CR} &= \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+ \setminus \mathcal{LR}.\end{aligned}$$

Clearly, outside the liquidation region $\mathcal{LR}$, it is never optimal to liquidate the assets at the available discounted value. Moreover, the smallest optimal stopping time θ^*_{ixy} verifies

$$\theta^*_{ixy} = \inf\left\{u \geq 0 \,|\, \left(L^i_u, X^x_u, Y^y_u\right) \in \mathcal{LR}\right\}.$$

We define the $(i,x)-$sections for every $(i,x) \in \{0,...,m\} \times \mathbb{R}$ by

$$\mathcal{LR}_{(i,x)} = \{y \geq 0 \,|\, v(i,x,y) = h(x,y)\} \text{ and } \mathcal{CR}_{(i,x)} = \mathbb{R}^+ \setminus \mathcal{LR}_{(i,x)}.$$

Proposition 3.3 (Properties of liquidation region)

i) *$\mathcal{LR}$ is closed in $\{0,...,m\} \times \mathbb{R} \times (0,+\infty)$,*

ii) *Let $(i,x) \in \{0,...,m\} \times \mathbb{R}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] = U(e^x)$, then, for all $y \in \mathbb{R}^+$, $v(i,x,y) = U(e^x)$ and $\mathcal{LR}_{(i,x)} = \{0\}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})] < U(e^x)$, then there exists $x_0 \in \mathbb{R}$ such that $\mathcal{LR}_{(i,x_0)} \setminus \{0\} \neq \emptyset$ and $\bar{y}^*(i,x) := \sup \mathcal{LR}_{(i,x)} < +\infty$.*

Proof: The proof follows some ideas presented in [22].

i) For all $i \in \{0,...,m\}$, v_i, U and f are continuous, then $\mathcal{LR} = \bigcup_{i=0}^m \left[v_i - h\right]^{-1}(0)$ is a closed set.

ii) Let $(i,x) \in \{0,...,m\} \times \mathbb{R}$. If $\mathbb{E}^{i,x}[U(e^{X_\tau})] = U(e^x)$, it follows from Proposition 3.1 that $v_i(x,y) = U(e^x)$, for all $y \geq 0$. As $f < 1$ on $(0,+\infty)$ and U is increasing, we obviously have $\mathcal{LR}_{(i,x)} = \{0\}$.

Now, we assume that $\mathbb{E}^{i,x}[U(e^{X^x_\tau})] < U(e^x)$ and that for all $x_0 \in \mathbb{R}$, $\mathcal{LR}_{(i,x_0)} \setminus \{0\} = \emptyset$. Let $y \in (0,+\infty)$. We find

$$\begin{aligned}T_t(i,x,y) &:= \mathbb{E}^{i,x,y}\left[h\left(X_t, Y_t\right) \mathbb{1}_{t \leq \tau \wedge \xi_y} + U(e^{X_\tau})\, \mathbb{1}_{t > \tau \wedge \xi_y}\right] \\ &\geq \mathbb{E}^{i,x,y}\left[h(X_t, Y_t)\, \mathbb{1}_{t \leq \tau \wedge \xi_y} + v(L_\tau, X_\tau, Y_\tau)\, \mathbb{1}_{t > \tau \wedge \xi_y}\right].\end{aligned}$$

Therefore, letting t going to $+\infty$, we have

$$\liminf_{t \to +\infty} T_t(i,x,y) \geq \mathbb{E}^{i,x,y}\left[v(L_{\tau \wedge \xi_y}, X_{\tau \wedge \xi_y}, Y_{\tau \wedge \xi_y})\right] = v(i,x,y).$$

The last equality comes from the fact that τ is almost surely finite and that the process $(v(L_t, X_t, Y_t))_{0\leq t}$ is a martingale up to time $\xi_y = \inf\{t \geq 0 : Y_t^y = 0\}$. Indeed, we have $\xi_y = \theta^*_{ixy}$ since we have assumed that for all $x_0 \in \mathbb{R}$, $\mathcal{LR}_{(i,x_0)} \setminus \{0\} = \emptyset$. On the other hand, from the above assumption, we derive the following relation:

$$\limsup_{t\to+\infty} T_t(i,x,y) = \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] < U(e^x).$$

Therefore we have proved that, for all $y > 0$,

$$h(x,y) < v(i,x,y) \leq \liminf_{t\to+\infty} T_t(i,x,y) \leq \limsup_{t\to+\infty} T_t(i,x,y) \leq \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right].$$

Since $f(0) = 1$, by taking y going to 0, we obtain the following contradiction $U(e^x) \leq \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] < U(e^x)$.

Finally, we recall that U is increasing and $\lim_{y\to+\infty} f(y) = 0$. Therefore, we have

$$\lim_{y\to+\infty} v(i,x,y) \geq \mathbb{E}^{i,x}\left[U(e^{X_\tau})\right] > U(0) = \lim_{y\to+\infty} h(x,y).$$

It obviously follows that $\bar{y}^*(i,x) := \sup \mathcal{LR}_{(i,x)} < +\infty$.

□

4 Logarithmic utility

Throughout this section, we assume that the diffusion processes X and Y are governed by the following SDE, which are particular cases of (2.1) and (2.4)

$$\begin{aligned} dX_t &= \mu dt + \sigma(X_t) dB_t;\ X_0 = x \\ dY_t &= \kappa\left(\beta - Y_t\right) dt + \gamma\sqrt{Y_t} dW_t;\ Y_0 = y \end{aligned} \tag{D-1}$$

where μ, κ, β and γ are constant. We first notice that the supermeanvalued assumption combined with the logarithmic utility function implies that $\mu \leq 0$. Moreover, if $\mu = 0$, we have seen that $v(i,x,y) = U(e^x)$ and $\mathcal{LR}_{(i,x)} = \{0\}$ (see Proposition 3.3), so we shall assume throughout this section that $\mu < 0$.

The following theorem shows that in the logarithmic case, we can reduce the dimension of the problem by factoring out the x-variable. For this purpose, we define $\mathcal{T}_{L,W}$ the set of stopping times with respect to the filtration generated by (L, W), and the differential operator $\overline{\mathcal{L}}\phi(y) := \frac{1}{2}\gamma^2 y \frac{\partial^2\phi}{\partial y^2} + \kappa(\beta - y)\frac{\partial\phi}{\partial y} + \mu$, for $\phi \in C^2(\mathbb{R}^+)$.

Theorem 4.2 *For $(i,y) \in \{1,...,m\} \times \mathbb{R}^+$ we define the function:*

$$w(i,y) = \sup_{\theta\in\mathcal{T}_{L,W}} \mathbb{E}^{i,y}[\mu(\theta \wedge \tau) + \ln\left(f(Y_\theta)\right) 1\!\!1_{\{\theta\leq\tau\}}].$$

Then,

$$v(i,x,y) = x + w(i,y) \text{ on } \{0,...,m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with w the unique viscosity solution to the system of equations:

$$\min\Big[-\overline{\mathcal{L}}w(i,y)+\lambda_i w(i,y)-\sum_{j\neq i}\vartheta_{i,j}\left(w(j,\,y)-w(i,\,y)\right)\,,\,w(i,y)-g(y)\Big]=0, \qquad (4.26)$$

where $g(y) := \ln(f(y))$ Moreover, the functions $w(i,.)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{CR}_{(i,x)} \cup \mathrm{Int}(\mathcal{LR}_{(i,x)})$.

Proof: We first notice that

$$v(i,x,y)=\sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,x,y}[X_{\theta\wedge\tau}+\ln\left(f(Y_\theta)\right)\mathbb{1}_{\{\theta\leq\tau\}}] \text{ on } \{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+.$$

Moreover, for $(i,x,x',y) \in \{0,...,m\} \times \mathbb{R}^2 \times \mathbb{R}^+$, we have

$$v(i,x',y)-v(i,x,y)\leq\sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,x',y}[X_{\theta\wedge\tau}|L_0=i]-\mathbb{E}^{i,x,y}[X_{\theta\wedge\tau}|L_0=i]=x'-x.$$

On the other hand, we have

$$v(i,x',y)-v(i,x,y)\geq\mathbb{E}^{i,x',y}[X_{\theta^*_{ixy}\wedge\tau}|L_0=i]-\mathbb{E}^{i,x,y}[X_{\theta^*_{ixy}\wedge\tau}|L_0=i]=x'-x.$$

It follows that there exists a function w defined on $\{0,...,m\} \times \mathbb{R}^+$ such that $v(i,x,y) = x + w(i,y)$. Then, $\theta^*_{ixy} := \inf\left\{u \geq 0 \,|\, \left(L^i_u, X^x_u, Y^y_u\right) \in \mathcal{LR}\right\} = \inf\{t \geq 0 : \ w(L^i_t, Y^y_t) = \ln(f(Y^y_t))\} := \theta^*_{iy}$, belongs to $\mathcal{T}_{L,W}$. Hence, we have

$$w(i,y)=\sup_{\theta\in\mathcal{T}_{L,W}}\mathbb{E}^{i,y}[\mu(\theta\wedge\tau)+\ln\left(f(Y_\theta)\right)\mathbb{1}_{\{\theta\leq\tau\}}].$$

We deduce from Theorem 3.1 that $(w_i)_{0\leq i\leq m}$ is the unique continuous viscosity solution of the system of equation (4.26). We conclude the proof by asserting that the fact that $w(i,\cdot)$ is of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{CR}_{(i,x)} \cup \mathrm{Int}(\mathcal{LR}_{(i,x)})$ for all $i \in \{1,...,m\}$ can be established by following the proof of Proposition 3.3 in [17]. □

Remark 4.3 From Theorem 4.2, we notice that the (i,x)-sections of the liquidation region $\mathcal{LR}_{(i,x)}$ do not depend on x. For convenience we denote them by $\mathcal{LR}_{(i,.)}$ in this section. In the same way, we write $\mathcal{CR}_{(i,.)} := \mathcal{CR}_{(i,x)}$.

4.1 Liquidation region

In the logarithmic case, the liquidation region can be characterized in more details.

Proposition 4.4 *Let $i \in \{0,...,m\}$ and set*

$$\hat{y}_i=\inf\{y\geq0:\ \mathcal{H}_i g(y)\geq0\} \text{ with } \mathcal{H}_i g(y)=\overline{\mathcal{L}}g(y)-\lambda_i g(y)+\sum_{j\neq i}\vartheta_{i,j}\left(w(j,\,y)-g(y)\right).$$

There exists $y^*_i \geq 0$ such that $[0, y^*_i] = \mathcal{LR}_{(i,.)} \cap [0, \hat{y}_i]$. Moreover, $w(i,\cdot) - g(\cdot)$ is non-decreasing on $[y^*_i, \hat{y}_i]$.

Proof: Let $y_i^* = \inf\{y \geq 0 : \; w(i,y) > g(y)\}$. Notice that

$$\mathcal{H}_i g(0) = \mu + \kappa\beta g'(0) < 0,$$

since $\mu < 0$ and $g'(0) < 0$. Hence, $\hat{y}_i > 0$. As we have $\mathcal{H}_i w(i,y) \leq 0$ on $\mathbb{R}^+$, we have $y_i^* \leq \hat{y}_i$. If $y_i^* = \hat{y}_i$, the result is obvious so we shall assume that $y_i^* < \hat{y}_i$. For all $z \in (y_i^*, \hat{y}_i) \cap \mathcal{CR}_{(i,.)}$, we have

$$\frac{\gamma^2 z}{2}\frac{\partial^2 w}{\partial y^2}(i,z) = -\kappa(\beta - z)\frac{\partial w}{\partial x}(i,z) + \lambda_i w(i,z) - \sum_{j\neq i} \vartheta_{i,j}\left(w(j,\, z) - w(i,\, z)\right) - \mu.$$

Therefore, if we set $d_i = w(i,\cdot) - g(\cdot)$, we find

$$\frac{\gamma^2 z}{2} d_i''(z) \geq -\kappa(\beta - z)d_i'(z) + \left(\lambda_i + \sum_{j\neq i}\vartheta_{i,j}\right) d_i(x) - \sup_{z'\in[y_i^*,z]} \mathcal{H}_i g(z'). \tag{4.27}$$

From the definition of y_i^*, there exists a sequence $(z_n)_{n\geq 0}$ taking values in $(y_i^*, \hat{y}_i) \cap \mathcal{CR}_{(i,.)}$ and such that $\lim_{n\to+\infty} z_n = y_i^*$. It follows from the smooth fit property and from (4.27) that

$$\lim_{z\searrow y_i^*} \frac{\gamma^2 z}{2} d_i''(z) \geq -\mathcal{H}_i g(y_i^*) > 0.$$

It implies that $y_i^* < \xi_i$ where $\xi_i := \inf\{z > y_i^* : \; d_i(z) = 0 \text{ or } d_i'(z) < 0\}$. Assume that $\xi_i < \hat{y}_i$. As d_i is increasing on (y_i^*, ξ_i), we have $d_i(\xi_i) > 0$ and $d_i'(\xi_i) = 0$. However, it leads to a contradiction because (4.27) implies that $d_i''(\xi_i) > 0$. □

When $\overline{\mathcal{L}}g(y)$ is non-decreasing in y, the previous result can be specified further.

Proposition 4.5 *Assume that the function $y \mapsto \overline{\mathcal{L}}g(y)$ is non-decreasing on $\mathbb{R}^+$, then for all $i \in \{0, ..., m\}$, $w(i,\cdot) - g(\cdot)$ is non-decreasing on $\mathbb{R}^+$ and we have $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$, with $y_i^* > 0$.*

Proof: Let $i \in \{0, ..., m\}$ and $0 \leq y < z$. We introduce the following stopping time

$$\theta_{yz} = \inf\{t \geq 0 : \; Y_t^y = Y_t^z\}.$$

As g is non increasing, we have

$$\begin{aligned} w(i,z) - w(i,y) &\geq \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}}) - g(Y^y_{\theta^*_{iy}})\right)\mathbb{1}_{\{\theta^*_{iy} < \tau\}}\Big] \\ &= \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}\wedge\theta_{yz}}) - g(Y^y_{\theta^*_{iy}\wedge\theta_{yz}})\right)\mathbb{1}_{\{\theta^*_{iy} < \tau\}}\Big] \\ &\geq \mathbb{E}\Big[\left(g(Y^z_{\theta^*_{iy}\wedge\theta_{yz}}) - g(Y^y_{\theta^*_{iy}\wedge\theta_{yz}})\right)\Big]. \end{aligned}$$

Applying Itô formula, it follows from the fact that $\overline{\mathcal{L}}g$ is non decreasing that

$$w(i,z) - w(i,y) \geq g(z) - g(y) + \mathbb{E}\Big[\int_0^{\theta^*_{iy}\wedge\theta_{yz}} \overline{\mathcal{L}}g(Y_u^z) - \overline{\mathcal{L}}g(Y_u^y)\, du\Big] \geq g(z) - g(y).$$

□

Remark 4.4 If we set $f(y) = e^{-y}$ on $\mathbb{R}^+$, we have $\overline{\mathcal{L}}g(y) = \kappa(y - \beta)$ so the assumption of Proposition 4.5 is satisfied.

4.2 Logarithmic utility with no-switch (case $\vartheta_{i,j} = 0\ \forall i \neq j$)

Fix $i \leq m$ and assume that $\vartheta_{i,j} = 0\ \forall i \neq j$. In this case, an explicit solution of the HJB system can be obtained.

Proposition 4.6 *If $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$ then y_i^* is the solution of the following equation*

$$\frac{g(y_i^*) - \frac{\mu}{\lambda_i}}{g'(y_i^*)} = -\frac{\gamma^2}{2\lambda_i} \frac{\Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)}{\Psi\left(\frac{\lambda_i}{\kappa} + 1, \frac{2\kappa\beta}{\gamma^2} + 1, \frac{2\kappa}{\gamma^2} y_i^*\right)}, \tag{4.28}$$

and $w(i, \cdot)$ is given by

$$w(i,y) = \begin{cases} g(y) & y \leq y_i^* \\ \dfrac{g(y_i^*) - \frac{\mu}{\lambda_i}}{\Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)} \Psi\left(\dfrac{\lambda_i}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\mu}{\lambda_i} & y > y_i^*, \end{cases} \tag{4.29}$$

where Ψ denotes the confluent hypergeometric function of second kind (see Appendix B).

Proof: We start analyzing the following differential equation:

$$-\frac{1}{2}\gamma^2 y \phi''(y) - \kappa(\beta - y)\phi'(y) + \lambda_i \phi(y) - \mu = 0\,. \tag{4.30}$$

The solution is $\eta_i(y) = a_i \Phi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y\right) + b_i \Psi\left(\frac{\lambda_i}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y\right) + \frac{\mu}{\lambda_i}$, where $(a_i, b_i) \in \mathbb{R}^2$ and Φ and Ψ are respectively the confluent hypergeometric functions of the first and second kind, see Appendix B. As $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$, there exists $a_i, b_i \in \mathbb{R}$ such that

$$w(i,y) = \begin{cases} g(y) & \text{for} \quad 0 \leq y \leq y_i^* \\ \eta_i(y) & \text{for} \quad y > y_i^*. \end{cases}$$

From Proposition 3.1, $w(i, \cdot)$ is non-increasing on $[0, +\infty)$, as such $\lim_{y \to \infty} w(i, y)$ exists. The coefficient a_i of the confluent hypergeometric function of first kind is then equal to zero, since Φ does not admit a limit, see Appendix B.

From Theorem 4.2, $w(i, \cdot)$ is $C^1([0, +\infty))$. Using the continuity of $w(i, \cdot)$ and $w'(i, \cdot)$ at y_i^*, we obtain two conditions which depend linearly on the parameter b_i, as such, we obtain relation (4.28). For explicit details on the derivatives of the confluent hypergeometric functions, we refer to Appendix B.

□

4.3 Logarithmic utility with switch between two regimes

Now, we assume that there are two regimes (i.e., $m = 1$) and $\vartheta_{0,1}\vartheta_{1,0} \neq 0$. We also assume that, for both $i = 0, 1$, there exists $y_i^* > 0$ such that $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$.
Let Λ be the matrix

$$\Lambda = \begin{pmatrix} \lambda_0 + \vartheta_{0,1} & -\vartheta_{0,1} \\ -\vartheta_{1,0} & \lambda_1 + \vartheta_{1,0} \end{pmatrix}. \tag{4.31}$$

As $\vartheta_{0,1}\vartheta_{1,0} > 0$ it is easy to check that Λ has two eigenvalues $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_1 < \widetilde{\lambda}_0$. Let $\widetilde{\Lambda} = P^{-1}\Lambda P$ be the diagonal matrix with diagonal $(\widetilde{\lambda}_0, \widetilde{\lambda}_1)$. The transition matrix P is denoted by

$$P = \begin{pmatrix} p_0^0 & p_1^0 \\ p_0^1 & p_1^1 \end{pmatrix}. \tag{4.32}$$

Without loss of generality, we shall assume that $p_0^0 + p_1^0 = 1 = p_0^1 + p_1^1$, indeed $(1, -1)$ is not an eigenvector of Λ as $\lambda_0 > \lambda_1$.

Proposition 4.7 *With the above assumptions, we obtain $y_0^* \leq y_1^*$.*

Proof: Assume that $y_1^* < y_0^*$ and set $d(y) := w(0, y) - w(1, y)$ on $\mathbb{R}^+$. We obviously have $d'(y_1^*) \leq 0$ and we set $\hat{y} := \inf\{y > y_1^* : \ d'(y) = 0\}$.
As we have $\lim_{y\to+\infty} w(1, y) = \dfrac{\mu}{\lambda_1} < \dfrac{\mu}{\lambda_0} = \lim_{y\to+\infty} w(0, y) < 0$, we know that $\hat{y} < +\infty$. From Proposition 4.4, we know that the function $y \mapsto w(1, y) - g(y)$ is increasing on $(y_1^*, \hat{y}_1)$. Moreover, for $y \leq y_0^*$, we have

$$\begin{aligned} 0 &\geq \mathcal{H}_0 g(y) \\ &= \mathcal{H}_1 g(y) + \vartheta_{0,1}(w(1, y) - g(y)) - (\lambda_0 - \lambda_1) g(y) \\ &\geq \mathcal{H}_1 g(y). \end{aligned}$$

Therefore, we find $y_0^* < \hat{y}$ and $\hat{y} \in \mathcal{CR}_{(0,\cdot)} \cap \mathcal{CR}_{(1,\cdot)}$. We then obtain

$$\begin{aligned} 0 &= \mathcal{H}_1 w(1, \hat{y}) - \mathcal{H}_0 w(0, \hat{y}) \\ &= \frac{\gamma^2 \hat{y}}{2} d''(\hat{y}) - \lambda_1 w(1, \hat{y}) + \lambda_0 w(0, \hat{y})) - (\vartheta_{0,1} + \vartheta_{1,0}) d(\hat{y}). \end{aligned}$$

Hence, we have

$$\frac{\gamma^2 \hat{y}}{2} d''(\hat{y}) = (\lambda_0 + \vartheta_{0,1} + \vartheta_{1,0}) d(\hat{y}) - (\lambda_0 - \lambda_1) w(1, \hat{y})) < 0,$$

which leads to a contradiction. □

As before, the value function can be written in terms of the confluent hypergeometric functions.

Proposition 4.8 *The function w is given by*

$$w(0, y) = \begin{cases} g(y) & y \in [0, y_0^*] \\[2ex] \widehat{c}\Phi\left(\dfrac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \widehat{d}\Psi\left(\dfrac{\lambda_0 + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) & y \in (y_0^*, y_1^*] \\ \quad + \mathcal{I}\left(\dfrac{2\kappa}{\gamma^2}, \beta, -2\dfrac{\lambda_0 + \vartheta_{0,1}}{\gamma^2}, 2\dfrac{\vartheta_{0,1} g(\cdot) + \mu}{\gamma^2}\right)(y) & \\[2ex] p_0^0\left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} x\right) + \dfrac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^0\left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} x\right) + \dfrac{\mu}{\widetilde{\lambda}_1}\right] & \end{cases} \tag{4.33}$$

$$w(1,y) = \begin{cases} g(y) & y \in [0, y_1^*] \\ p_0^1 \left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] & y \in (y_1^*, \infty) \\ \qquad + p_1^1 \left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2}y\right) + \frac{\mu}{\widetilde{\lambda}_1}\right], \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind, and $\mathcal{I}$ is a particular solution to the non-homogeneous confluent differential equation. Moreover, $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $w(0,y)$ and $w(1,y)$ belong to $C^1(\mathbb{R}^+)$.

Proof: We have

$$-\overline{\mathcal{L}}w(i,y) - \sum_{j \neq i} \vartheta_{i,j}\big(w(j,y) - w(i,y)\big) + \lambda_i w(i,y) - \mu = 0 \qquad \forall i = 0,1 \quad \forall y > y_i^* \,.$$

From Proposition 4.7, we have $y_0^* \leq y_1^*$. We may therefore distinguish two regions:

- $\mathcal{CR}_0 \cap \mathcal{LR}_1$, the region where is optimal to liquidate when we are in the regime 1 and not to liquidate when we are in the regime 0. This region is the interval $(y_0^*, y_1^*]$, which may be empty when $y_1^* = y_0^*$.
- $\mathcal{CR}_0 \cap \mathcal{CR}_1$ which corresponds to $(y_1^*, +\infty)$, where it is never optimal to liquidate regardless of the liquidity state.

We start with an analysis of the region $\mathcal{CR}_0 \cap \mathcal{LR}_1$. For all $y \in \mathcal{CR}_0 \cap \mathcal{LR}_1$ we have

$$\begin{aligned} w(1,y) &= g(y) \\ \overline{\mathcal{L}}w(0,y) &= \vartheta_{0,1}\big(w(0,y) - g(y)\big) + \lambda_0 w(0,y) - \mu \,. \end{aligned}$$

The function $w(0,\cdot)$ is solution of a non-homogeneous ordinary differential equation. The general solution is a linear combination of the two confluent hypergeometric functions and a particular solution in order to verify the non-homogeneous part. This particular solution may be obtained with the usual method of variation of parameters. See Appendix B in this regard. A straightforward computation shows that the expression for $w(1,\cdot)$ as given in (4.33) satisfies the ODE.
We now analyze the region $\mathcal{CR}_0 \cap \mathcal{CR}_1$. For all $y \in \mathcal{CR}_0 \cap \mathcal{CR}_1$ we have

$$\begin{aligned} \overline{\mathcal{L}}w(1,y) &= -\vartheta_{1,0}\big(w(0,y) - w(1,y)\big) + \lambda_1 w(1,y) - \mu \\ \overline{\mathcal{L}}w(0,y) &= -\vartheta_{0,1}\big(w(1,y) - w(0,y)\big) + \lambda_0 w(0,y) - \mu \,. \end{aligned}$$

We recall that the operator $\overline{\mathcal{L}}$ does not depend on the liquidity state i. Then, we consider the two linear combinations $\widehat{w}(0,y) = p_0^0 w(0,y) + p_1^0 w(1,y)$ and $\widehat{w}(1,y) = p_0^1 w(0,y) + p_1^1 w(1,y)$. As such, the pair $(\widehat{w}(0,y), \widehat{w}(1,y))$ satisfies

$$\begin{cases} \overline{\mathcal{L}}\widehat{w}(1,y) &= \widetilde{\lambda}_1 \widehat{w}(1,y) - \mu \\ \overline{\mathcal{L}}\widehat{w}(0,y) &= \widetilde{\lambda}_0 \widehat{w}(0,y) - \mu. \end{cases}$$

The above two ODEs are independent and are of the confluent hypergeometric kind. The general solution is a linear combination of the two confluent hypergeometric functions plus a particular solution, which could be chosen as a constant. Moreover, since the value function is decreasing in y and therefore admits a limit when y goes to infinity, the coefficient of the confluent hypergeometric function of the first kind must be zero since this function does not have a limit when y goes to infinity. We therefore obtain the expressions for $w(0,\cdot)$ and $w(1,\cdot)$ on the interval (y_1^*, ∞) as written in (4.33).

Finally, the two functions $w(0,\cdot)$ and $w(1,\cdot)$ belong to C^1, so that the free parameters $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ may be chosen in order to preserve the continuity and the differentiability of the two functions at points y_0^* and y_1^*. □

Corollary 4.1 *Assume $f(y) = e^{-y}$, we have*

$$\mathcal{I}\left(\frac{2\kappa}{\gamma^2}, \beta, -2\frac{\lambda_0 + \vartheta_{0,1}}{\gamma^2}, 2\frac{\vartheta_{0,1} g(\cdot) + \mu}{\gamma^2}\right)(y) = \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0 + \vartheta_{0,1}} - \frac{\vartheta_{0,1}}{\kappa + \lambda_0 + \vartheta_{0,1}} y. \quad (4.34)$$

The explicit system of equations satisfied by $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ is linear with respect to $(\widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ and is detailed in Appendix B (see (B.49)*).*

4.4 Numerical Simulation

In Figure 1, we represent the value functions in the two-regime case, for the cases $\mu = -.05$ and $\mu = -0.3$. In Figures 2, 3, 4 and 5, we present a sensitivity analysis for the parameters μ, λ, β, and $\vartheta_{0,1}$, respectively. We observe a number of properties that we can expect intuitively:

1. The continuation region is larger (i.e., y_0^* and y_1^* decrease) as μ and/or λ increases (Figures 2 and 3), or β decreases (Figure 4),
2. $y_0^* \to y_1^*$ as $\lambda_0 \to \lambda_1$ (Figure 3),
3. $y_1^* \to y_0^*$ as $\vartheta_{1,0}$ increases (Figure 5), when regime 0 is absorbing (i.e., $\vartheta_{0,1} = 0$).

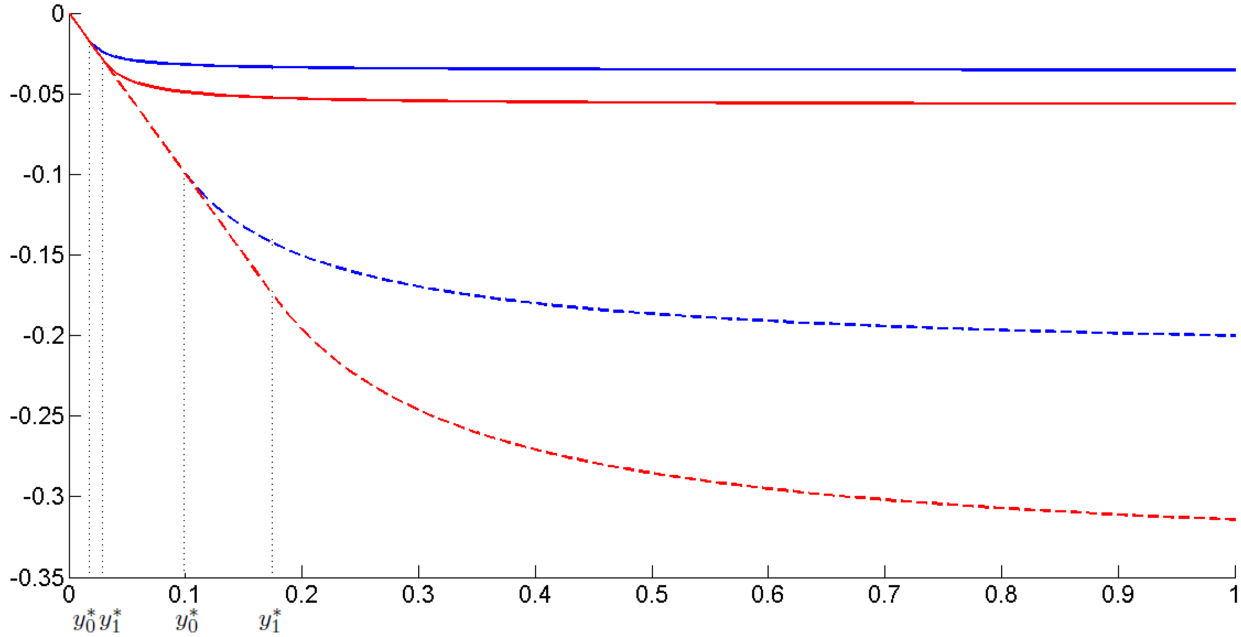

Figure 1: Value functions in the two-regime case, for the cases $\mu = -0.05$ (solid line) and $\mu = -0.3$ (dashed line). Regime 0 is presented in blue and regime 1 in red. The parameters used are $\lambda_0 = 2, \lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 1, \kappa = 1, \beta = 0.25, \gamma = 0.5$. The liquidation region are indicated by dashed lines. In the case $\mu = -0.5$, $y_0^* = 0.0172$ and $y_1^* = 0.0288$. In the case $\mu = -0.3$, $y_0^* = 0.0983$ and $y_1^* = 0.1742$.

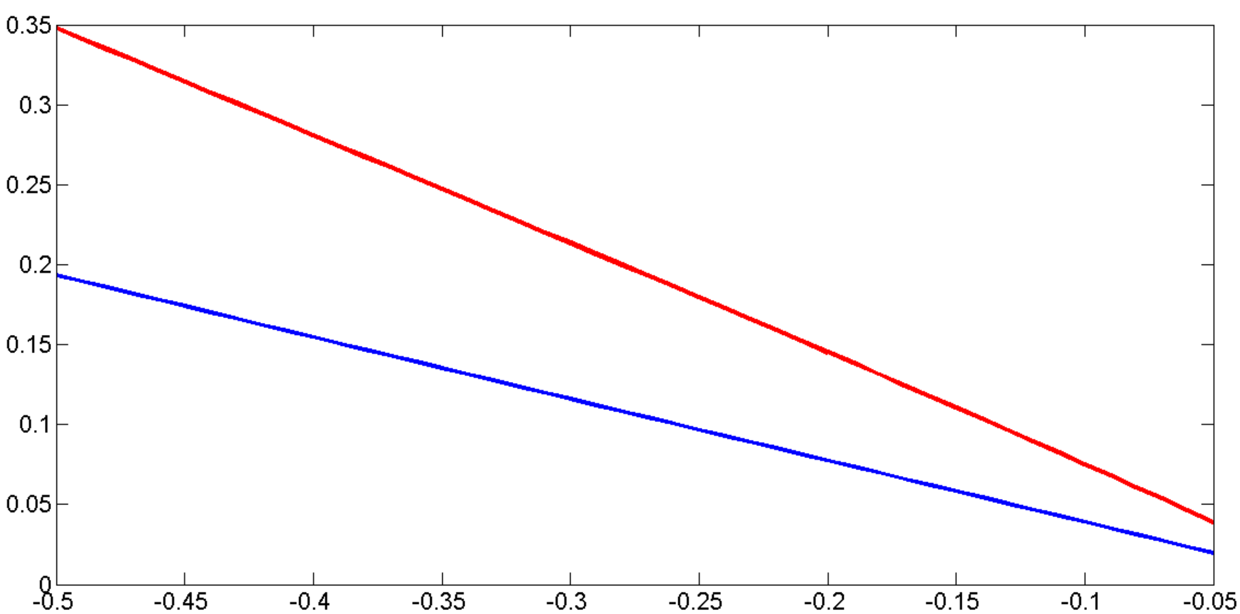

Figure 2: Critical points y_0^* and y_1^*, in terms of μ. y_0^* is presented in blue and y_1^* in red. The parameters used are $\lambda_0 = 2.5, \lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 0.3, \beta = 0.25, \kappa = 1, \gamma = 0.5$.

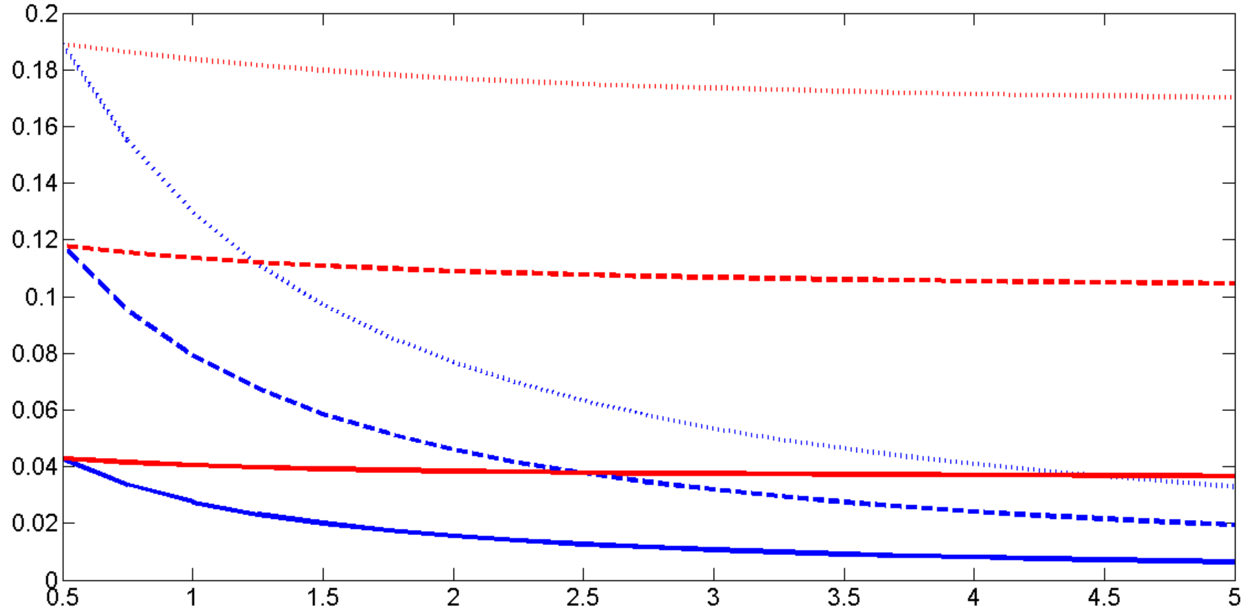

Figure 3: Critical points y_0^* and y_1^*, in terms of λ_0 for the cases $\mu = -0.05$ (solid line), $\mu = -0.15$ (dashed line) and $\mu = -.25$ (dotted line). y_0^* is presented in blue and y_1^* in red. The parameters used are $\lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 0.3, \beta = 0.25, \kappa = 1, \gamma = 0.5$.

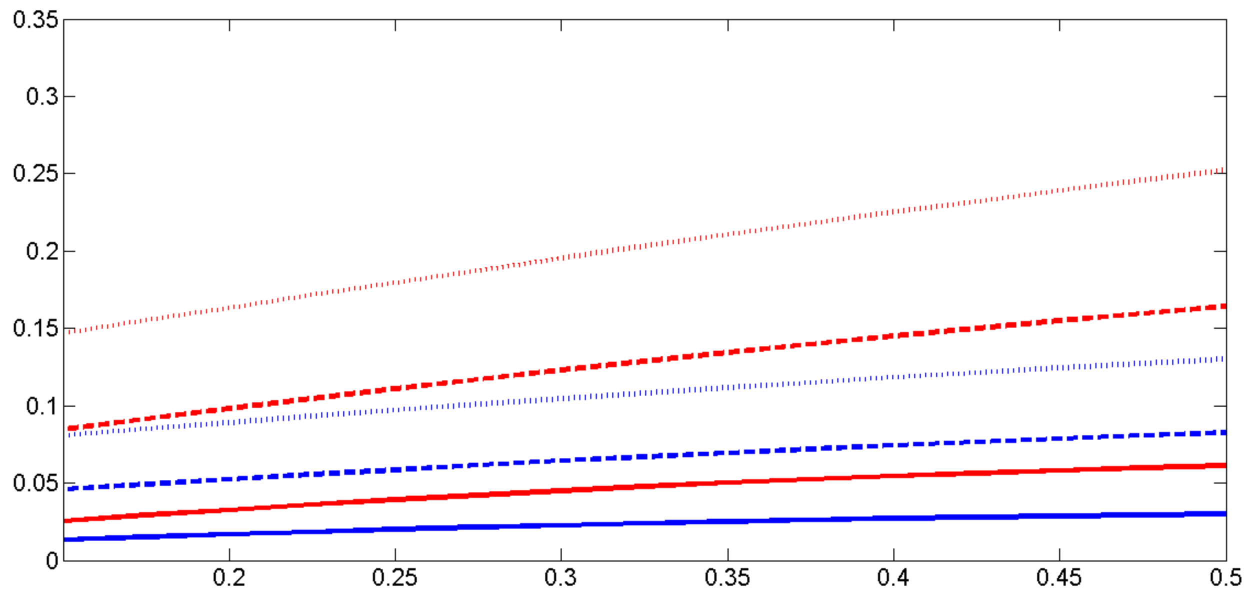

Figure 4: Critical points y_0^* and y_1^*, in terms of β for the cases $\mu = -0.05$ (solid line), $\mu = -0.15$ (dashed line) and $\mu = -.25$ (dotted line). y_0^* is presented in blue and y_1^* in red. The parameters used are $\lambda_0 = 2.5, \lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 0.3, \kappa = 1, \gamma = 0.5$.

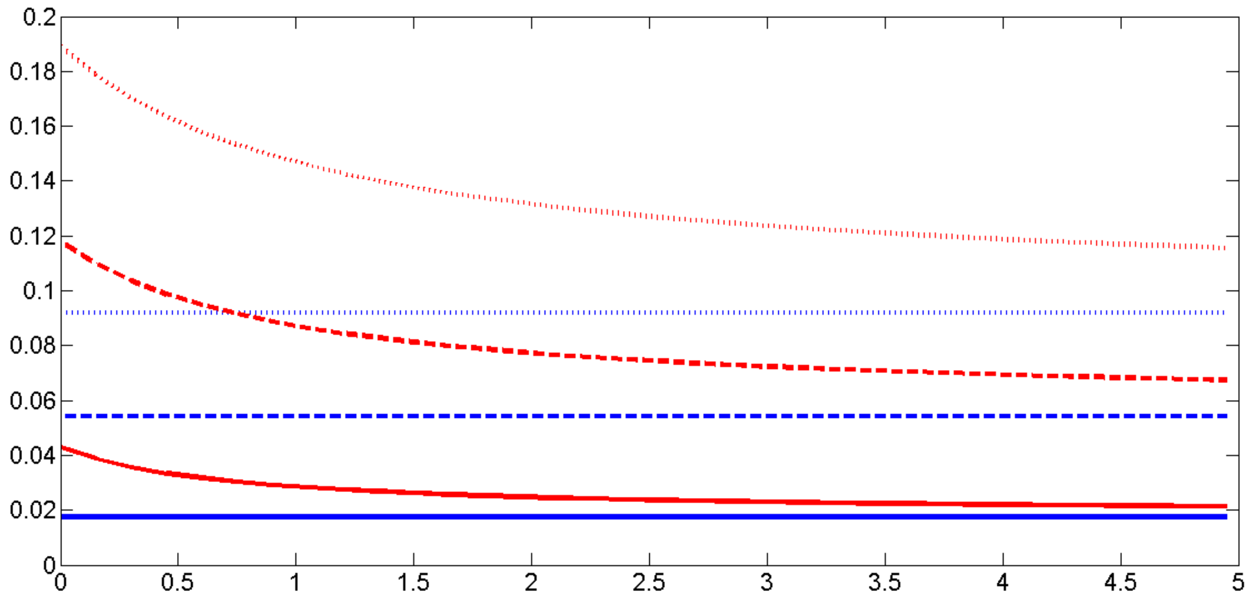

Figure 5: Critical points y_0^* and y_1^*, in terms of $\vartheta_{1,0}$ for the cases $\mu = -0.05$ (solid line), $\mu = -0.15$ (dashed line) and $\mu = -.25$ (dotted line). y_0^* is presented in blue and y_1^* in red. The parameters used are $\lambda_0 = 2.5, \lambda_1 = 0.5, \vartheta_{0,1} = 0, \beta = 0.25, \kappa = 1, \gamma = 0.5$.

5 Power utility

Throughout this section, we assume that $U(s) = s^a$ with $0 < a \leq 1$ and that μ and σ are constant. The diffusion processes X and Y are then governed by the following SDE, which are particular cases of (2.1) and (2.4),

$$\begin{aligned} dX_t &= \mu dt + \sigma dB_t \\ dY_t &= \kappa\left(\beta - Y_t\right) dt + \gamma\sqrt{Y_t} dW_t. \end{aligned} \tag{D-2}$$

We first notice that the supermeanvalued assumption implies that $\mu a + \frac{\sigma^2 a^2}{2} \leq 0$. If $\mu a + \frac{\sigma^2 a^2}{2} = 0$, we have seen that $v(i,x,y) = U(e^x)$ and $\mathcal{LR}_{(i,x)} = \{0\}$ (see Proposition 3.3). We shall then assume throughout this section that $\mu a + \frac{\sigma^2 a^2}{2} < 0$.

Recall that $\mathcal{T}_{L,W}$ is the set of stopping times with respect to the filtration generated by (L, W). In the power utility case, the differential operator $\tilde{\mathcal{L}}$ is given by

$$\tilde{\mathcal{L}}\phi(y) = \frac{1}{2}\gamma^2 y \frac{\partial^2 \phi}{\partial y^2} + \left[\kappa(\beta - y) + \rho\sigma\gamma a\sqrt{y}\right]\frac{\partial \phi}{\partial y} + \left[\frac{\sigma^2 a^2}{2} + \mu a\right]\phi(y).$$

Theorem 5.3 *For $(i, y) \in \{1, ..., m\} \times \mathbb{R}^+$, define*

$$u(i,y) = \sup_{\theta \in \mathcal{T}_{L,W}} \mathbb{E}^{i,y}\left[e^{(\mu a + (1-\rho^2)\frac{\sigma^2 a^2}{2})(\theta\wedge\tau) + \rho\sigma a W_{\theta\wedge\tau}}\left(1\!\!1_{\{\theta>\tau\}} + g(Y_\theta)1\!\!1_{\{\theta\leq\tau\}}\right)\right].$$

Then,

$$v(i,x,y) = e^{ax}u(i,y) \text{ on } \{1,...,m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with u the unique viscosity solution of the system of equations:

$$\min\Big[-\tilde{\mathcal{L}}u(i,y)-\lambda_i(1-u(i,y))-\sum_{j\neq i}\vartheta_{i,j}\left(u(j,\,y)-u(i,\,y)\right)\,,\,u(i,y)-g(y)\Big]=0 \quad (5.35)$$

where $g(y):=(f(y))^a$. Moreover, the functions $u(i,.)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{CR}_{(i,x)}\cup\mathrm{Int}(\mathcal{LR}_{(i,x)})$.

Proof: We first notice that

$$v(i,x,y)=\sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,x,y}[e^{aX_{\theta\wedge\tau}}\left\{[(f(Y_\theta))^a-1]\,\mathbb{1}_{\{\theta\leq\tau\}}+1\right\}] \text{ on } \{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+.$$

Moreover, for $(i,x,x',y)\in\{0,...,m\}\times\mathbb{R}^2\times\mathbb{R}^+$, we have

$$e^{-ax'}v(i,x',y)-e^{-ax}v(i,x,y)=0.$$

Indeed, if we set $\hat{B}=\frac{1}{\sqrt{1-\rho^2}}(B-\rho W)$, we have

$$\begin{aligned}
e^{-ax}v(i,x,y) &= \sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,y}[e^{\mu a(\theta\wedge\tau)+\sigma aB_{\theta\wedge\tau}}\left(\mathbb{1}_{\{\theta>\tau\}}+g(Y_\theta)\mathbb{1}_{\{\theta\leq\tau\}}\right)]\\
&= \sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,y}[e^{\mu a(\theta\wedge\tau)+\rho\sigma aW_{\theta\wedge\tau}+\sqrt{1-\rho^2}\sigma a\hat{B}_{\theta\wedge\tau}}\left(\mathbb{1}_{\{\theta>\tau\}}+g(Y_\theta)\mathbb{1}_{\{\theta\leq\tau\}}\right)]\\
&= \sup_{\theta\in\mathcal{T}}\mathbb{E}^{i,y}[e^{(\mu a+(1-\rho^2)\frac{\sigma^2a^2}{2})(\theta\wedge\tau)+\rho\sigma aW_{\theta\wedge\tau}}\left(\mathbb{1}_{\{\theta>\tau\}}+g(Y_\theta)\mathbb{1}_{\{\theta\leq\tau\}}\right)].
\end{aligned}$$

It follows that there exists a function u defined on $\{0,...,m\}\times\mathbb{R}^+$ such that $v(i,x,y)=e^{ax}u(i,y)$ and $\theta^*_{ixy}=\inf\{t\geq 0:\ u(L^i_t,Y^y_t)=(f(Y^y_t))^a\}:=\theta^*_{iy}$, belongs to the set of stopping times with respect to the filtration generated by (L,X), denoted by $\mathcal{T}_{L,W}$. Hence, we have $v(i,x,y)=e^{ax}u(i,y)$ where

$$u(i,y)=\sup_{\theta\in\mathcal{T}_{L,W}}\mathbb{E}^{i,y}[e^{(\mu a+(1-\rho^2)\frac{\sigma^2a^2}{2})(\theta\wedge\tau)+\rho\sigma aW_{\theta\wedge\tau}}\left(\mathbb{1}_{\{\theta>\tau\}}+g(Y_\theta)\mathbb{1}_{\{\theta\leq\tau\}}\right)].$$

We deduce from Theorem 3.1 that $(u(i,\cdot))_{0\leq i\leq m}$ are the unique continuous viscosity solutions of the system of equations (5.35). We conclude the proof by asserting that $u(i,\cdot)$ is of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{CR}_{(i,x)}\cup\mathrm{Int}(\mathcal{LR}_{(i,x)})$ for all $i\in\{1,...,m\}$. It can be established by following the proof of Proposition 3.3 in [17]. □

In Proposition 5.9, 5.10, and 5.11, we give results on the liquidation region and the explicit solution of the HJB equation that are similar to those presented in the previous section. We shall omit the proofs as they may be obtained using the same arguments. We begin by the next proposition which summarizes some criteria implying that the liquidation region is an interval.

Proposition 5.9 *(Liquidation region)*
Let $i\in\{0,...,m\}$ and set $\hat{y}_i=\inf\{y\geq 0:\ \mathcal{H}_ig(y)\geq 0\}$ with $\mathcal{H}_ig(y)=\tilde{\mathcal{L}}g(y)+\lambda_i(1-g(y))+\sum_{j\neq i}\vartheta_{i,j}\left(u(j,\,y)-g(y)\right)$.
There exists $y^*_i\geq 0$ such that for all $x\in\mathbb{R}$, $[0,y^*_i]=\mathcal{LR}_{(i,x)}\cap[0,\hat{y}_i]$. Moreover, $u(i,\cdot)-g(\cdot)$ is non-decreasing on $[y^*_i,\hat{y}_i]$.
Assume that the function $y\to\tilde{\mathcal{L}}g(y)$ is non decreasing on $\mathbb{R}^+$, then for all $i\in\{0,...,m\}$, $u(i,\cdot)-g(\cdot)$ is non decreasing on $\mathbb{R}^+$. Especially, for all $x\in\mathbb{R}$, $[0,y^*_i]=\mathcal{LR}_{(i,x)}$.

As before, the value function can be written in terms of the confluent hypergeometric functions.

Proposition 5.10 *(Power utility with no-switch)*
Let $(i,x) \in \{1,...,m\} \times \mathbb{R}$. We assume that $\rho = 0$, $\vartheta_{i,j} = 0$ for all $j \neq i$ and that there exists $y_i^* \geq 0$ such that $\mathcal{LR}_{(i,x)} = [0, y_i^*]$. If we set $\lambda_i^{(a)} := \lambda_i - \frac{\sigma^2}{2}a^2 - \mu a$, then y_i^* is the solution of

$$\frac{g(y_i^*) - \frac{\lambda_i}{\lambda_i^{(a)}}}{g'(y_i^*)} = -\frac{\gamma^2}{2\lambda_i^{(a)}} \frac{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)}{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa} + 1, \frac{2\kappa\beta}{\gamma^2} + 1, \frac{2\kappa}{\gamma^2} y_i^*\right)}. \tag{5.36}$$

The function $u(i,\cdot)$ is given by

$$u(i,y) = \begin{cases} g(y) & y \leq y_i^* \\ \dfrac{g(y_i^*) - \frac{\lambda_i}{\lambda_i^{(a)}}}{\Psi\left(\frac{\lambda_i^{(a)}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_i^*\right)} \Psi\left(\dfrac{\lambda^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\lambda}{\lambda^{(a)}} & y > y_i^*. \end{cases} \tag{5.37}$$

where Ψ denotes the confluent hypergeometric function of second kind (see Appendix B).

Proposition 5.11 *(The two-regime case)*
Assume that $m = 1$, $\rho = 0$ and, for all $i \in \{0,1\}$, $\mathcal{LR}_{(i,s)} = [0, y_i^*]$. We then have $y_0^* \leq y_1^*$ and the function u is given by

$$u(0,y) = \begin{cases} g(y) & y \in [0, y_0^*] \\ \widehat{c}\Phi\left(\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \widehat{d}\Psi\left(\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) & y \in]y_0^*, y_1^*] \\ \quad + \mathcal{I}\left(\dfrac{2\kappa}{\gamma^2}, \beta, -2\dfrac{\lambda_0^{(a)} + \vartheta_{0,1}}{\gamma^2}, 2\dfrac{\vartheta_{0,1} g(\cdot) + \lambda_0}{\gamma^2}\right)(y) & \\ p_0^0\left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\widetilde{\lambda}_0}{\widetilde{\lambda}_0^{(a)}}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^0\left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\widetilde{\lambda}_1}{\widetilde{\lambda}_1^{(a)}}\right] & \end{cases} \tag{5.38}$$

$$u(1,y) = \begin{cases} g(y) & x \in [0, y_1^*] \\ p_0^1\left[\widehat{e}\Psi\left(\dfrac{\widetilde{\lambda}_0^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\widetilde{\lambda}_0}{\widetilde{\lambda}_0^{(a)}}\right] & y \in (y_1^*, \infty) \\ \quad + p_1^1\left[\widehat{f}\Psi\left(\dfrac{\widetilde{\lambda}_1^{(a)}}{\kappa}, \dfrac{2\kappa\beta}{\gamma^2}, \dfrac{2\kappa}{\gamma^2} y\right) + \dfrac{\widetilde{\lambda}_1}{\widetilde{\lambda}_1^{(a)}}\right], & \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind and $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ are such that $u(0,y)$ and $u(1,y)$ belong to $C^1(\mathbb{R}^+)$.

A Proof of comparison principle

Proof of lemma 3.5: In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0,1)$:

$$\max_{i\in\{0,\dots,m\}} \sup_{\mathbb{R}\times\mathbb{R}^+} (u_i - w_i^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0.

We argue by contradiction and suppose that there exist some $\gamma \in (0,1)$ and $i \in \{0,\dots,m\}$, s.t.

$$\theta := \max_{j\in\{0,\dots,m\}} \sup_{\mathbb{R}\times\mathbb{R}^+} (u_j - w_j^\gamma) = \sup_{\mathbb{R}\times\mathbb{R}^+} (u_i - w_i^\gamma) > 0. \tag{A.39}$$

Let $z = (x,y)$. Notice that $u_i(z) - w_i^\gamma(z)$ goes to $-\infty$ when $|z|$ goes to infinity, as pointed out in Remark 3.2. We also have $\lim_{y\downarrow 0} u_i(x,y) - \lim_{y\downarrow 0} w_i^\gamma(x,y) \leq 0$ by assumption. Hence, by continuity of the functions u_i and w_i^γ, there exists $z_0 \in \mathbb{R}\times(0,\infty)$ s.t.

$$\theta = u_i(z_0) - w_i^\gamma(z_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned} \Phi_\varepsilon(z,z') &= u_i(z) - w_i^\gamma(z') - \phi_\varepsilon(z,z'), \\ \phi_\varepsilon(z,z') &= \frac{1}{4}|z-z_0|^4 + \frac{1}{2\varepsilon}|z-z'|^2, \end{aligned}$$

for all $z, z' \in \mathbb{R}\times(0,\infty)$. By standard arguments of comparison principles, the function Φ_ε attains a maximum in $(z_\varepsilon, z'_\varepsilon) \in (\mathbb{R}\times(0,\infty))^2$, which converges (up to a subsequence) to (z_0, z_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon\to 0} \frac{|z_\varepsilon - z'_\varepsilon|^2}{\varepsilon} = 0. \tag{A.40}$$

Applying Theorem 3.2 of [5], we obtain the existence of 2×2 matrices $M_\varepsilon = (M_{\varepsilon jl})_{1\leq j,l\leq 2}$, $M'_\varepsilon = (M'_{\varepsilon jl})_{1\leq j,l\leq 2}$ such that:

$$\begin{aligned} (p_\varepsilon, M_\varepsilon) &\in J^{2,+}u_i(z_\varepsilon), \\ (p'_\varepsilon, M'_\varepsilon) &\in J^{2,-}w_i^\gamma(z'_\varepsilon), \end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -M'_\varepsilon \end{pmatrix} \leq D^2_{z,z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon) + \varepsilon\left(D^2_{z,z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon)\right)^2, \tag{A.41}$$

where

$$\begin{aligned} p_\varepsilon = (p_{\varepsilon j})_{1\leq j\leq 2} &= D_z\phi_\varepsilon(z_\varepsilon, z'_\varepsilon), \\ p'_\varepsilon = (p'_{\varepsilon j})_{1\leq j\leq 2} &= -D_{z'}\phi_\varepsilon(z_\varepsilon, z'_\varepsilon). \end{aligned}$$

By writing the viscosity subsolution property of u_i and the strict viscosity supersolution property (3.25) of w_i^γ, we have the following inequalities:

$$\min\Big[- p_{\varepsilon 1}\mu(x_\varepsilon) - p_{\varepsilon 2}\alpha(y_\varepsilon) - \frac{1}{2}\sigma^2(x_\varepsilon)M_{\varepsilon 11} - \rho\gamma(y_\varepsilon)\sigma(x_\varepsilon)M_{\varepsilon 12}$$
$$-\frac{1}{2}\gamma^2(y_\varepsilon)M_{\varepsilon 22} - \mathcal{G}_i u(.,x_\varepsilon,y_\varepsilon) - \mathcal{J}_i u(i,x_\varepsilon,y_\varepsilon)\,,\, u_i(x_\varepsilon,y_\varepsilon) - U(e^{x_\varepsilon}f(y_\varepsilon))\Big] \leq 0, \quad \text{(A.42)}$$
$$\min\Big[- p'_{\varepsilon 1}\mu(x'_\varepsilon) - p'_{\varepsilon 2}\alpha(y'_\varepsilon) - \frac{1}{2}\sigma^2(x'_\varepsilon)M'_{\varepsilon 11} - \rho\gamma(y'_\varepsilon)\sigma(x'_\varepsilon)M'_{\varepsilon 12} - \frac{1}{2}\gamma^2(y'_\varepsilon)M'_{\varepsilon 22}$$
$$-\mathcal{G}_i w^\gamma(.,x'_\varepsilon,y'_\varepsilon) - \mathcal{J}_i w^\gamma(i,x'_\varepsilon,y'_\varepsilon)\,,\, w_i^\gamma(x'_\varepsilon,y'_\varepsilon) - U(e^{x'_\varepsilon}f(y'_\varepsilon))\Big] \geq \delta. \quad \text{(A.43)}$$

We then distinguish the following two cases :

⋆ *Case 1 :* $u_i(x_\varepsilon,y_\varepsilon) - U(e^{x_\varepsilon}f(y_\varepsilon)) \leq 0$ in (A.42).
From the continuity of u_i and by sending $\varepsilon \to 0$, this implies

$$u_i(x_0,y_0) - U(e^{x_0}f(y_0)) \leq 0. \quad \text{(A.44)}$$

On the other hand, from (A.43), we also have

$$w_i^\gamma(x'_\varepsilon,y'_\varepsilon) - U(e^{x'_\varepsilon}f(y'_\varepsilon)) \geq \delta,$$

which implies, by sending $\varepsilon \to 0$ and using the continuity of w_i :

$$w_i^\gamma(x_0,y_0) - U(e^{x_0}f(y_0)) \geq \delta. \quad \text{(A.45)}$$

Combining (A.44) and (A.45), we obtain

$$\theta = u_i(z_0) - w_i^\gamma(z_0) \quad \leq \quad -\delta,$$

which is a contradiction.

⋆ *Case 2 :* $\big[- p_{\varepsilon 1}\mu(x_\varepsilon) - p_{\varepsilon 2}\alpha(y_\varepsilon) - \frac{1}{2}\sigma^2(x_\varepsilon)M_{\varepsilon 11} - \rho\gamma(y_\varepsilon)\sigma(x_\varepsilon)M_{\varepsilon 12}$
$- \frac{1}{2}\gamma^2(y_\varepsilon)M_{\varepsilon 22} - \mathcal{G}_i u(.,x_\varepsilon,y_\varepsilon) - \mathcal{J}_i u(i,x_\varepsilon,y_\varepsilon)\big] \leq 0$ in (A.42)

From (A.43), we have

$$\Big[- p'_{\varepsilon 1}\mu(x'_\varepsilon) - p'_{\varepsilon 2}\alpha(y'_\varepsilon) - \frac{1}{2}\sigma^2(x'_\varepsilon)M'_{\varepsilon 11} - \rho\gamma(y'_\varepsilon)\sigma(x'_\varepsilon)M'_{\varepsilon 12} - \frac{1}{2}\gamma^2(y'_\varepsilon)M'_{\varepsilon 22}$$
$$-\mathcal{G}_i w^\gamma(.,x'_\varepsilon,y'_\varepsilon) - \mathcal{J}_i w^\gamma(i,x'_\varepsilon,y'_\varepsilon)\big] \geq \delta.$$

Combining the two above inequalities and using relation (A.41) and the continuity of u_i and w_i^γ, we obtain the required contradiction : $\delta \leq 0$. This ends the proof. □

B Confluent Hypergeometric Functions

In this appendix, we discuss the solution of the following class of ordinary differential equation:

$$yf''(y) + a(b-y)f'(y) + cf(y) + l(y) = 0\,. \quad \text{(B.46)}$$

We refer mainly to [21] for more complete details in the resolution of this type of equations. We start analyzing the associated homogeneous ED

$$y f_0''(y) + a(b-y) f_0'(y) + c f_0(y) = 0\,.$$

Let $J(A, C, y)$ be the solution of the confluent hypergeometric differential equation

$$y J''(A,C,y) + (C-y) J'(A,C,y) - A J(A,C,y) = 0\,,$$

then, it is easy to verify that

$$f_0(y) = J\Big(-\frac{c}{a}, ab, ay\Big)\,.$$

In the rest of this appendix, we will assume that $a, b, c \neq 0$ and $\frac{c}{a} \notin \mathbb{N}$. In the other cases, the solution is either polynomial and exponential functions or a linear combination of confluent hypergeometric functions of first kind. A direct application of the separation of variable method gives the following solution

$$f(y) = k f_0(y) + \mathcal{I}(a,b,c,l(\cdot))(y) \tag{B.47}$$

where

$$\begin{aligned} h_0(y) &= exp\Big\{\int^y \frac{a(b-z)f_0(z) + 2z f_0'(z)}{z f_0(z)} dz\Big\} \\ h_1(y) &= -\int^y \frac{l(z)}{z h_0(z)} dz \\ \mathcal{I}(a,b,c,l(\cdot))(y) &= \int^y (k_0 + h_1(z))\, h_0(z) dz \end{aligned}$$

with k and k_0 constant.

$$f(y) = J\Big(-\frac{c}{a}, ab, ay\Big) - \frac{d}{c}\,. \tag{B.48}$$

The general solution J of the confluent hypergeometric differential equation is generally written as a linear combination of the Kummer function Φ and the Tricomi function Ψ.

We summarize here some properties of functions Φ and Ψ:

$$\begin{aligned} \Phi'(A,C,y) &= \frac{A}{C}\Phi(A+1, C+1, y) \\ \Psi'(A,C,y) &= -A\Psi(A+1, C+1, y) \\ \lim_{y\to 0}\Phi(A,C,y) &= 1 \\ \lim_{y\to 0}\Psi(A,C,y) &\simeq \begin{cases} \frac{\Gamma(1-C)}{\Gamma(A-C+1)} & \text{if } C<1 \\ \frac{\Gamma(C-1)}{\Gamma(A)} y^{1-C} & \text{if } C>1 \end{cases} \\ \lim_{y\to\infty}\Phi(A,C,y) & \quad \text{does not exist} \\ \lim_{y\to\infty}\Psi(A,C,y) &\simeq y^{-A} \end{aligned}$$

B.1 System of equations verified by the parameters of the value function

The following system of equations provides us with the values $(y_0^*, y_1^*, \widehat{c}, \widehat{d}, \widehat{e}, \widehat{f})$ needed to define completely the value function in Proposition 4.8.

The first equation is derived from the continuity of $w(0, y)$ at y_0^*:

$$\begin{aligned} 0 \;=\; & \frac{\kappa+\lambda_0}{\kappa+\lambda_0+\vartheta_{0,1}} y_0^* + \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_0^*\right) \\ & + \widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_0^*\right) + \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0+\vartheta_{0,1}}. \end{aligned} \tag{B.49a}$$

The second equation is obtained from the continuity of the derivative of $w(0, y)$ at y_0^*:

$$\begin{aligned} 0 \;=\; & \frac{\kappa+\lambda_0}{\kappa+\lambda_0+\vartheta_{0,1}} + \widehat{c}\frac{\lambda_0+\vartheta_{0,1}}{\kappa\beta}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_0^*\right) \\ & - 2\widehat{d}\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_0^*\right). \end{aligned} \tag{B.49b}$$

The third equation is obtained from the continuity of $w(0, y)$ at y_1^*:

$$\begin{aligned} 0 \;=\; & \frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}} y_1^* - \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ & - \widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) - \frac{\mu - \kappa\beta\frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}}}{\lambda_0+\vartheta_{0,1}} \\ & + p_0^0\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] + p_1^0\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_1}\right]. \end{aligned} \tag{B.49c}$$

The fourth equation is obtained from the continuity of the derivative of $w(0, y)$ at y_1^*:

$$\begin{aligned} 0 \;=\; & \frac{\vartheta_{0,1}}{\kappa+\lambda_0+\vartheta_{0,1}} - \widehat{c}\frac{\lambda_0+\vartheta_{0,1}}{\kappa\beta}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ & + 2\widehat{d}\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ & - 2p_0^0\widehat{e}\frac{\widetilde{\lambda}_0}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ & - 2p_1^0\widehat{f}\frac{\widetilde{\lambda}_1}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right). \end{aligned} \tag{B.49d}$$

The fifth equation gives the continuity of $w(1, y)$ at y_1^*:

$$\begin{aligned} 0 \;=\; & y_1^* + p_0^1\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_0}\right] \\ & + p_1^1\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}, \frac{2\kappa\beta}{\gamma^2}, \frac{2\kappa}{\gamma^2} y_1^*\right) + \frac{\mu}{\widetilde{\lambda}_1}\right]. \end{aligned} \tag{B.49e}$$

The sixth equation gives the continuity of the derivative of $w(1, y)$ at y_1^*:

$$\begin{aligned} 1 \;=\; & 2p_0^1\widehat{e}\frac{\widetilde{\lambda}_0}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right) \\ & + 2p_1^1\widehat{f}\frac{\widetilde{\lambda}_1}{\gamma^2}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa}+1, \frac{2\kappa\beta}{\gamma^2}+1, \frac{2\kappa}{\gamma^2} y_1^*\right). \end{aligned} \tag{B.49f}$$

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