Explicit solution to an optimal switching problem in the two-regime case*

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Abstract

This paper considers the problem of determining the optimal sequence of stopping times for a diffusion process subject to regime switching decisions. This is motivated in the economics literature, by the investment problem under uncertainty for a multi-activity firm involving opening and closing decisions. We use a viscosity solutions approach combined with the smooth-fit property, and explicitly solve the problem in the two regime case when the state process is of geometric Brownian nature. The results of our analysis take several qualitatively different forms, depending on model parameter values.

Key words : Optimal switching, system of variational inequalities, viscosity solutions, smooth-fit principle.

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1 Introduction

The theory of optimal stopping and its generalization, thoroughly studied in the seventies, has received a renewed interest with a variety of applications in economics and finance. These applications range from asset pricing (American options, swing options) to firm investment and real options. We refer to [4] for a classical and well documented reference on the subject.

In this paper, we consider the optimal switching problem for a one dimensional stochastic process X. The diffusion process X may take a finite number of regimes that are switched at stopping time decisions. For example in the firm's investment problem under uncertainty, a company (oil tanker, electricity station ....) manages several production activities operating in different modes or regimes representing a number of different economic outlooks (e.g. state of economic growth, open or closed production activity, ...). The process X is the price of input or output goods of the firm and its dynamics may differ according to the regimes. The firm's project yields a running payoff that depends on the commodity price X and on the regime choice. The transition from one regime to another one is realized sequentially at time decisions and incurs certain fixed costs. The problem is to find the switching strategy that maximizes the expected value of profits resulting from the project.

Optimal switching problems were studied by several authors, see [1] or [10]. These control problems lead via the dynamic programming principle to a system of variational inequalities. Applications to option pricing, real options and investment under uncertainty were considered by [2], [5] and [7]. In this last paper, the drift and volatility of the state process depend on an uncontrolled finite-state Markov chain, and the author provides an explicit solution to the optimal stopping problem with applications to Russian options. In [2], an explicit solution is found for a resource extraction problem with two regimes (open or closed field), a linear profit function and a price process following a geometric Brownian motion. In [5], a similar model is solved with a general profit function in one regime and equal to zero in the other regime. In both models [2], [5], there is no switching in the diffusion process : changes of regimes only affect the payoff functions. Their method of resolution is to construct a solution to the dynamic programming system by guessing a priori the form of the strategy, and then validate a posteriori the optimality of their candidate by a verification argument. Our model combines regime switchings both on the diffusion process and on the general profit functions. We use a viscosity solutions approach for determining the solution to the system of variational inequalities. In particular, we derive directly the smooth-fit property of the value functions and the structure of the switching regions. Explicit solutions are provided in the following cases : $\star$ the drift and volatility terms of the diffusion take two different regime values, and the profit functions are identical of power type, $\star$ there is no switching on the diffusion process, and the two different profit functions satisfy a general condition, including typically power functions. We also consider the cases for which both switching costs are positive, and for which one of the two is negative. This last case is interesting in applications where a firm chooses between an open or closed activity, and may regain a fraction of its opening costs when it decides to close. The results of our analysis take several qualitatively different forms, depending on model parameter values, essentially the payoff functions and the switching costs.

The paper is organized as follows. We formulate in Section 2 the optimal switching problem. In Section 3, we state the system of variational inequalities satisfied by the value functions in the viscosity sense. The smooth-fit property for this problem, proved in [9], plays an important role in our subsequent analysis. We also state some useful properties on the switching regions. In Section 4, we explicitly solve the problem in the two-regimes case when the state process is of geometric Brownian nature.

2 Formulation of the optimal switching problem

We consider a stochastic system that can operate in d modes or regimes. The regimes can be switched at a sequence of stopping times decided by the operator (individual, firm, ...). The indicator of the regimes is modeled by a cadlag process I_t valued in $\mathbb{I}_d = \{1, \ldots, d\}$. The stochastic system X (commodity price, salary, ...) is valued in $\mathbb{R}_+^* = (0, \infty)$ and satisfies the s.d.e.

$$dX_t \;\; = \;\; b_{I_t} X_t dt + \sigma_{I_t} X_t dW_t, \tag{2.1}$$

where W is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions. $b_i \in \mathbb{R}$, and $\sigma_i > 0$ are the drift and volatility of the system X once in regime $I_t = i$ at time t.

A strategy decision for the operator is an impulse control α consisting of a double sequence $\tau_1, \ldots, \tau_n, \ldots, \kappa_1, \ldots, \kappa_n, \ldots$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where τ_n are stopping times, $\tau_n < \tau_{n+1}$ and $\tau_n \to \infty$ a.s., representing the switching regimes time decisions, and κ_n are $\mathcal{F}_{\tau_n}$-measurable valued in $\mathbb{I}_d$, and representing the new value of the regime at time $t = \tau_n$. We denote by $\mathcal{A}$ the set of all such impulse controls. Now, for any initial condition $(x, i) \in (0, \infty) \times \mathbb{I}_d$, and any control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$, there exists a unique strong solution valued in $(0, \infty) \times \mathbb{I}_d$ to the controlled stochastic system :

$$X_0 \;\; = \;\; x, \quad I_{0^-} \;\; = \;\; i, \tag{2.2}$$

$$dX_t \;\; = \;\; b_{\kappa_n} X_t dt + \sigma_{\kappa_n} X_t dW_t, \quad I_t \;\; = \;\; \kappa_n, \quad \tau_n \leq t < \tau_{n+1}, \quad n \geq 0. \tag{2.3}$$

Here, we set $\tau_0 = 0$ and $\kappa_0 = i$. We denote by $(X^{x,i}, I^i)$ this solution (as usual, we omit the dependence in α for notational simplicity). We notice that $X^{x,i}$ is a continuous process and I^i is a cadlag process, possibly with a jump at time 0 if $\tau_1 = 0$ and so $I_0 = \kappa_1$.

We are given a running profit function $f : \mathbb{R}_+ \times \mathbb{I}_d \to \mathbb{R}$ and we set $f_i(.) = f(., i)$ for $i \in \mathbb{I}_d$. We assume that for each $i \in \mathbb{I}_d$, the function f_i is nonnegative and is Hölder continuous on $\mathbb{R}_+$: there exists $\gamma_i \in (0, 1]$ s.t.

$$|f_i(x) - f_i(\hat{x})| \;\; \leq \;\; C|x - \hat{x}|^{\gamma_i}, \quad \forall x, \hat{x} \in \mathbb{R}_+, \tag{2.4}$$

for some positive constant C. Without loss of generality (see Remark 2.1), we may assume that $f_i(0) = 0$. We also assume that for all $i \in \mathbb{I}_d$, the conjugate of f_i is finite on $(0, \infty)$:

$$\tilde{f}_i(y) \;\; := \;\; \sup_{x \geq 0} [f_i(x) - xy] \; < \; \infty, \quad \forall y > 0. \tag{2.5}$$

The cost for switching from regime i to j is a constant equal to g_{ij}, with the convention $g_{ii} = 0$, and we assume the triangular condition :

$$g_{ik} \quad < \quad g_{ij} + g_{jk}, \quad j \neq i, k. \tag{2.6}$$

This last condition means that it is less expensive to switch directly in one step from regime i to k than in two steps via an intermediate regime j. Notice that a switching cost g_{ij} may be negative, and condition (2.6) for $i = k$ prevents arbitrage by switching back and forth, i.e.

$$g_{ij} + g_{ji} \quad > \quad 0, \quad i \neq j \in \mathbb{I}_d. \tag{2.7}$$

The expected total profit of running the system when initial state is (x, i) and using the impulse control $\alpha = (\tau_n, \kappa_n)_{n \geq 1} \in \mathcal{A}$ is

$$J_i(x, \alpha) \quad = \quad E\left[\int_0^\infty e^{-rt} f(X_t^{x,i}, I_t^i) dt - \sum_{n=1}^\infty e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right].$$

Here $r > 0$ is a positive discount factor, and we use the convention that $e^{-r\tau_n(\omega)} = 0$ when $\tau_n(\omega) = \infty$. We also make the standing assumption :

$$r \quad > \quad b := \max_{i \in \mathbb{I}_d} b_i. \tag{2.8}$$

The objective is to maximize this expected total profit over all strategies α. Accordingly, we define the value functions

$$v_i(x) \quad = \quad \sup_{\alpha \in \mathcal{A}} J_i(x, \alpha), \quad x \in \mathbb{R}_+^*, \ i \in \mathbb{I}_d. \tag{2.9}$$

We shall see in the next section that under (2.5) and (2.8), the expectation defining $J_i(x)$ is well-defined and the value function v_i is finite.

Remark 2.1 The initial values $f_i(0)$ of the running profit functions received by the firm manager (the controller) before any decision are considered as included into the switching costs when changing of regime. This means that w.l.o.g. we may assume that $f_i(0) = 0$. Indeed, for any profit function f_i, and by setting $\tilde{f}_i = f_i - f_i(0)$, we have for all $x > 0$, $\alpha \in \mathcal{A}$,

$$\begin{aligned}
J_i(x, \alpha) \quad &= \quad E\left[\sum_{n=1}^\infty \int_{\tau_{n-1}}^{\tau_n} e^{-rt} f(X_t^{x,i}, \kappa_{n-1}) dt - \sum_{n=1}^\infty e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right] \\
&= \quad E\left[\sum_{n=1}^\infty \int_{\tau_{n-1}}^{\tau_n} e^{-rt} \left(\tilde{f}(X_t^{x,i}, \kappa_{n-1}) + f_{\kappa_{n-1}}(0)\right) dt \ - \sum_{n=1}^\infty e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}\right] \\
&= \quad E\left[\sum_{n=1}^\infty \int_{\tau_{n-1}}^{\tau_n} e^{-rt} \tilde{f}(X_t^{x,i}, \kappa_{n-1}) dt + \frac{f_{\kappa_0}(0)}{r}\right. \\
&\qquad \left. - \sum_{n=1}^\infty e^{-r\tau_n} \left(g_{\kappa_{n-1},\kappa_n} + \frac{f_{\kappa_n}(0) - f_{\kappa_{n-1}}(0)}{r}\right)\right] \\
&= \quad \frac{f_i(0)}{r} + E\left[\int_0^\infty e^{-rt} \tilde{f}(X_t^{x,i}, I_t^i) dt - \sum_{n=1}^\infty e^{-r\tau_n} \tilde{g}_{\kappa_{n-1},\kappa_n}\right],
\end{aligned}$$

with modified switching costs that take into account the possibly different initial values of the profit functions :

$$\tilde{g}_{ij} \;\; = \;\; g_{ij} + \frac{f_j(0) - f_i(0)}{r}.$$

3 System of variational inequalities, switching regions and viscosity solutions

We first state the linear growth property and the boundary condition on the value functions.

Lemma 3.1 *We have for all $i \in \mathbb{I}_d$:*

$$\max_{j\in\mathbb{I}_d}[-g_{ij}] \;\leq\; v_i(x) \;\;\leq\; \frac{xy}{r-b} \;+\; \max_{j\in\mathbb{I}_d}\frac{\tilde{f}_j(y)}{r} + \max_{j\in\mathbb{I}_d}[-g_{ij}],\; \forall x>0,\; \forall y>0. \quad (3.1)$$

In particular, we have $v_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$.

Proof. By considering the particular strategy $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\kappa}_n)$ of immediate switching from the initial state (x,i) to state (x,j), $j \in \mathbb{I}_d$ (eventually equal to i), at cost g_{ij} and then doing nothing, i.e. $\tilde{\tau}_1 = 0$, $\tilde{\kappa}_1 = j$, $\tilde{\tau}_n = \infty$, $\tilde{\kappa}_n = j$ for all $n \geq 2$, we have

$$J_i(x,\tilde{\alpha}) \;\; = \;\; E\Big[\int_0^\infty e^{-rt} f_j(\tilde{X}_t^{x,j})dt - g_{ij}\Big],$$

where $\tilde{X}^{x,j}$ denotes the geometric brownian in regime j starting from x at time 0. Since f_j is nonnegative, and by the arbitrariness of j, we get the lower bound in (3.1).

Given an initial state $(X_0, I_{0^-}) = (x,i)$ and an arbitrary impulse control $\alpha = (\tau_n,\kappa_n)$, we get from the dynamics (2.2)-(2.3), the following explicit expression of $X^{x,i}$:

$$\begin{aligned} X_t^{x,i} \;\; &= \;\; xY_t(i) \\ &:= \;\; x\left(\prod_{l=0}^{n-1} e^{b_{\kappa_l}(\tau_{l+1}-\tau_l)} Z^{\kappa_l}_{\tau_l,\tau_{l+1}}\right) e^{b_{\kappa_n}(t-\tau_n)} Z^{\kappa_n}_{\tau_n,t}, \quad \tau_n \leq t < \tau_{n+1}, \quad n\in\mathbb{N}, \end{aligned} \quad (3.2)$$

where

$$Z^j_{s,t} \;\; = \;\; \exp\left(\sigma_j(W_t - W_s) - \frac{\sigma_j^2}{2}(t-s)\right), \quad 0 \leq s \leq t, \quad j\in\mathbb{I}_d. \quad (3.3)$$

Here, we used the convention that $\tau_0 = 0$, $\kappa_0 = i$, and the product term from l to $n-1$ in (3.2) is equal to 1 when $n=1$. We then deduce the inequality $X_t^{x,i} \leq xe^{bt}M_t$, for all t, where

$$M_t \;\; = \;\; \left(\prod_{l=0}^{n-1} Z^{\kappa_l}_{\tau_l,\tau_{l+1}}\right) Z^{\kappa_n}_{\tau_n,t}, \quad \tau_n \leq t < \tau_{n+1}, \quad n\in\mathbb{N}. \quad (3.4)$$

Now, we notice that (M_t) is a martingale obtained by continuously patching the martingales $(Z^{\kappa_{n-1}}_{\tau_{n-1},t})$ and $(Z^{\kappa_n}_{\tau_n,t})$ at the stopping times τ_n, $n \geq 1$. In particular, we have $E[M_t] = M_0$ $= 1$ for all t.

We set $\tilde{f}(y) = \max_{j\in\mathbb{I}_d} \tilde{f}_i(y)$, $y > 0$, and we notice by definition of $\tilde{f}_i$ in (2.5) that $f(X_t^{x,i}, I_t^i) \leq yX_t^{x,i} + \tilde{f}(y)$ for all t, y. Moreover, we show by induction on N that for all $N \geq 1$, $\tau_1 \leq \ldots \leq \tau_N$, $\kappa_0 = i$, $\kappa_n \in \mathbb{I}_d$, $n = 1, \ldots, N$:

$$-\sum_{n=1}^{N} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \quad \leq \quad \max_{j\in\mathbb{I}_d}[-g_{ij}], \quad a.s.$$

Indeed, the above assertion is obviously true for $N = 1$. Suppose now it holds true at step N. Then, at step $N+1$, we distinguish two cases : If $g_{\kappa_N,\kappa_{N+1}} \geq 0$, then we have $-\sum_{n=1}^{N+1} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \leq -\sum_{n=1}^{N} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n}$ and we conclude by the induction hypothesis at step N. If $g_{\kappa_N,\kappa_{N+1}} < 0$, then by (2.6), and since $\tau_N \leq \tau_{N+1}$, we have $-e^{-r\tau_N} g_{\kappa_{N-1},\kappa_N} - e^{-r\tau_{N+1}} g_{\kappa_N,\kappa_{N+1}} \leq e^{-r\tau_N} g_{\kappa_{N-1},\kappa_{N+1}}$, and so $-\sum_{n=1}^{N+1} e^{-r\tau_n} g_{\kappa_{n-1},\kappa_n} \leq -\sum_{n=1}^{N} e^{-r\tau_n} g_{\tilde{\kappa}_{n-1},\tilde{\kappa}_n}$, with $\tilde{\kappa}_n = \kappa_n$ for $n = 1, \ldots, N-1$, $\tilde{\kappa}_N = \kappa_{N+1}$. We then conclude by the induction hypothesis at step N.

It follows that

$$\begin{aligned}
J_i(x,\alpha) &\leq E\left[\int_0^\infty e^{-rt}\left(yxe^{bt}M_t + \tilde{f}(y)\right)dt + \max_{j\in\mathbb{I}_d}[-g_{ij}]\right] \\
&= \int_0^\infty e^{-(r-b)t} yxE[M_t]dt + \int_0^\infty e^{-rt}\tilde{f}(y)dt + \max_{j\in\mathbb{I}_d}[-g_{ij}] \\
&= \frac{xy}{r-b} + \frac{\tilde{f}(y)}{r} + \max_{j\in\mathbb{I}_d}[-g_{ij}].
\end{aligned}$$

From the arbitrariness of α, this shows the upper bound for v_i.

By sending x to zero and then y to infinity into the r.h.s. of (3.1), and recalling that $\tilde{f}_i(\infty) = f_i(0) = 0$ for $i \in \mathbb{I}_d$, we conclude that v_i goes to $\max_{j\in\mathbb{I}_d}[-g_{ij}]$ when x tends to zero. □

We next show the Hölder continuity of the value functions.

Lemma 3.2 *For all $i \in \mathbb{I}_d$, v_i is Hölder continuous on $(0,\infty)$:*

$$|v_i(x) - v_i(\hat{x})| \quad \leq \quad C|x-\hat{x}|^\gamma, \quad \forall x, \hat{x} \in (0,\infty), \quad \textit{with} \;\; |x-\hat{x}| \leq 1,$$

for some positive constant C, and where $\gamma = \min_{i\in\mathbb{I}_d} \gamma_i$ of condition (2.4).

Proof. By definition (2.9) of v_i and under condition (2.4), we have for all $x, \hat{x} \in (0,\infty)$, with $|x-\hat{x}| \leq 1$:

$$\begin{aligned}
|v_i(x) - v_i(\hat{x})| &\leq \sup_{\alpha\in\mathcal{A}} |J_i(x,\alpha) - J_i(\hat{x},\alpha)| \\
&\leq \sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|f(X_t^{x,i}, I_t^i) - f(X_t^{\hat{x},i}, I_t^i)\right| dt\right] \\
&\leq C\sup_{\alpha\in\mathcal{A}} E\left[\int_0^\infty e^{-rt}\left|X_t^{x,i} - X_t^{\hat{x},i}\right|^{\gamma_{I_t^i}} dt\right] \\
&= C\sup_{\alpha\in\mathcal{A}} \int_0^\infty E\left[e^{-rt}|x-\hat{x}|^{\gamma_{I_t^i}}|Y_t(i)|^{\gamma_{I_t^i}} dt\right] \\
&\leq C|x-\hat{x}|^\gamma \sup_{\alpha\in\mathcal{A}} \int_0^\infty e^{-(r-b)t} E|M_t|^{\gamma_{I_t^i}} dt \qquad (3.5)
\end{aligned}$$

by (3.2) and (3.4). For any $\alpha = (\tau_n, \kappa_n)_n \in \mathcal{A}$, by the independence of $(Z^{\kappa_n}_{\tau_n,\tau_{n+1}})_n$ in (3.3), and since

$$E\left[\left|Z^{\kappa_n}_{\tau_n,\tau_{n+1}}\right|^{\gamma_{\kappa_n}}\middle|\mathcal{F}_{\tau_n}\right] \;=\; E\left[\exp\left(\gamma_{\kappa_n}(\gamma_{\kappa_n}-1)\frac{\sigma^2_{\kappa_n}}{2}(\tau_{n+1}-\tau_n)\right)\middle|\mathcal{F}_{\tau_n}\right] \;\leq\; 1, \quad a.s.,$$

we clearly see that $E|M_t|^{\gamma_{I^i_t}} \leq 1$ for all $t \geq 0$. We thus conclude with (3.5). □

The dynamic programming principle combined with the notion of viscosity solutions are known to be a general and powerful tool for characterizing the value function of a stochastic control problem via a PDE representation, see [6]. We recall the definition of viscosity solutions for a P.D.E in the form

$$H(x, v, D_x v, D^2_{xx} v) \;=\; 0, \quad x \in \mathcal{O}, \tag{3.6}$$

where $\mathcal{O}$ is an open subset in $\mathbb{R}^n$ and H is a continuous function and noninceasing in its last argument (with respect to the order of symmetric matrices).

Definition 3.1 *Let v be a continuous function on $\mathcal{O}$. We say that v is a viscosity solution to* (3.6) *on $\mathcal{O}$ if it is*

(i) a viscosity supersolution to (3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local minimum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \;\geq\; 0.$$

and

(ii) a viscosity subsolution to (3.6) *on $\mathcal{O}$: for any $\bar{x} \in \mathcal{O}$ and any C^2 function φ in a neighborhood of $\bar{x}$ s.t. $\bar{x}$ is a local maximum of $v - \varphi$, we have :*

$$H(\bar{x}, v(\bar{x}), D_x\varphi(\bar{x}), D^2_{xx}\varphi(\bar{x})) \;\leq\; 0.$$

Remark 3.1 **1.** By misuse of notation, we shall say that v is viscosity supersolutin (resp. subsolution) to (3.6) by writing :

$$H(x, v, D_x v, D^2_{xx} v) \;\geq\; (\text{resp. } \leq) \;\; 0, \quad x \in \mathcal{O}, \tag{3.7}$$

2. We recall that if v is a smooth C^2 function on $\mathcal{O}$, supersolution (resp. subsolution) in the classical sense to (3.7), then v is a viscosity supersolution (resp. subsolution) to (3.7).
3. There is an equivalent formulation of viscosity solutions, which is useful for proving uniqueness results, see [3] :
(i) A continuous function v on $\mathcal{O}$ is a viscosity supersolution to (3.6) if

$$H(x, v(x), p, M) \;\geq\; 0, \quad \forall x \in \mathcal{O},\ \forall (p, M) \in J^{2,-}v(x).$$

(ii) A continuous function v on $\mathcal{O}$ is a viscosity subsolution to (3.6) if

$$H(x, v(x), p, M) \;\leq\; 0, \quad \forall x \in \mathcal{O},\ \forall (p, M) \in J^{2,+}v(x).$$

Here $J^{2,+}v(x)$ is the second order superjet defined by :

$$J^{2,+}v(x) = \Big\{(p,M) \in \mathbb{R}^n \times S^n : \limsup_{\substack{x' \to x \\ x \in \mathcal{O}}} \frac{v(x') - v(x) - p.(x'-x) - \frac{1}{2}(x'-x).M(x'-x)}{|x'-x|^2} \leq 0 \Big\},$$

S^n is the set of symmetric $n \times n$ matrices, and $J^{2,-}v(x) = -J^{2,+}(-v)(x)$.

In the sequel, we shall denote by $\mathcal{L}_i$ the second order operator associated to the diffusion X when we are in regime i : for any C^2 function φ on $(0,\infty)$,

$$\mathcal{L}_i\varphi = \frac{1}{2}\sigma_i^2 x^2 \varphi'' + b_i x \varphi'.$$

We then have the following PDE characterization of the value functions v_i by means of viscosity solutions.

Theorem 3.1 *The value functions v_i, $i \in \mathbb{I}_d$, are the unique viscosity solutions with linear growth condition on $(0,\infty)$ and boundary condition $v_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$ to the system of variational inequalities :*

$$\min\Big\{rv_i - \mathcal{L}_i v_i - f_i\ ,\ v_i - \max_{j\neq i}(v_j - g_{ij})\Big\} = 0, \quad x \in (0,\infty), \quad i \in \mathbb{I}_d. \tag{3.8}$$

This means
(1) Viscosity property : for each $i \in \mathbb{I}_d$, v_i is a viscosity solution to

$$\min\Big\{rv_i - \mathcal{L}_i v_i - f_i\ ,\ v_i - \max_{j\neq i}(v_j - g_{ij})\Big\} = 0, \quad x \in (0,\infty). \tag{3.9}$$

(2) Uniqueness property : if w_i, $i \in \mathbb{I}_d$, are viscosity solutions with linear growth conditions on $(0,\infty)$ and boundary conditions $w_i(0^+) = \max_{j\in\mathbb{I}_d}[-g_{ij}]$ to the system of variational inequalities (3.8) *, then $v_i = w_i$ on $(0,\infty)$.*

Proof. (1) The viscosity property follows from the dynamic programming principle and is proved in [9].
(2) Uniqueness results for switching problems has been proved in [10] in the finite horizon case under different conditions. For sake of completeness, we provide in Appendix a proof of comparison principle in our infinite horizon context, which implies the uniqueness result. □

Remark 3.2 For fixed $i \in \mathbb{I}_d$, we also have uniqueness of viscosity solution to equation (3.9) in the class of continuous functions with linear growth condition on $(0,\infty)$ and given boundary condition on 0. In the next section, we shall use either uniqueness of viscosity solutions to the system (3.8) or for fixed i to equation (3.9), for the identification of an explicit solution in the two-regimes case $d = 2$.

We shall also combine the uniqueness result for the viscosity solutions with the smoooth-fit property on the value functions that we state below.

For any regime $i \in \mathbb{I}_d$, we introduce the switching region :

$$\mathcal{S}_i = \left\{ x \in (0,\infty) \ : v_i(x) = \max_{j \neq i} (v_j - g_{ij})(x) \right\}.$$

$\mathcal{S}_i$ is a closed subset of $(0,\infty)$ and corresponds to the region where it is optimal for the operator to change of regime. The complement set $\mathcal{C}_i$ of $\mathcal{S}_i$ in $(0,\infty)$ is the so-called continuation region :

$$\mathcal{C}_i = \left\{ x \in (0,\infty) \ : v_i(x) > \max_{j \neq i} (v_j - g_{ij})(x) \right\},$$

where the operator remains in regime i. In this open domain, the value function v_i is smooth C^2 on $\mathcal{C}_i$ and satisfies in a classical sense :

$$rv_i(x) - \mathcal{L}_i v_i(x) - f_i(x) = 0, \quad x \in \mathcal{C}_i.$$

As a consequence of the condition (2.6), we have the following elementary partition property of the switching regions, see Lemma 4.2 in [9] :

$$\mathcal{S}_i = \cup_{j \neq i} \mathcal{S}_{ij}, \quad i \in \mathbb{I}_d,$$

where

$$\mathcal{S}_{ij} = \{ x \in \mathcal{C}_j \ : v_i(x) = (v_j - g_{ij})(x) \}.$$

$\mathcal{S}_{ij}$ represents the region where it is optimal to switch from regime i to regime j and stay here for a moment, i.e. without changing instantaneously from regime j to another regime. The following Lemma gives some partial information about the structure of the switching regions.

Lemma 3.3 *For all $i \neq j$ in $\mathbb{I}_d$, we have*

$$\mathcal{S}_{ij} \subset Q_{ij} := \{ x \in \mathcal{C}_j \ : (\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0 \}.$$

Proof. Let $x \in \mathcal{S}_{ij}$. By setting $\varphi_j = v_j - g_{ij}$, this means that x is a minimum of $v_i - \varphi_j$ with $v_i(x) = \varphi_j(x)$. Moreover, since x lies in the open set $\mathcal{C}_j$ where v_j is smooth, we have that φ_j is C^2 in a neighborhood of x. By the supersolution viscosity property of v_i to the PDE (3.8), this yields :

$$r\varphi_j(x) - \mathcal{L}_i \varphi_j(x) - f_i(x) \geq 0. \tag{3.10}$$

Now recall that for $x \in \mathcal{C}_j$, we have

$$rv_j(x) - \mathcal{L}_j v_j(x) - f_j(x) = 0,$$

so that by substituting into (3.10), we obtain :

$$(\mathcal{L}_j - \mathcal{L}_i) v_j(x) + (f_j - f_i)(x) - r g_{ij} \geq 0,$$

which is the required result. □

We quote the smooth fit property on the value functions, proved in [9].

Theorem 3.2 *For all $i \in \mathbb{I}_d$, the value function v_i is continuously differentiable on* $(0, \infty)$.

Remark 3.3 In a given regime i, the variational inequality satisfied by the value function v_i is a free-boundary problem as in optimal stopping problem, which divides the state space into the switching region (stopping region in pure optimal stopping problem) and the continuation region. The main difficulty with regard to optimal stopping problems for proving the smooth-fit property through the boundaries of the switching regions, comes from the fact that the switching region for the value function v_i depends also on the other value functions v_j. The method in [9] use viscosity solutions arguments and the condition of one-dimensional state space is critical for proving the smooth-fit property. The crucial conditions in this paper require that the diffusion coefficient in any regime of the system X is strictly positive on the interior the the state space, which is the case here since $\sigma_i > 0$ for all $i \in \mathbb{I}_d$, and a triangular condition (2.6) on the switching costs. Under these conditions, on a point x of the switching region $\mathcal{S}_i$ for regime i, there exists some $j \neq i$ s.t. $x \in \mathcal{S}_{ij}$, i.e. $v_i(x) = v_j(x) - g_{ij}$, and the C^1 property of the value functions is written as : $v_i'(x) = v_j'(x)$ since g_{ij} is constant.

The next result provides suitable conditions for determining a viscosity solution to the variational inequality type arising in our switching problem.

Lemma 3.4 *Fix $i \in \mathbb{I}_d$. Let $\mathcal{C}$ be an open set in $(0, \infty)$, $\mathcal{S} = (0, \infty) \setminus \mathcal{C}$ supposed to be an union of a finite number of closed intervals in $(0, \infty)$, and w, h two continuous functions on $(0, \infty)$, with $w = h$ on $\mathcal{S}$ such that*

$$w \quad \text{is } C^1 \quad \text{on } \partial\mathcal{S} \tag{3.11}$$

$$w \quad \geq \quad h \quad \text{on } \mathcal{C}, \tag{3.12}$$

w is C^2 on $\mathcal{C}$, solution to

$$rw - \mathcal{L}_i w - f_i \quad = \quad 0 \quad \text{on } \mathcal{C}, \tag{3.13}$$

and w is a viscosity supersolution to

$$rw - \mathcal{L}_i w - f_i \quad \geq \quad 0 \quad \text{on } \operatorname{int}(\mathcal{S}). \tag{3.14}$$

Here $\operatorname{int}(\mathcal{S})$ is the interior of $\mathcal{S}$ and $\partial\mathcal{S} = \mathcal{S} \setminus \operatorname{int}(\mathcal{S})$ its boundary. Then, w is a viscosity solution to

$$\min\{rw - \mathcal{L}_i w - f_i, w - h\} \quad = \quad 0 \quad \text{on } (0, \infty). \tag{3.15}$$

Proof. Take some $\bar{x} \in (0, \infty)$ and distinguish the following cases :

$\star$ $\bar{x} \in \mathcal{C}$. Since $w = v$ is C^2 on $\mathcal{C}$ and satisfies $rw(\bar{x}) - \mathcal{L}_i w(\bar{x}) - f_i(\bar{x}) = 0$ by (3.13), and recalling $w(\bar{x}) \geq h(\bar{x})$ by (3.12), we obtain the classical solution property, and so a fortiori the viscosity solution property (3.15) of w at $\bar{x}$.

$\star$ $\bar{x} \in \mathcal{S}$. Then $w(\bar{x}) = h(\bar{x})$ and the viscosity subsolution property is trivial at $\bar{x}$. It remains to show the viscosity supersolution property at $\bar{x}$. If $\bar{x} \in \operatorname{int}(\mathcal{S})$, this follows directly from (3.14). Suppose now $\bar x \in \partial\mathcal{S}$, and to fix the idea, we consider that $\bar x$ is on the left-boundary of $\mathcal{S}$ so that from the assumption on the form of $\mathcal{S}$, there exists $\varepsilon > 0$ s.t. $(\bar x - \varepsilon, \bar x) \subset \mathcal{C}$ on which w is smooth C^2 (the same argument holds true when $\bar x$ is on the right-boundary of $\mathcal{S}$). Take some smooth C^2 function φ s.t. $\bar x$ is a local minimum of $w - \varphi$. Since w is C^1 by (3.11), we have $\varphi'(\bar x) = w'(\bar x)$. We may also assume w.l.o.g (by taking ε small enough) that $(w-\varphi)(\bar x) \le (w-\varphi)(x)$ for $x \in (\bar x - \varepsilon, \bar x)$. Moreover, by Taylor's formula, we have :

$$w(\bar x - \eta) \;=\; w(\bar x) - \eta \int_0^1 w'(\bar x - t\eta)dt, \qquad \varphi(\bar x - \eta) \;=\; \varphi(\bar x) - \eta \int_0^1 \varphi'(\bar x - t\eta)dt,$$

so that

$$\int_0^1 \varphi'(\bar x - t\eta) - w'(\bar x - t\eta)\,dt \;\;\ge\;\; 0, \quad \forall\, 0 < \eta < \varepsilon.$$

Since $\varphi'(\bar x) = w'(\bar x)$, this last inequality is written as

$$\int_0^1 \frac{\varphi'(\bar x - t\eta) - \varphi'(\bar x)}{\eta} - \frac{w'(\bar x - t\eta) - w'(\bar x)}{\eta}\,dt \;\;\ge\;\; 0, \quad \forall\, 0 < \eta < \varepsilon, \tag{3.16}$$

Now, from (3.13), we have $rw(x) - \mathcal{L}_i w(x) - f_i(x) = 0$ for $x \in (\bar x - \varepsilon, \bar x)$. By sending x towards $\bar x$ into this last equality, this shows that $w''(\bar x^-) = \lim_{x \nearrow \bar x} w''(x)$ exists, and

$$rw(\bar x) - b_i \bar x w'(\bar x) - \frac{1}{2}\sigma_i^2 \bar x^2 w''(\bar x^-) - f_i(\bar x) \;\;=\;\; 0. \tag{3.17}$$

Moreover, by sending η to zero into (3.16), we obtain :

$$\int_0^1 t[-\varphi''(\bar x) + w''(\bar x^-)]dt \;\;\ge\;\; 0,$$

and so $\varphi''(\bar x) \le w''(\bar x^-)$. By substituting into (3.17), and recalling that $w'(\bar x) = \varphi'(\bar x)$, we then obtain :

$$rw(\bar x) - \mathcal{L}_i\varphi(\bar x) - f_i(\bar x) \;\;\ge\;\; 0,$$

which is the required supersolution inequality, and ends the proof. □

Remark 3.4 Since $w = h$ on $\mathcal{S}$, relation (3.14) means equivalently that h is a viscosity supersolution to

$$rh - \mathcal{L}_i h - f_i \;\;\ge\;\; 0 \quad \text{on } \operatorname{int}(\mathcal{S}). \tag{3.18}$$

Practically, Lemma 3.4 shall be used as follows in the next section : we consider two C^1 functions v and h on $(0,\infty)$ s.t.

$$\begin{aligned} v(x) \;=\; h(x),\; v'(x) \;&=\; h'(x), \quad x \in \partial\mathcal{S} \\ v \;&\ge\; h \quad \text{on } \mathcal{C}, \end{aligned}$$

v is C^2 on $\mathcal{C}$, solution to

$$rv - \mathcal{L}_i v - f_i \;=\; 0 \quad \text{on } \mathcal{C},$$

and h is a viscosity supersolution to (3.18). Then, the function w defined on $(0,\infty)$ by :

$$w(x) \;=\; \begin{cases} v(x), & x \in \mathcal{C} \\ h(x), & x \in \mathcal{S} \end{cases}$$

satisfies the conditions of Lemma 3.4 and is so a viscosity solution to (3.15). This Lemma combined with uniqueness viscosity solution result may be viewed as an alternative to the classical verification approach in the identification of the value function. Moreover, with our viscosity solutions approach, we shall see in subsection 4.2 that Lemma 3.3 and smooth-fit property of the value functions in Theorem 3.2 provide a direct derivation for the structure of the switching regions and then of the solution to our problem.

4 Explicit solution in the two regime case

In this section, we consider the case where $d = 2$. In this two-regimes case, we know from Theorem 3.1 that the value functions v_i, $i = 1, 2$, are the unique continuous viscosity solutions with linear growth condition on $(0,\infty)$, and boundary conditions $v_i(0^+) = (-g_{ij})_+$ $:= \max(-g_{ij}, 0)$, $j \neq i$, to the system :

$$\min\{rv_1 - \mathcal{L}_1 v_1 - f_1, v_1 - (v_2 - g_{12})\} \;=\; 0 \tag{4.1}$$

$$\min\{rv_2 - \mathcal{L}_2 v_2 - f_2, v_2 - (v_1 - g_{21})\} \;=\; 0. \tag{4.2}$$

Moreover, the switching regions are :

$$\mathcal{S}_i \;=\; \mathcal{S}_{ij} \;=\; \{x > 0 \;: v_i(x) = v_j(x) - g_{ij}\}, \quad i, j = 1, 2,\; i \neq j.$$

We set

$$\underline{x}_i^* \;=\; \inf \mathcal{S}_i \;\in\; [0,\infty] \qquad \bar{x}_i^* \;=\; \sup \mathcal{S}_i \;\in\; [0,\infty],$$

with the usual convention that $\inf \emptyset = \infty$.

Let us also introduce some other notations. We consider the second order o.d.e for $i = 1, 2$:

$$rv - \mathcal{L}_i v - f_i \;=\; 0, \tag{4.3}$$

whose general solution (without second member f_i) is given by :

$$v(x) \;=\; Ax^{m_i^+} + Bx^{m_i^-},$$

for some constants A, B, and where

$$m_i^- \;=\; -\frac{b_i}{\sigma_i^2} + \frac{1}{2} - \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;<\; 0$$

$$m_i^+ \;=\; -\frac{b_i}{\sigma_i^2} + \frac{1}{2} + \sqrt{\left(-\frac{b_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}} \;>\; 1.$$

We also denote

$$\hat{V}_i(x) \;=\; E\left[\int_0^\infty e^{-rt} f_i(\hat{X}_t^{x,i})dt\right],$$

with $\hat{X}^{x,i}$ the solution to the s.d.e. $d\hat{X}_t = b_i\hat{X}_t dt + \sigma_i \hat{X}_t dW_t$, $\hat{X}_0 = x$. Actually, $\hat{V}_i$ is a particular solution to ode (4.3), with boundary condition $\hat{V}_i(0^+) = f_i(0) = 0$. It corresponds to the reward function associated to the no switching strategy from initial state (x,i), and so $\hat{V}_i \leq v_i$.

Remark 4.1 If $g_{ij} > 0$, then from (2.7), we have $v_i(0^+) = 0 > (-g_{ji})_+ - g_{ij} = v_j(0^+) - g_{ij}$. Therefore, by continuity of the value functions on $(0,\infty)$, we get $\underline{x}_i^* > 0$.

We now give the explicit solution to our problem in the following two situations :
⋆ the diffusion operators are different and the running profit functions are identical.
⋆ the diffusion operators are identical and the running profit functions are different

We also consider the cases for which both switching costs are positive, and for which one of the two is negative, the other being then positive according to (2.7). This last case is interesting in applications where a firm chooses between an open or closed activity, and may regain a fraction of its opening costs when it decides to close.

4.1 Identical profit functions with different diffusion operators

In this subsection, we suppose that the running functions are identical in the form :

$$f_1(x) \;=\; f_2(x) \;=\; x^\gamma, \quad 0 < \gamma < 1, \tag{4.4}$$

and the diffusion operators are different. A straightforward calculation shows that under (4.4), we have

$$\hat{V}_i(x) \;=\; K_i x^\gamma, \quad \text{with } \; K_i \;=\; \frac{1}{r - b_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} \;>\; 0, \;\; i = 1,2.$$

We show that the structure of the switching regions depends actually only on the sign of $K_2 - K_1$, and of the sign of the switching costs g_{12} and g_{21}. More precisely, we have the following explicit result.

Theorem 4.1 *Let i, $j = 1,2$, $i \neq j$.*
1) If $K_i = K_j$, then

$$\begin{aligned} v_i(x) &= \hat{V}_i(x) + (-g_{ij})_+, \qquad x \in (0,\infty), \\ \mathcal{S}_i &= \begin{cases} \emptyset & \text{if } \; g_{ij} > 0 \\ (0,\infty) & \text{if } \; g_{ij} \leq 0. \end{cases} \end{aligned}$$

It is always optimal to switch from regime i to j if the corresponding swiching cost is nonpositive, and never optimal to switch otherwise.

2) If $K_j > K_i$, then we have the following situations depending on the switching costs :

a) $g_{ij} \leq 0$ *: we have* $\mathcal{S}_i = (0,\infty)$, $\mathcal{S}_j = \emptyset$, *and*

$$v_i \ = \ \hat{V}_j - g_{ij}, \qquad v_j \ = \ \hat{V}_j.$$

b) $g_{ij} > 0$ *:*

- *if* $g_{ji} \geq 0$, *then* $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ *with* $\underline{x}_i^* \in (0,\infty)$, $\mathcal{S}_j = \emptyset$, *and*

$$v_i(x) \ = \ \begin{cases} Ax^{m_i^+} + \hat{V}_i(x), & x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases} \tag{4.5}$$

$$v_j(x) \ = \ \hat{V}_j(x), \quad x \in (0,\infty) \tag{4.6}$$

where the constants A *and* $\underline{x}_i^*$ *are determined by the continuity and smooth-fit conditions of* v_i *at* $\underline{x}_i^*$, *and explicitly given by :*

$$\underline{x}_i^* \ = \ \left(\frac{m_i^+}{m_i^+ - \gamma} \frac{g_{ij}}{K_j - K_i} \right)^{\frac{1}{\gamma}} \tag{4.7}$$

$$A \ = \ (K_j - K_i) \frac{\gamma}{m_i^+} (\underline{x}_i^*)^{\gamma - m_i^+}. \tag{4.8}$$

When we are in regime i,*it is optimal to switch to regime* j *whenever the state process* X *exceeds the threshold* $\underline{x}_i^*$, *while when we are in regime* j, *it is optimal never to switch.*

- *if* $g_{ji} < 0$, *then* $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ *with* $\underline{x}_i^* \in (0,\infty)$, $\mathcal{S}_j = (0, \bar{x}_j^*]$, *and*

$$v_i(x) \ = \ \begin{cases} Ax^{m_i^+} + \hat{V}_i(x), & x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases} \tag{4.9}$$

$$v_j(x) \ = \ \begin{cases} v_i(x) - g_{ji}, & x \leq \bar{x}_j^* \\ Bx^{m_j^-} + \hat{V}_j(x), & x > \bar{x}_i^* \end{cases} \tag{4.10}$$

where the constants A, B *and* $\bar{x}_j^* < \underline{x}_i^*$ *are determined by the continuity and smooth-fit conditions of* v_i *and* v_j *at* $\underline{x}_i^*$ *and* $\bar{x}_j^*$, *and explicitly given by :*

$$\overline{x}_j \ = \ \left[\frac{-m_j^-(g_{ji} + g_{ij} y^{m_i^+})}{(K_i - K_j)(\gamma - m_j^-)(1 - y^{m_i^+ - \gamma})} \right]^{\frac{1}{\gamma}}$$

$$\underline{x}_i \ = \ \frac{\overline{x}_j}{y}$$

$$B \ = \ \frac{(K_i - K_j)(m_i^+ - \gamma)\underline{x}_i^{\gamma - m_j^-} + m_i^+ g_{ij} \underline{x}_i^{-m_j}}{m_i^+ - m_j^-}$$

$$A \ = \ B \underline{x}_i^{m_j^- - m_i^+} - (K_i - K_j) \underline{x}_i^{\gamma - m_i^+} - g_{ij} \underline{x}_i^{-m_i^+}$$

with y *solution in* $\left(0, \left(-\frac{g_{ji}}{g_{ij}}\right)^{\frac{1}{m_i^+}}\right)$ *to the equation :*

$$\begin{aligned} & m_i^+(\gamma - m_j^-) \left(1 - y^{m_i^+ - \gamma}\right) \left(g_{ij} y^{m_j^-} + g_{ji}\right) \\ & + m_j^-(m_i^+ - \gamma) \left(1 - y^{m_j^- - \gamma}\right) \left(g_{ij} y^{m_i^+} + g_{ji}\right) \quad = 0 \end{aligned}$$

When we are in regime i, it is optimal to switch to regime j whenever the state process X exceeds the threshold $\underline{x}_i^*$, while when we are in regime j, it is optimal to switch to regime i for values of the state process X under the threshold $\bar{x}_j^*$.

Economic interpretation.
In the particular case where $\sigma_1 = \sigma_2$, then $K_2 - K_1 > 0$ means that regime 2 provides a higher expected return b_2 than the one b_1 of regime 1 for the same volatility coefficient σ_i. Moreover, if the switching cost g_{21} from regime 2 to regime 1 is nonnegative, it is intuitively clear that one has always interest to stay in regime 2, which is formalized by the property that $\mathcal{S}_2 = \emptyset$. However, if one receives some gain compensation to switch from regime 2 to regime 1, i.e. the corresponding cost g_{21} is negative, then one has interest to change of regime for small values of the current state. This is formalized by the property that $\mathcal{S}_2$ $= (0, \bar{x}_2^*]$. On the other hand, in regime 1, one has interest to switch to regime 2, for all current values of the state if the corresponding switching cost g_{12} is nonpositive, or from a certain threshold $\underline{x}_1^*$ if the switching cost g_{12} is positive. A similar interpretation holds when $b_1 = b_2$, and $K_2 - K_1 > 0$, i.e. $\sigma_2 < \sigma_1$. Theorem 4.1 extends these results for general coefficients b_i and σ_i, and show that the critical parameter value determining the form of the optimal strategy is given by the sign of $K_2 - K_1$ and the switching costs. The different optimal strategy structures are depicted in Figure I.

Proof of Theorem 4.1.
1) If $K_i = K_j$, then $\hat{V}_i = \hat{V}_j$. We consider the smooth functions $w_i = \hat{V}_i + (-g_{ij})_+$ for i, j $= 1, 2$ and $j \neq i$. Since $\hat{V}_i$ are solution to (4.3), we see that w_i satisfy :

$$
\begin{aligned}
rw_i - \mathcal{L}w_i - f_i &= r(-g_{ij})_+ && (4.11)\\
w_i - (w_j - g_{ij}) &= g_{ij} + (-g_{ij})_+ - (-g_{ji})_+. && (4.12)
\end{aligned}
$$

Notice that the l.h.s of (4.11) and (4.12) are both nonnegative by (2.7). Moreover, if $g_{ij} >$ 0, then the l.h.s. of (4.11) is zero, and if $g_{ij} \leq 0$, then $g_{ji} > 0$ and the l.h.s. of (4.12) is zero. Therefore, w_i, $i = 1, 2$ is solution to the system :

$$\min\left\{rw_i - \mathcal{L}_i w_i - f_i, w_i - (w_j - g_{ij})\right\} = 0.$$

Since $\hat{V}_i(0^+) = 0$, we have $w_i(0^+) = (-g_{ij})_+$. Moreover, w_i satisfy like $\hat{V}_i$ a linear growth condition. Therefore, from uniquenes of solution to the PDE system (4.1)-(4.2), we deduce that $v_i = w_i$. As observed above, if $g_{ij} \leq 0$, then the l.h.s. of (4.12) is zero, and so $\mathcal{S}_i =$ $(0, \infty)$. Finally, if $g_{ij} > 0$, then the l.h.s. of (4.12) is positive, and so $\mathcal{S}_i = \emptyset$.

2) We now suppose w.l.o.g. that $K_2 > K_1$.
a) Consider first the case where $g_{12} \leq 0$, and so $g_{21} > 0$. We set $w_1 = \hat{V}_2 - g_{12}$ and $w_2 =$ $\hat{V}_2$. Then, by construction, we have $w_1 = w_2 - g_{12}$ on $(0, \infty)$, and by definition of $\hat{V}_1$ and $\hat{V}_2$:

$$rw_1(x) - \mathcal{L}_1 w_1(x) - f_1(x) = \frac{K_2 - K_1}{K_1} x^\gamma - rg_{12} > 0, \quad \forall\, x > 0.$$

On the other hand, we also have $rw_2 - \mathcal{L}_2 w_2 - f_2 = 0$ on $(0, \infty)$, and $w_2 \geq w_1 - g_{21}$ since $g_{12} + g_{21} \geq 0$. Hence, w_1 and w_2 are smooth (hence viscosity) solutions to the system

Figure I

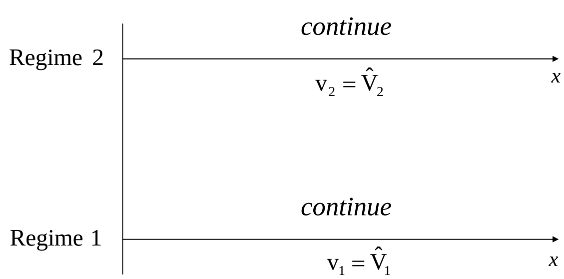

Figure I.1.a: $f_1 = f_2,\ K_1 = K_2,\ g_{12} > 0,\ g_{21} > 0$

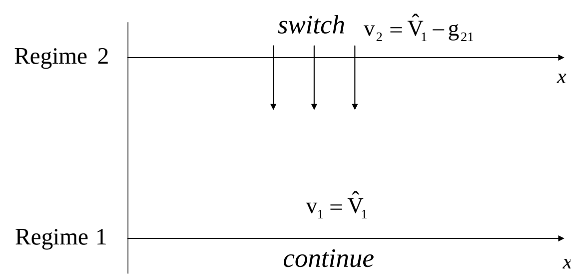

Figure I.1.b: $f_1 = f_2,\ K_1 = K_2,\ g_{12} > 0,\ g_{21} \le 0$

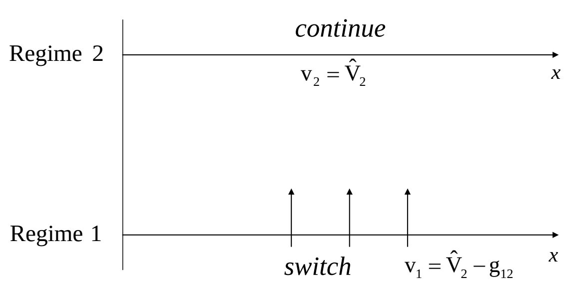

Figure I.2.a: $f_1 = f_2,\ K_2 > K_1,\ g_{12} \le 0$

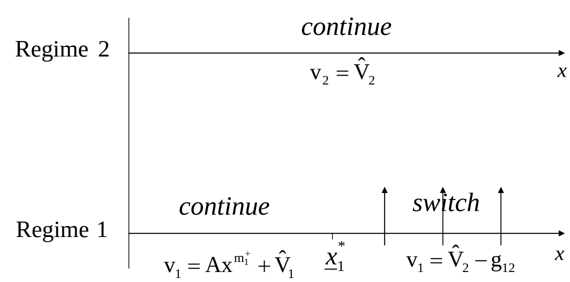

Figure I.2.bi: $f_1 = f_2,\ K_2 > K_1,\ g_{12} > 0,\ g_{21} \ge 0$

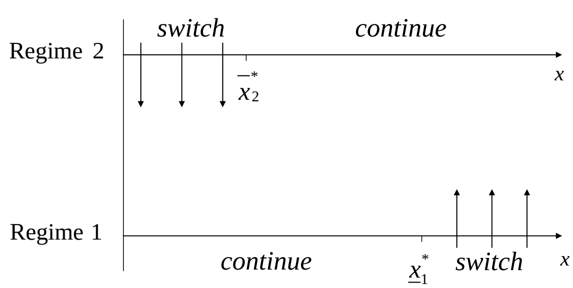

Figure I.2.bii: $f_1 = f_2,\ K_2 > K_1,\ g_{12} > 0,\ g_{21} < 0$

(4.1)-(4.2), with linear growth conditions and boundary conditions $w_1(0^+) = V_1(0^+) - g_{12}$ $= (-g_{12})_+$, $w_2(0^+) = \hat{V}_2(0^+) = 0 = (-g_{21})_+$. By uniqueness result of Theorem 3.1, we deduce that $v_1 = w_1$, $v_2 = w_2$, and thus $\mathcal{S}_1 = (0,\infty)$, $\mathcal{S}_2 = \emptyset$.

b) Consider now the case where $g_{12} > 0$. We already know from Remark 4.1 that $\underline{x}_1^* > 0$, and we claim that $\underline{x}_1^* < \infty$. Otherwise, v_1 should be equal to $\hat{V}_1$. Since $v_1 \geq v_2 - g_{12} \geq \hat{V}_2 - g_{12}$, this would imply $(\hat{V}_2 - \hat{V}_1)(x) = (K_2 - K_1)x^\gamma \leq g_{12}$ for all $x > 0$, an obvious contradiction. By definition of $\underline{x}_1^*$, we have $(0, \underline{x}_1^*) \subset \mathcal{C}_1$. We shall prove actually the equality : $(0, \underline{x}_1^*) = \mathcal{C}_1$, i.e. $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$. On the other hand, the form of $\mathcal{S}_2$ will depend on the sign of g_{21}.

• *Case* : $g_{21} \geq 0$.

We shall prove that $\mathcal{C}_2 = (0,\infty)$, i.e. $\mathcal{S}_2 = \emptyset$. To this end, let us consider the function

$$w_1(x) = \begin{cases} Ax^{m_1^+} + \hat{V}_1(x), & 0 < x < x_1 \\ \hat{V}_2(x) - g_{12}, & x \geq x_1, \end{cases}$$

where the positive constants A and x_1 satisfy

$$Ax_1^{m_1^+} + \hat{V}_1(x_1) = \hat{V}_2(x_1) - g_{12} \tag{4.13}$$

$$Am_1^+ x_1^{m_1^+ - 1} + \hat{V}_1'(x_1) = \hat{V}_2'(x_1), \tag{4.14}$$

and are explicitly determined by :

$$(K_2 - K_1)x_1^\gamma = \frac{m_1^+}{m_1^+ - \gamma} g_{12} \tag{4.15}$$

$$A = (K_2 - K_1)\frac{\gamma}{m_1^+} x_1^{\gamma - m_1^+}. \tag{4.16}$$

Notice that by construction, w_1 is C^2 on $(0, x_1) \cup (x_1, \infty)$, and C^1 on x_1.

⋆ By using Lemma 3.4, we now show that w_1 is a viscosity solution to

$$\min\Big\{ rw_1 - \mathcal{L}_1 w_1 - f_1, w_1 - (\hat{V}_2 - g_{12})\Big\} = 0, \quad \text{on } (0,\infty). \tag{4.17}$$

We first check that

$$w_1(x) \geq \hat{V}_2(x) - g_{12}, \quad \forall\, 0 < x < x_1, \tag{4.18}$$

i.e.

$$G(x) := Ax^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) + g_{12} \geq 0, \quad \forall\, 0 < x < x_1.$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of G is positive, i.e. G is strictly convex. By (4.14), we have $G'(x_1) = 0$ and so G' is negative, i.e. G is strictly decreasing on $(0, x_1)$. Now, by (4.13), we have $G(x_1) = 0$ and thus G is positive on $(0, x_1)$, which proves (4.18).

By definition of w_1 on $(0, x_1)$, we have in the classical sense

$$rw_1 - \mathcal{L}_1 w_1 - f_1 = 0, \quad \text{on } (0, x_1). \tag{4.19}$$

We now check that

$$rw_1 - \mathcal{L}_1 w_1 - f_1 \;\; \geq \;\; 0, \quad \text{ on } (x_1, \infty), \tag{4.20}$$

holds true in the classical sense, and so a fortiori in the viscosity sense. By definition of w_1 on (x_1, ∞), and K_1, we have for all $x > x_1$,

$$rw_1(x) - \mathcal{L}_1 w_1(x) - f_1(x) \;\; = \;\; \frac{K_2 - K_1}{K_1} x^\gamma - r g_{12}, \quad \forall x > x_1,$$

so that (4.20) is satisfied iff $\frac{K_2 - K_1}{K_1} x_1^\gamma - r g_{12} \geq 0$ or equivalently by (4.15) :

$$\frac{m_1^+}{m_1^+ - \gamma} \;\; \geq \;\; rK_1 \; = \; \frac{r}{r - b_1\gamma + \frac{1}{2}\sigma_1^2 \gamma(1-\gamma)} \tag{4.21}$$

Now, since $\gamma < 1 < m_1^+$, and by definition of m_1^+, we have

$$\frac{1}{2}\sigma_1^2 m_1^+ (\gamma - 1) \;\; < \;\; \frac{1}{2}\sigma_1^2 m_1^+ (m_1^+ - 1) \; = \; r - b_1 m_1^+,$$

which proves (4.21) and thus (4.20).

Relations (4.13)-(4.14), (4.18)-(4.19)-(4.20) mean that conditions of Lemma 3.4 are satisfied with $\mathcal{C} = (0, x_1)$, $h = \hat{V}_2 - g_{12}$, and we thus get the required assertion (4.17).

$\star$ On the other hand, we check that

$$\hat{V}_2(x) \;\; \geq \;\; w_1(x) - g_{21}, \quad \forall x > 0, \tag{4.22}$$

which amounts to show

$$H(x) \; := \; A x^{m_1^+} + \hat{V}_1(x) - \hat{V}_2(x) - g_{21} \;\; \leq \;\; 0, \quad \forall \, 0 < x < x_1.$$

Since $A > 0$, $0 < \gamma < 1 < m_1^+$, $K_2 - K_1 > 0$, a direct derivation shows that the second derivative of H is positive, i.e. H is strictly convex. By (4.14), we have $H'(x_1) = 0$ and so H' is negative, i.e. H is strictly decreasing on $(0, x_1)$. Now, we have $H(0) = -g_{21} \leq 0$ and thus H is negative on $(0, x_1)$, which proves (4.22). Recalling that $\hat{V}_2$ is solution to $r\hat{V}_2 - \mathcal{L}_2 \hat{V}_2 - f_2 = 0$ on $(0, \infty)$, we deduce obviously from (4.22) that $\hat{V}_2$ is a classical, hence a viscosity solution to :

$$\min\Big\{ r\hat{V}_2 - \mathcal{L}_2 \hat{V}_2 - f_2, \hat{V}_2 - (w_1 - g_{21}) \Big\} \;\; = \;\; 0, \quad \text{ on } \; (0, \infty). \tag{4.23}$$

$\star$ Since $w_1(0^+) = 0 = (-g_{12})_+$, $\hat{V}_2(0^+) = 0 = (-g_{21})_+$, and w_1, $\hat{V}_2$ satisfy a linear growth condition, we deduce from (4.17), (4.23), and uniqueness to the PDE system (4.1)-(4.2), that

$$v_1 \; = \; w_1, \qquad v_2 \; = \; \hat{V}_2, \quad \text{ on } \; (0, \infty).$$

This proves $\underline{x}_1^* = x_1$, $\mathcal{S}_1 = [x_1, \infty)$ and $\mathcal{S}_2 = \emptyset$.

- *Case :* $g_{21} < 0$.

We shall prove that $\mathcal{S}_2 = (0, \bar{x}_2^*]$. To this end, let us consider the functions

$$
\begin{aligned}
w_1(x) &= \begin{cases} Ax^{m_1^+} + \hat{V}_1(x), & x < \underline{x}_1 \\ w_2(x) - g_{12}, & x \geq \underline{x}_1 \end{cases} \\
w_2(x) &= \begin{cases} w_1(x) - g_{21}, & x \leq \bar{x}_2 \\ Bx^{m_2^-} + \hat{V}_2(x), & x > \bar{x}_2, \end{cases}
\end{aligned}
$$

where the positive constants A, B, $\underline{x}_1 > \bar{x}_2$, solution to

$$
\begin{aligned}
A\underline{x}_1^{m_1^+} + \hat{V}_1(\underline{x}_1) &= w_2(\underline{x}_1) - g_{12} = B\underline{x}_1^{m_2^-} + \hat{V}_2(\underline{x}_1) - g_{12} && (4.24) \\
Am_1^+\underline{x}_1^{m_1^+-1} + \hat{V}_1'(\underline{x}_1) &= w_2'(\underline{x}_1) = Bm_2^-\underline{x}_1^{m_2^--1} + \hat{V}_2'(\underline{x}_1) && (4.25) \\
A\bar{x}_2^{m_1^+} + \hat{V}_1(\bar{x}_2) - g_{21} &= w_1(\bar{x}_2) - g_{21} = B\bar{x}_2^{m_2^-} + \hat{V}_2(\bar{x}_2) && (4.26) \\
Am_1^+\bar{x}_2^{m_1^+-1} + \hat{V}_1'(\bar{x}_2) &= w_1'(\bar{x}_2) = Bm_2^-\bar{x}_2^{m_2^--1} + \hat{V}_2'(\bar{x}_2), && (4.27)
\end{aligned}
$$

exist and are explicitly determined after some calculations by

$$
\begin{aligned}
\overline{x}_2 &= \left[\frac{-m_2^-(g_{21} + g_{12}y^{m_1^+})}{(K_1 - K_2)(\gamma - m_2^-)(1 - y^{m_1^+-\gamma})}\right]^{\frac{1}{\gamma}} && (4.28) \\
\underline{x}_1 &= \frac{\overline{x}_2}{y} && (4.29) \\
B &= \frac{(K_1 - K_2)(m_1^+ - \gamma)\underline{x}_1^{\gamma - m_2^-} + m_1^+ g_{12}\underline{x}_1^{-m_2}}{m_1^+ - m_2^-} && (4.30) \\
A &= B\underline{x}_1^{m_2^- - m_1^+} - (K_1 - K_2)\underline{x}_1^{\gamma - m_1^+} - g_{12}\underline{x}_1^{-m_1^+}, && (4.31)
\end{aligned}
$$

with y solution in $\left(0, \left(-\frac{g_{21}}{g_{12}}\right)^{\frac{1}{m_1^+}}\right)$ to the equation :

$$
\begin{aligned}
&m_1^+(\gamma - m_2^-)\left(1 - y^{m_1^+-\gamma}\right)\left(g_{12}y^{m_2^-} + g_{21}\right) \\
&+ m_2^-(m_1^+ - \gamma)\left(1 - y^{m_2^--\gamma}\right)\left(g_{12}y^{m_1^+} + g_{21}\right) = 0. && (4.32)
\end{aligned}
$$

Using (2.7), we have $y < \left(-\frac{g_{21}}{g_{12}}\right)^{\frac{1}{m_1^+}} < 1$. As such, $0 < \bar{x}_2 < \underline{x}_1$. Furthermore, by using (4.29) and the equation (4.32) satisfied by y, we may easily check that A and B are positive constants.

Notice that by construction, w_1 (resp. w_2) is C^2 on $(0, \underline{x}_1) \cup (\underline{x}_1, \infty)$ (resp. $(0, \bar{x}_2) \cup (\bar{x}_2, \infty)$) and C^1 at $\underline{x}_1$ (resp. $\bar{x}_2$).

$\star$ By using Lemma 3.4, we now show that w_i, $i = 1, 2$, is a viscosity solution to the system :

$$
\min\{rw_i - \mathcal{L}_i w_i - f_i, w_i - (w_j - g_{ij})\} = 0, \quad \text{on } (0, \infty),\ i, j = 1, 2,\ j \neq i. \quad (4.33)
$$

Since the proof is similar for both w_i, $i = 1, 2$, we only prove the result for w_1. We first check that

$$
w_1 \geq w_2 - g_{12}, \quad \forall\, 0 < x < \underline{x}_1. \quad (4.34)
$$

From the definition of w_1 and w_2 and using the fact that $g_{12}+g_{21}>0$, it is straightforward to see that

$$w_1 \geq w_2 - g_{12}, \quad \forall\, 0<x\leq \overline{x}_2. \tag{4.35}$$

Now, we need to prove that

$$G(x) := Ax^{m_1^+} + \hat{V}_1(x) - Bx^{m_2^-} - \hat{V}_2(x) + g_{12} \geq 0, \quad \forall\, \overline{x}_2 < x < \underline{x}_1. \tag{4.36}$$

We have $G(\bar{x}_2) = g_{12}+g_{21} > 0$ and $G(\underline{x}_1)=0$. Suppose that there exists some $x_0 \in (\bar{x}_2, \underline{x}_1)$ such that $G(x_0)=0$. We then deduce that there exists $x_3 \in (\bar{x}_0, \underline{x}_1)$ such that $G'(x_3)=0$. As such, the equation $G'(x)=0$ admits at least three solutions in $[\bar{x}_2, \underline{x}_1] : \{\bar{x}_2, x_3, \underline{x}_1\}$. However, a straightforward study of the function G shows that G' can take the value zero at most at two points in $(0,\infty)$. This leads to a contradiction, proving therefore (4.36).

By definition of w_1, we have in the classical sense

$$rw_1 - \mathcal{L}_1 w_1 - f = 0, \quad \text{on } (0, \underline{x}_1). \tag{4.37}$$

We now check that

$$rw_1 - \mathcal{L}_1 w_1 - f \geq 0, \quad \text{on } (\underline{x}_1, \infty) \tag{4.38}$$

holds true in the classical sense, and so a fortiori in the viscosity sense. By definition of w_1 on (x_1,∞), and K_1, we have for all $x > \underline{x}_1$,

$$H(x) := rw_1(x) - \mathcal{L}_1 w_1(x) - f(x) = \frac{K_2-K_1}{K_1}x^{\gamma} + m_2^- LBx^{m_2^-} - rg_{12}, \quad \forall x > \underline{x}_1, \tag{4.39}$$

where $L = \frac{1}{2}(\sigma_2^2 - \sigma_1^2)(m_2^- - 1) + b_2 - b_1$.

We distinguish two cases :

- First, if $L \geq 0$, the function H would be non-decreasing on $(0,\infty)$ with $\lim_{x\to 0^+} H(x) = -\infty$ and $\lim_{x\to\infty} H(x) = +\infty$. As such, it suffices to show that $H(\underline{x}_1) \geq 0$. From (4.24)-(4.25), we have

$$H(\underline{x}_1) \quad = \quad (K_2-K_1)\left[\frac{m_1^+ - m_2^-}{K_1} - (m_1^+ - \gamma)m_2^- L\right] - rg_{12} + m_1^+ m_2^- g_{12} L.$$

Using relations (4.21), (4.24), (4.25), (4.29) and the definition of m_1^+ and m_2^-, we then obtain

$$H(\underline{x}_1) \quad = \quad \frac{m_1^+(m_1^+ - m_2^-)}{K_1(m_1^+ - \gamma)} - r \geq \frac{m_1^+}{K_1(m_1^+ - \gamma)} - r \geq 0.$$

- Second, if $L < 0$, it suffices to show that

$$\frac{K_2-K_1}{K_1}x^{\gamma} - rg_{12} \geq 0, \quad \forall\, x > \underline{x}_1,$$

which is rather straightforward from (4.21) and (4.29) .

Relations (4.34), (4.37) (4.38) and the regularity of w_i, $i=1,2$, as constructed, mean that conditions of Lemma 3.4 are satisfied and we thus get the required assertion (4.33).

$\star$ Since $w_1(0^+) = 0 = (-g_{12})_+$, $w_2(0^+) = -g_{21} = (-g_{21})_+$, and w_1, $\hat{V}_2$ satisfy a linear growth condition, we deduce from (4.33) and uniqueness to the PDE system (4.1)-(4.2), that

$$v_1 \;=\; w_1, \qquad v_2 \;=\; w_2, \quad \text{on } (0,\infty).$$

This proves $\underline{x}_1^* = \underline{x}_1$, $\mathcal{S}_1 = [x_1,\infty)$ and $\bar{x}_2^* = \bar{x}_2$, $\mathcal{S}_2 = (0,\bar{x}_2]$.

4.2 Identical diffusion operators with different profit functions

In this subsection, we suppose that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, i.e. $b_1 = b_2 = b$, $\sigma_1 = \sigma_2 = \sigma > 0$. We then set $m^+ = m_1^+ = m_2^+$, $m^- = m_1^- = m_2^-$, and $\hat{X}^x = \hat{X}^{x,1} = \hat{X}^{x,2}$. Notice that in this case, the set Q_{ij}, i, $j = 1,2$, $i \neq j$, introduced in Lemma 3.3, satisfies :

$$\begin{aligned} Q_{ij} &= \{x \in \mathcal{C}_j \;: (f_j - f_i)(x) - rg_{ij} \geq 0\} \\ &\subset \;\hat{Q}_{ij} := \{x > 0 \;: (f_j - f_i)(x) - rg_{ij} \geq 0\}\,. \end{aligned} \tag{4.40}$$

Once we are given the profit functions f_i, f_j, the set $\hat{Q}_{ij}$ can be explicitly computed. Moreover, we prove in the next key Lemma that the structure of $\hat{Q}_{ij}$, when it is connected, determines the same structure for the switching region $\mathcal{S}_i$.

Lemma 4.1 *Let $i,j = 1,2$, $i \neq j$.*
1) Assume that

$$\sup_{x>0}(\hat{V}_j - \hat{V}_i)(x) \;>\; g_{ij}. \tag{4.41}$$

- *If there exists $0 < \underline{x}_{ij} < \infty$ such that*

$$\hat{Q}_{ij} \;=\; [\underline{x}_{ij},\infty), \tag{4.42}$$

then $0 < \underline{x}_i^* < \infty$ and

$$\mathcal{S}_i \;=\; [\underline{x}_i^*,\infty).$$

- *If $g_{ij} \leq 0$ and there exists $0 < \bar{x}_{ij} < \infty$ such that*

$$\hat{Q}_{ij} \;=\; (0,\bar{x}_{ij}], \tag{4.43}$$

then $0 < \bar{x}_i^* < \infty$ and

$$\mathcal{S}_i \;=\; (0,\bar{x}_i^*].$$

2) If there exist $0 < \underline{x}_{ij} < \bar{x}_{ij} < \infty$ such that

$$\hat{Q}_{ij} \;=\; [\underline{x}_{ij},\bar{x}_{ij}]. \tag{4.44}$$

Then $0 < \underline{x}_i^* < \bar{x}_i^* < \infty$ and

$$\mathcal{S}_i \;=\; [\underline{x}_i^*,\bar{x}_i^*].$$

3) If $g_{ij} \leq 0$ and $\hat{Q}_{ij} = (0,\infty)$, then $\mathcal{S}_i = (0,\infty)$ and $\mathcal{S}_j = \emptyset$.

Proof. 1) • Consider the case of condition (4.42). Since $\mathcal{S}_i \subset \hat{Q}_{ij}$ by Lemma 3.3, this implies $\underline{x}_i^* := \inf \mathcal{S}_i \geq \underline{x}_{ij} > 0$. We now claim that $\underline{x}_i^* < \infty$. On the contrary, the switching region $\mathcal{S}_i$ would be empty, and so v_i would satisfy on $(0, \infty)$:

$$rv_i - \mathcal{L}v_i - f_i \quad = \quad 0, \quad \text{on } (0, \infty).$$

Then, v_i would be on the form :

$$v_i(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_i(x), \quad x > 0.$$

Since $0 \leq v_i(0^+) < \infty$ and v_i is a nonnegative function satisfying a linear growth condition, and using the fact that $m^- < 0$ and $m^+ > 1$, we deduce that v_i should be equal to $\hat{V}_i$. Now, since we have $v_i \geq v_j - g_{ij} \geq \hat{V}_j - g_{ij}$, this would imply :

$$\hat{V}_j(x) - \hat{V}_i(x) \quad \leq \quad g_{ij}, \quad \forall x > 0.$$

This contradicts condition (4.41) and so $0 < \underline{x}_i^* < \infty$.

By definition of $\underline{x}_i^*$, we already know that $(0, \underline{x}_i^*) \subset \mathcal{C}_i$. We prove actually the equality, i.e. $\mathcal{S}_i = [\underline{x}_i^*, \infty)$ or $v_i(x) = v_j(x) - g_{ij}$ for all $x \geq \underline{x}_i^*$. Consider the function

$$w_i(x) \quad = \quad \begin{cases} v_i(x), & 0 < x < \underline{x}_i^* \\ v_j(x) - g_{ij}, & x \geq \underline{x}_i^* \end{cases}$$

We now check that w_i is a viscosity solution of

$$\min\{rw_i - \mathcal{L}w_i - f_i \ , \ w_i - (v_j - g_{ij})\} \quad = \quad 0 \quad \text{on} \quad (0, \infty). \tag{4.45}$$

From Theorem 3.2, the function w_i is C^1 on $(0, \infty)$ and in particular at $\underline{x}_i^*$ where $w_i'(\underline{x}_i^*) = v_i'(\underline{x}_i^*) = v_j'(\underline{x}_i^*)$. We also know that $w_i = v_i$ is C^2 on $(0, \underline{x}_i^*) \subset \mathcal{C}_i$, and satisfies $rw_i - \mathcal{L}w_i - f_i = 0$, $w_i \geq (v_j - g_{ij})$ on $(0, \underline{x}_i^*)$. Hence, from Lemma 3.4, we only need to check the viscosity supersolution property of w_i to :

$$rw_i - \mathcal{L}w_i - f_i \quad \geq \quad 0, \quad \text{on} \quad (\underline{x}_i^*, \infty). \tag{4.46}$$

For this, take some point $\bar{x} > \underline{x}_i^*$ and some smooth test function φ s.t. $\bar{x}$ is a local minimum of $w_i - \varphi$. Then, $\bar{x}$ is a local minimum of $v_j - (\varphi + g_{ij})$, and by the viscosity solution property of v_j to its Bellman PDE, we have

$$rv_j(\bar{x}) - \mathcal{L}\varphi(\bar{x}) - f_j(\bar{x}) \quad \geq \quad 0.$$

Now, since $\underline{x}_i^* \geq \underline{x}_{ij}$, we have $\bar{x} > \underline{x}_{ij}$ and so by (4.42), $\bar{x} \in \hat{Q}_{ij}$. Hence,

$$(f_j - f_i)(\bar{x}) - rg_{ij} \quad \geq \quad 0.$$

By adding the two previous inequalities, we also obtain the required supersolution inequality :

$$rw_i(\bar{x}) - \mathcal{L}\varphi(\bar{x}) - f_i(\bar{x}) \quad \geq \quad 0,$$

and so (4.45) is proved.

Since $w_i(0^+) = v_i(0^+)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (4.45), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \geq \underline{x}_i^*$, which shows that $\mathcal{S}_i = [\underline{x}_i^*, \infty)$.

• The case of condition (4.43) is dealt by same arguments as above : we first observe that $0 < \bar{x}_i^* := \sup \mathcal{S}_i < \infty$ under (4.41), and then show with Lemma 3.4 that the function

$$w_i(x) \quad = \quad \begin{cases} v_j(x) - g_{ij}, & 0 < x < \bar{x}_i^* \\ v_i(x), & x \geq \bar{x}_i^* \end{cases}$$

is a viscosity solution to

$$\min\{rw_i - \mathcal{L}w_i - f_i \ , \ w_i - (v_j - g_{ij})\} \quad = \quad 0 \quad \text{on} \quad (0, \infty).$$

Then, under the condition that $g_{ij} \leq 0$, we see that $g_{ji} > 0$ by (2.7), and so $v_i(0^+) = -g_{ij}$ $= (-g_{ji})_+ - g_{ij} = v_j(0^+) - g_{ij} = w_i(0^+)$. From uniqueness of viscosity solution to PDE (4.45), we conclude that $v_i = w_i$, and so $\mathcal{S}_i = (0, \bar{x}_i^*]$.

2) By Lemma 3.3 and (4.40), the condition (4.44) implies $0 < \underline{x}_{ij} \leq \underline{x}_i^* \leq \bar{x}_i^* \leq \bar{x}_{ij} < \infty$. We claim that $\underline{x}_i^* < \bar{x}_i^*$. Otherwise, $\mathcal{S}_i = \{\bar{x}_i^*\}$ and v_i would satisfy $rv_i - \mathcal{L}v_i - f_i = 0$ on $(0, \bar{x}_i^*) \cup (\bar{x}_i^*, \infty)$. By continuity and smooth-fit condition of v_i at $\bar{x}_i^*$, this implies that v_i satisfies actually

$$rv_i - \mathcal{L}v_i - f_i \quad = \quad 0, \quad x \in (0, \infty),$$

and so is in the form :

$$v_i(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_i(x), \quad x \in (0, \infty)$$

Since $0 \leq v_i(0^+) < \infty$ and v_i is nonnegative function satisfying a linear growth condition, this implies $A = B = 0$. Therefore, v_i is equal to $\hat{V}_i$, which also means that $\mathcal{S}_i = \emptyset$, a contradiction.

We now prove that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$. Let us consider the function

$$w_i(x) \quad = \quad \begin{cases} v_i(x), & x \in (0, \underline{x}_i^*) \cup (\bar{x}_i^*, \infty) \\ v_j(x) - g_{ij}, & x \in [\underline{x}_i^*, \bar{x}_i^*], \end{cases}$$

which is C^1 on $(0, \infty)$ and in particular on $\underline{x}_i^*$ and $\bar{x}_i^*$ from Theorem 3.2. Hence, by similar arguments as in case 1), using Lemma 3.4, we then show that w_i is a viscosity solution of

$$\min\{rw_i - \mathcal{L}w_i - f_i \ , \ w_i - (v_j - g_{ij})\} \quad = \quad 0. \tag{4.47}$$

Since $w_i(0^+) = v_i(0^+)$ and w_i satisfies a linear growth condition, and from uniqueness of viscosity solution to PDE (4.47), we deduce that w_i is equal to v_i. In particular, we have $v_i(x) = v_j(x) - g_{ij}$ for $x \in [\underline{x}_i^*, \bar{x}_i^*]$, which shows that $\mathcal{S}_i = [\underline{x}_i^*, \bar{x}_i^*]$.

3) Suppose that $g_{ij} \leq 0$ and $\hat{Q}_{ij} = (0,\infty)$. We shall prove that $\mathcal{S}_i = (0,\infty)$ and $\mathcal{S}_j = \emptyset$. To this end, we consider the smooth functions $w_i = \hat{V}_j - g_{ij}$ and $w_j = \hat{V}_j$. Then, recalling the ode satisfied by $\hat{V}_j$, and inequality (2.7), we get :

$$rw_j - \mathcal{L}w_j - f_j \;=\; 0, \qquad w_j - (w_i - g_{ji}) \;=\; g_{ij} + g_{ji} \;\geq\; 0.$$

Therefore w_j is a smooth (and so a viscosity) solution to :

$$\min\left[rw_j - \mathcal{L}w_j - f_j, w_j - (w_i - g_{ji})\right] \;=\; 0 \qquad \text{on } (0,\infty).$$

On the other hand, by definition of $\hat{Q}_{ij}$, which is supposed equal to $(0,\infty)$, we have :

$$\begin{aligned} rw_i(x) - \mathcal{L}w_i(x) - f_i(x) &= r\hat{V}_j(x) - \mathcal{L}\hat{V}_j(x) - f_j(x) + f_j(x) - f_i(x) - rg_{ij} \\ &= f_j(x) - f_i(x) - rg_{ij} \;\geq\; 0, \quad \forall x > 0. \end{aligned}$$

Moreover, by construction we have $w_i = w_j - g_{ij}$. Therefore w_i is a smooth (and so a viscosity) solution to :

$$\min\left[rw_i - \mathcal{L}w_i - f_i, w_i - (w_j - g_{ij})\right] \;=\; 0 \qquad \text{on } (0,\infty).$$

Notice also that $g_{ji} > 0$ by (2.7) and since $g_{ij} \leq 0$. Hence, $w_i(0^+) = -g_{ij} = (-g_{ij})_+ = v_i(0^+)$, $w_j(0^+) = 0 = (-g_{ji})_+ = v_j(0^+)$. From uniqueness result of Theorem 3.1, we deduce that $v_i = w_i$, $v_j = w_j$, which proves that $\mathcal{S}_i = (0,\infty)$, $\mathcal{S}_j = \emptyset$. □

We shall now provide explicit solutions to the switching problem under general assumptions on the running profit functions, which include several interesting cases for applications :

(HF) There exists $\hat{x} \in \mathbb{R}_+$ s.t the function $F := f_2 - f_1$ is decreasing on $(0,\hat{x})$, increasing on $[\hat{x},\infty)$, and $F(\infty) := \lim_{x\to\infty} F(x) > 0$, $g_{12} > 0$.

Under **(HF)**, there exists some $\bar{x} \in \mathbb{R}_+$ ($\bar{x} > \hat{x}$ if $\hat{x} > 0$ and $\bar{x} = 0$ if $\hat{x} = 0$) from which F is positive : $F(x) > 0$ for $x > \bar{x}$. Economically speaking, condition **(HF)** means that the profit in regime 2 is "better" than profit in regime 1 from a certain level $\bar{x}$, eventually equal to zero, and the improvement becomes then better and better. Moreover, since profit in regime 2 is better than the one in regime 1, it is natural to assume that the corresponding switching cost g_{12} from regime 1 to 2 should be positive. However, we shall consider both cases where g_{21} is positive and nonpositive. Notice that $F(\hat{x}) < 0$ if $\hat{x} > 0$, $F(\hat{x}) = 0$ if $\hat{x} = 0$, and we do not assume necessarily $F(\infty) = \infty$.

Example 4.1 A typical example of different running profit functions satisfying **(HF)** is given by

$$f_i(x) \;=\; k_i x^{\gamma_i}, \;\; i = 1,2, \quad \text{with } 0 < \gamma_1 < \gamma_2 < 1, \;\; k_1 \in \mathbb{R}_+, \; k_2 > 0. \tag{4.48}$$

In this case, $\hat{x} = (\frac{k_1\gamma_1}{k_2\gamma_2})^{\frac{1}{\gamma_2-\gamma_1}}$, and $\lim_{x\to\infty} F(x) = \infty$.

Another example of profit functions of interest in applications is the case where the profit function in regime 1 is $f_1 = 0$, and the other f_2 is increasing. In this case, assumption **(HF)** is satisfied with $\hat{x} = 0$.

The next proposition states the form of the switching regions in regimes 1 and 2, depending on the parameter values.

Proposition 4.1 *Assume that* **(HF)** *holds.*
1) (i) If $rg_{12} \geq F(\infty)$, then $\underline{x}_1^* = \infty$, i.e. $\mathcal{S}_1 = \emptyset$.
(ii) If $rg_{12} < F(\infty)$, then $\underline{x}_1^* \in (0,\infty)$ and $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$.

2) (i) If $rg_{21} \geq -F(\hat{x})$, then $\mathcal{S}_2 = \emptyset$.
(ii) If $0 < rg_{21} < -F(\hat{x})$, then $0 < \underline{x}_2^* < \bar{x}_2^* < \underline{x}_1^*$, and $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$.
(iii) If $g_{21} \leq 0$ and $-F(\infty) < rg_{21} < -F(\hat{x})$, then $0 = \underline{x}_2^* < \bar{x}_2^* < \underline{x}_1^*$, and $\mathcal{S}_2 = (0, \bar{x}_2^*]$.
(iv) If $rg_{21} \leq -F(\infty)$, then $\mathcal{S}_2 = (0,\infty)$.

Proof. 1) From Lemma 3.3, we have

$$\hat{Q}_{12} = \{x > 0 \ : F(x) \geq rg_{12}\}. \tag{4.49}$$

Since $g_{12} > 0$, and $f_i(0) = 0$, we have $F(0) = 0 < rg_{12}$. Under **(HF)**, we then distinguish the two following cases :
(i) If $rg_{12} \geq F(\infty)$, then $\hat{Q}_{12} = \emptyset$, and so by Lemma 3.3 and (4.40), $\mathcal{S}_1 = \emptyset$.
(ii) If $rg_{12} < F(\infty)$, then there exists $\hat{x}_{12} \in (0,\infty)$ such that

$$\hat{Q}_{12} = [\underline{x}_{12}, \infty). \tag{4.50}$$

Moreover, since

$$(\hat{V}_2 - \hat{V}_1)(x) = E\left[\int_0^\infty e^{-rt} F(\hat{X}_t^x) dt\right], \quad \forall x > 0,$$

and F is lower-bounded, we obtain by Fatou's lemma :

$$\liminf_{x\to\infty} (\hat{V}_2 - \hat{V}_1)(x) \geq E\left[\int_0^\infty e^{-rt} F(\infty) dt\right] = \frac{F(\infty)}{r} > g_{12}.$$

Hence, conditions (4.41)-(4.42) with $i = 1$, $j = 2$, are satisfied, and we obtain the first assertion by Lemma 4.1 1).
2) From Lemma 3.3, we have

$$\hat{Q}_{21} = \{x > 0 \ : -F(x) \geq rg_{21}\}. \tag{4.51}$$

Under **(HF)**, we distinguish the following cases :

► (i1) If $rg_{21} > -F(\hat{x})$, then $\hat{Q}_{21} = \emptyset$, and so $\mathcal{S}_2 = \emptyset$.

(i2) If $rg_{21} = -F(\hat{x})$, then either $\hat{x} = 0$ and so $\mathcal{S}_2 = \hat{Q}_{21} = \emptyset$, or $\hat{x} > 0$, and so $\hat{Q}_{21} = \{\hat{x}\}$, $\mathcal{S}_2 \subset \{\hat{x}\}$. In this last case, v_2 satisfies $rv_2 - \mathcal{L}v_2 - f_2 = 0$ on $(0, \hat{x}) \cup (\hat{x}, \infty)$. By continuity and smooth-fit condition of v_2 at $\hat{x}$, this implies that v_2 satisfies actually

$$rv_2 - \mathcal{L}v_2 - f_2 \quad = \quad 0, \quad x \in (0, \infty),$$

and so is in the form :

$$v_2(x) \quad = \quad Ax^{m^+} + Bx^{m^-} + \hat{V}_2(x), \quad x \in (0, \infty)$$

Recalling that $0 \leq v_2(0^+) < \infty$ and v_2 is a nonnegative function satisfying a linear growth condition, this implies $A = B = 0$. Therefore, v_2 is equal to $\hat{V}_2$, which also means that $\mathcal{S}_2 = \emptyset$.

▶ If $rg_{21} < -F(\hat{x})$, we need to distinguish three subcases depending on g_{21} :

- If $g_{21} > 0$, then there exist $0 < \underline{x}_{21} < \hat{x} < \bar{x}_{21} < \infty$ such that

$$\hat{Q}_{21} \quad = \quad [\underline{x}_{21}, \bar{x}_{21}]. \tag{4.52}$$

  We then conclude with Lemma 4.1 2) for $i = 2$, $j = 1$.

- If $g_{21} \leq 0$ with $rg_{21} > -F(\infty)$, then there exists $\bar{x}_{21} < \infty$ s.t.

$$\hat{Q}_{21} \quad = \quad (0, \bar{x}_{21}].$$

  Moreover, we clearly have $\sup_{x>0}(\hat{V}_1 - \hat{V}_2)(x) > (\hat{V}_1 - \hat{V}_2)(0) = 0 \geq g_{21}$. Hence, conditions (4.41) and (4.43) with $i = 2$, $j = 1$ are satisfied, and we deduce from Lemma 4.1 1) that $\mathcal{S}_2 = (0, \bar{x}_2^*]$ with $0 < \bar{x}_2^* < \infty$.

- If $rg_{21} \leq -F(\infty)$, then $\hat{Q}_{21} = (0, \infty)$, and we deduce from Lemma 4.1 3) for $i = 2$, $j = 1$, that $\mathcal{S}_2 = (0, \infty)$.

Finally, in the two above subcases when $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$ or $(0, \bar{x}_2^*]$, we notice that $\bar{x}_2^* < \underline{x}_1^*$ since $\mathcal{S}_2 \subset \mathcal{C}_1 = (0, \infty) \setminus \mathcal{S}_1$, which is equal, from 1), either to $(0, \infty)$ when $\underline{x}_1^* = \infty$ or to $(0, \underline{x}_1^*)$. □

Remark 4.2 In our viscosity solutions approach, the structure of the switching regions is derived from the smooth fit property of the value functions, uniqueness result for viscosity solutions and Lemma 3.3. This contrasts with the classical verification approach where the structure of switching regions should be guessed ad-hoc and checked a posteriori by a verification argument.

Economic interpretation.
The previous proposition shows that, under **(HF)**, the switching region in regime 1 has two forms depending on the size of its corresponding positive switching cost : If g_{12} is larger than the "maximum net" profit $F(\infty)$ that one can expect by changing of regime (case 1) (i), which may occur only if $F(\infty) < \infty$), then one has no interest to switch of regime, and one always stay in regime 1, i.e. $\mathcal{C}_1 = (0, \infty)$. However, if this switching cost is smaller

than $F(\infty)$ (case 1) (ii), which always holds true when $F(\infty) = \infty$), then there is some positive threshold from which it is optimal to change of regime.

The structure of the switching region in regime 2 exhibits several different forms depending on the sign and size of its corresponding switching cost g_{21} with respect to the values $-F(\infty) < 0$ and $-F(\hat{x}) \geq 0$. If g_{21} is nonnegative larger than $-F(\hat{x})$ (case 2) (i)), then one has no interest to switch of regime, and one always stay in regime 2, i.e. $\mathcal{C}_2 = (0, \infty)$. If g_{21} is positive, but not too large (case 2) (ii)), then there exists some bounded closed interval, which is not a neighborhood of zero, where it is optimal to change of regime. Finally, when the switching cost g_{21} is negative, it is optimal to switch to regime 1 at least for small values of the state. Actually, if the negative cost g_{21} is larger than $-F(\infty)$ (case 2) (iii), which always holds true for negative cost when $F(\infty) = \infty$), then the switching region is a bounded neighborhood of 0. Moreover, if the cost is negative large enough (case 2) (iv), which may occur only if $F(\infty) < \infty$), then it is optimal to change of regime for every values of the state.

By combining the different cases for regimes 1 and 2, and observing that case 2) (iv) is not compatible with case 1) (ii) by (2.7), we then have a priori seven different forms for both switching regions. These forms reduce actually to three when $F(\infty) = \infty$. The various structures of the switching regions are depicted in Figure II.

Finally, we complete results of Proposition 4.1 by providing the explicit solutions for the value functions and the corresponding boundaries of the switching regions in the seven different cases depending on the model parameter values.

Theorem 4.2 *Assume that* **(HF)** *holds.*
1) *If* $rg_{12} < F(\infty)$ *and* $rg_{21} \geq -F(\hat{x})$, *then*

$$\begin{aligned} v_1(x) &= \begin{cases} Ax^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \\ v_2(x) &= \hat{V}_2(x), \end{aligned}$$

where the constants A *and* $\underline{x}_1^*$ *are determined by the continuity and smooth-fit conditions of* v_1 *at* $\underline{x}_1^*$ *:*

$$\begin{aligned} A(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= \hat{V}_2(\underline{x}_1^*) - g_{12} \\ Am^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= \hat{V}_2'(\underline{x}_1^*). \end{aligned}$$

In regime 1, it is optimal to switch to regime 2 whenever the state process X *exceeds the threshold* $\underline{x}_1^*$, *while when we are in regime 2, it is optimal never to switch.*
2) *If* $rg_{12} < F(\infty)$ *and* $0 < rg_{21} < -F(\hat{x})$, *then*

$$v_1(x) = \begin{cases} A_1x^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases} \tag{4.53}$$

$$v_2(x) = \begin{cases} A_2x^{m^+} + \hat{V}_2(x), & x < \underline{x}_2^* \\ v_1(x) - g_{21}, & \underline{x}_2^* \leq x \leq \bar{x}_2^* \\ B_2x^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases} \tag{4.54}$$

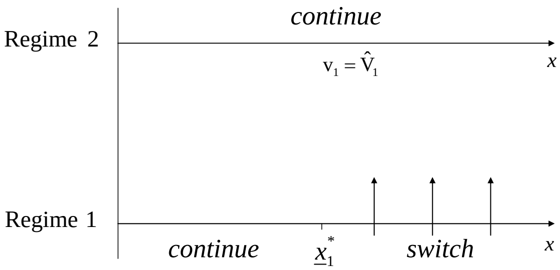

Figure II.1: $rg_{12} < F(\infty)$, $rg_{21} \geq -F(\hat{x})$

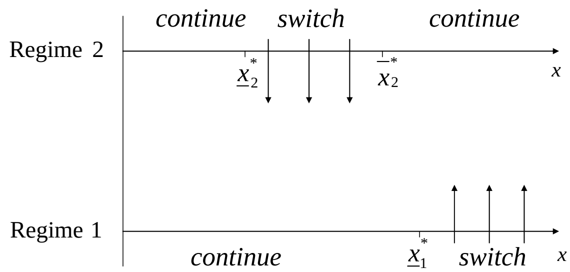

Figure II.2: $rg_{12} < F(\infty)$, $0 < rg_{21} < -F(\hat{x})$

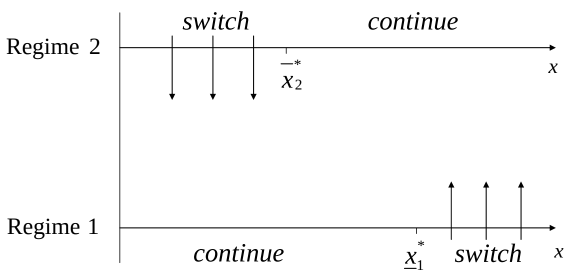

Figure II.3: $rg_{12} < F(\infty)$, $g_{21} \leq 0$, $-F(\infty) < rg_{21} < -F(\hat{x})$

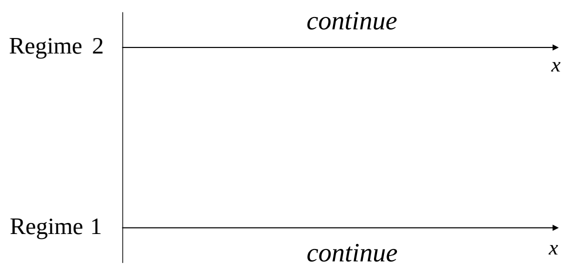

Figure II.4: $rg_{12} \geq F(\infty)$, $rg_{21} > -F(\hat{x})$

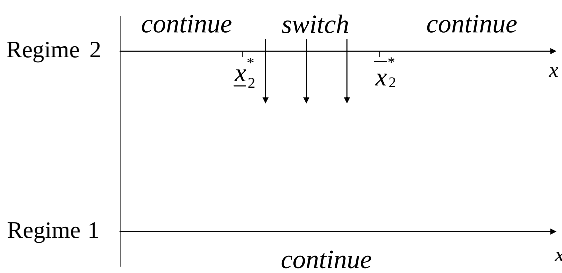

Figure II.5: $rg_{12} \geq F(\infty)$, $0 < rg_{21} < -F(\hat{x})$

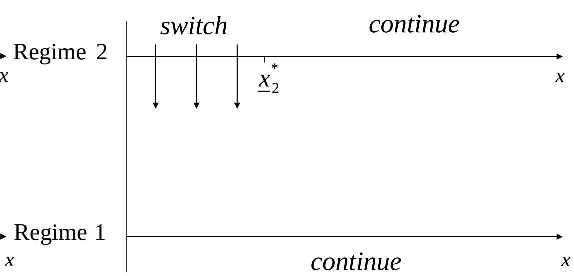

Figure II.6: $rg_{12} \geq F(\infty)$, $g_{21} \leq 0$, $F(\infty) < rg_{21} < -F(\hat{x})$

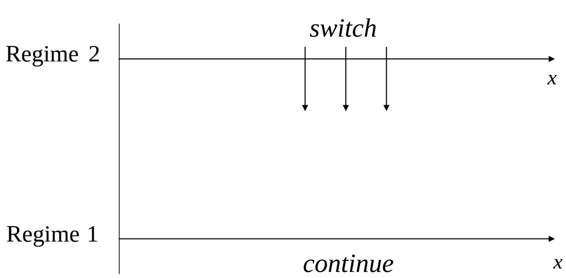

Figure II.7: $rg_{12} \geq F(\infty)$, $g_{21} \leq -F(\infty)$

where the constants A_1 and $\underline{x}_1^*$ are determined by the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$, and the constants A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$:

$$
\begin{aligned}
A_1(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= B_2(\underline{x}_1^*)^{m^-} + \hat{V}_2(\underline{x}_1^*) - g_{12} &\quad (4.55)\\
A_1 m^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= B_2 m^-(\underline{x}_1^*)^{m^--1} + \hat{V}_2'(\underline{x}_1^*) &\quad (4.56)\\
A_2(\underline{x}_2^*)^{m^+} + \hat{V}_2(\underline{x}_2^*) &= A_1(\underline{x}_2^*)^{m^+} + \hat{V}_1(\underline{x}_2^*) - g_{21} &\quad (4.57)\\
A_2 m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_2'(\underline{x}_2^*) &= A_1 m^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_1'(\underline{x}_2^*) &\quad (4.58)\\
A_1(\bar{x}_2^*)^{m^+} + \hat{V}_1(\bar{x}_2^*) - g_{21} &= B_2(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) &\quad (4.59)\\
A_1 m^+(\bar{x}_2^*)^{m^+-1} + \hat{V}_1'(\bar{x}_2^*) &= B_2 m^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). &\quad (4.60)
\end{aligned}
$$

In regime 1, it is optimal to switch to regime 2 whenever the state process X exceeds the threshold $\underline{x}_1^*$, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $\underline{x}_2^*$ and $\bar{x}_2^*$.

3) *If $rg_{12} < F(\infty)$ and $g_{21} \leq 0$ with $-F(\infty) < rg_{21} < -F(\hat{x})$, then*

$$
v_1(x) = \begin{cases} Ax^{m^+} + \hat{V}_1(x), & x < \underline{x}_1^* \\ v_2(x) - g_{12}, & x \geq \underline{x}_1^* \end{cases}
$$

$$
v_2(x) = \begin{cases} v_1(x) - g_{21}, & 0 < x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases}
$$

where the constants A and $\underline{x}_1^*$ are determined by the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$, and the constants B and $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\bar{x}_2^*$:

$$
\begin{aligned}
A(\underline{x}_1^*)^{m^+} + \hat{V}_1(\underline{x}_1^*) &= B(\underline{x}_1^*)^{m^-} + \hat{V}_2(\underline{x}_1^*) - g_{12}\\
Am^+(\underline{x}_1^*)^{m^+-1} + \hat{V}_1'(\underline{x}_1^*) &= Bm^-(\underline{x}_1^*)^{m^--1} + \hat{V}_2'(\underline{x}_1^*)\\
A(\bar{x}_2^*)^{m^+} + \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*)\\
Am^+(\bar{x}_2^*)^{m^+-1} + \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*).
\end{aligned}
$$

4) *If $rg_{12} \geq F(\infty)$ and $rg_{21} \geq -F(\hat{x})$, then $v_1 = \hat{V}_1$, $v_2 = \hat{V}_2$. It is optimal never to switch in both regimes 1 and 2.*

5) *If $rg_{12} \geq F(\infty)$ and $0 < rg_{21} < -F(\hat{x})$, then*

$$
\begin{aligned}
v_1(x) &= \hat{V}_1(x)\\
v_2(x) &= \begin{cases} Ax^{m^+} + \hat{V}_2(x), & x < \underline{x}_2^* \\ v_1(x) - g_{21}, & \underline{x}_2^* \leq x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases}
\end{aligned}
$$

where the constants A, B, $\underline{x}_2^*$, $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$:

$$
A(\underline{x}_2^*)^{m^+} + \hat{V}_2(\underline{x}_2^*) = \hat{V}_1(\underline{x}_2^*) - g_{21}
$$

$$\begin{aligned} Am^+(\underline{x}_2^*)^{m^+-1} + \hat{V}_2'(\underline{x}_2^*) &= \hat{V}_1'(\underline{x}_2^*) \\ \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) \\ \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). \end{aligned}$$

In regime 1, it is optimal never to switch, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies between $\underline{x}_2^*$ and $\bar{x}_2^*$.

6) *If $rg_{12} \geq F(\infty)$ and $g_{21} \leq 0$ with $-F(\infty) < rg_{21} < -F(\hat{x})$, then*

$$\begin{aligned} v_1(x) &= \hat{V}_1(x) \\ v_2(x) &= \begin{cases} v_1(x) - g_{21}, & 0 < x \leq \bar{x}_2^* \\ Bx^{m^-} + \hat{V}_2(x), & x > \bar{x}_2^*, \end{cases} \end{aligned}$$

where the constants B and $\bar{x}_2^*$ are determined by the continuity and smooth-fit conditions of v_2 at $\bar{x}_2^*$:

$$\begin{aligned} \hat{V}_1(\bar{x}_2^*) - g_{21} &= B(\bar{x}_2^*)^{m^-} + \hat{V}_2(\bar{x}_2^*) \\ \hat{V}_1'(\bar{x}_2^*) &= Bm^-(\bar{x}_2^*)^{m^--1} + \hat{V}_2'(\bar{x}_2^*). \end{aligned}$$

In regime 1, it is optimal never to switch, while when we are in regime 2, it is optimal to switch to regime 1 whenever the state process lies below $\bar{x}_2^*$.

7) *If $rg_{12} \geq F(\infty)$ and $rg_{21} \leq -F(\infty)$, then $v_1 = \hat{V}_1$ and $v_2 = v_1 - g_{12}$. In regime 1, it is optimal never to switch, while when we are in regime 2, it is always optimal to switch to regime 1.*

Proof. We prove the result only for the case **2)** since the other cases are dealt similarly and are even simpler. This case **2)** corresponds to the combination of cases 1) (ii) and 2) (ii) in Proposition 4.1. We then have $\mathcal{S}_1 = [\underline{x}_1^*, \infty)$, which means that $v_1 = v_2 - g_{12}$ on $[\underline{x}_1^*, \infty)$ and v_1 is solution to $rv_1 - \mathcal{L}v_1 - f_1 = 0$ on $(0, \underline{x}_1^*)$. Since $0 \leq v_1(0^+) < \infty$, v_1 should have the form expressed in (4.53). Moreover, $\mathcal{S}_2 = [\underline{x}_2^*, \bar{x}_2^*]$, which means that $v_2 = v_1 - g_{21}$ on $[\underline{x}_2^*, \bar{x}_2^*]$, and v_2 satisfies on $\mathcal{C}_2 = (0, \underline{x}_2^*) \cup (\bar{x}_2^*, \infty) : rv_2 - \mathcal{L}v_2 - f_2 = 0$. Recalling again that $0 \leq v_2(0^+) < \infty$ and v_2 satisfies a linear growth condition, we deduce that v_2 has the form expressed in (4.54). Finally, the constants A_1, $\underline{x}_1^*$, which characterize completely v_1, and the constants A_2, B_2, $\underline{x}_2^*$, $\bar{x}_2^*$, which characterize completely v_2, are determined by the six relations (4.55)-(4.56)-(4.57)-(4.58)-(4.59)-(4.60) resulting from the continuity and smooth-fit conditions of v_1 at $\underline{x}_1^*$ and v_2 at $\underline{x}_2^*$ and $\bar{x}_2^*$, and recalling that $\bar{x}_2^* < \underline{x}_1^*$. □

Remark 4.3 In the classical approach, for instance in the case **2)**, we construct a priori a candidate solution in the form (4.53)-(4.54), and we have to check the existence of a sixtuple solution to (4.55)-(4.56)-(4.57)-(4.58)-(4.59)-(4.60), which may be somewhat tedious! Here, by the viscosity solutions approach, and since we already state the smooth-fit C^1 property of the value functions, we know a priori the existence of a sixtuple solution to (4.55)-(4.56)-(4.57)-(4.58)-(4.59)-(4.60).

Appendix: proof of comparison principle

In this section, we prove a comparison principle for the system of variational inequalities (3.8). The comparison result in [10] for switching problems in finite horizon does not apply in our context. Inspired by [8], we first produce some suitable perturbation of viscosity supersolution to deal with the switching obstacle, and then follow the general viscosity solution technique, see e.g. [3].

Theorem 4.3 *Suppose u_i, $i \in \mathbb{I}_d$, are continuous viscosity subsolutions to the system of variational inequalities* (3.8) *on* $(0,\infty)$*, and w_i, $i \in \mathbb{I}_d$, are continuous viscosity supersolutions to the system of variational inequalities* (3.8) *on* $(0,\infty)$*, satisfying the boundary conditions $u_i(0^+) \leq w_i(0^+)$, $i \in I_d$, and the linear growth condition :*

$$|u_i(x)| + |w_i(x)| \quad \leq \quad C_1 + C_2 x, \quad \forall x \in (0,\infty), \ i \in \mathbb{I}_d, \tag{A.1}$$

for some positive constants C_1 and C_2. Then,

$$u_i \quad \leq \quad w_i, \quad \textit{on } (0,\infty), \ \ \forall i \in \mathbb{I}_d.$$

Proof. Step 1. Let u_i and w_i, $i \in \mathbb{I}_d$, as in Theorem 4.3. We first construct strict supersolutions to the system (3.8) with suitable perturbations of w_i, $i \in \mathbb{I}_d$. We set

$$h(x) \quad = \quad C_1' + C_2' x^p, \quad x > 0,$$

where C_1', $C_2' > 0$ and $p > 1$ are positive constants to be determined later. We then define for all $\lambda \in (0,1)$, the continuous functions on $(0,\infty)$ by :

$$w_i^\lambda = (1-\lambda) w_i + \lambda (h + \alpha_i), \quad i \in \mathbb{I}_d,$$

where $\alpha_i = \min_{j \neq i} g_{ji}$. We then see that for all $\lambda \in (0,1)$, $i \in \mathbb{I}_d$:

$$\begin{aligned} w_i^\lambda - \max_{j\neq i}(w_j^\lambda - g_{ij}) &= \lambda\alpha_i + (1-\lambda)w_i - \max_{j\neq i}[(1-\lambda)(w_j - g_{ij}) + \lambda\alpha_j - \lambda g_{ij}] \\ &\geq (1-\lambda)[w_i - \max_{j\neq i}(w_j - g_{ij})] + \lambda\left(\alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j)\right) \\ &\geq \lambda \min_{i\in\mathbb{I}_d}\left(\alpha_i + \min_{j\neq i}(g_{ij}-\alpha_j)\right) \\ &\geq \lambda\underline{\nu} \end{aligned} \tag{A.2}$$

where $\underline{\nu} := \min_{i\in\mathbb{I}_d}\left[\alpha_i + \min_{j\neq i}(g_{ij}-\alpha_j)\right]$ is a constant independent of i. We now check that $\underline{\nu}$ > 0, i.e. $\nu_i := \alpha_i + \min_{j\neq i}(g_{ij} - \alpha_j) > 0$, $\forall i \in \mathbb{I}_d$. Indeed, fix $i \in \mathbb{I}_d$, and let $k \in \mathbb{I}_d$ such that $\min_{j\neq i}(g_{ij} - \alpha_j) = g_{ik} - \alpha_k$ and set $\underline{i}$ such that $\alpha_i = \min_{j\neq i} g_{ji} = g_{\underline{i}i}$. We then have

$$\nu_i \quad = \quad g_{\underline{i}i} + g_{ik} - \min_{j\neq k} g_{jk} \ > \ g_{\underline{i}k} - \min_{j\neq k} g_{jk} \ \geq \ 0,$$

by (2.6) and thus $\underline{\nu} > 0$.

By definition of the Fenchel Legendre in (2.5), and by setting $\tilde{f}(1) = \max_{i\in\mathbb{I}_d} \tilde{f}_i(1)$, we have for all $i \in \mathbb{I}_d$,

$$f_i(x) \;\; \leq \;\; \tilde{f}(1) + x \; \leq \tilde{f}(1) + 1 + x^p, \quad \forall x > 0.$$

Moreover, recalling that $r > b := \max_i b_i$, we can choose $p > 1$ s.t.

$$\rho \; := \; r - pb - \frac{1}{2}\sigma^2 p(p-1) \;\; > \;\; 0,$$

where we set $\sigma := \max_i \sigma_i > 0$. By choosing

$$C_1' \; \geq \; \frac{2 + \tilde{f}(1)}{r} - \min_i \alpha_i, \qquad C_2' \; \geq \; \frac{1}{\rho},$$

we then have for all $i \in \mathbb{I}_d$,

$$\begin{aligned} rh(x) - \mathcal{L}_i h(x) - f_i(x) &= rC_1' + C_2' x^p[r - pb_i - \frac{1}{2}\sigma_i^2 p(p-1)] - f_i(x) \\ &\geq rC_1' + \rho C_2' x^p - f_i(x) \\ &\geq 1, \qquad \forall x > 0. \end{aligned} \tag{A.3}$$

From (A.2) and (A.3), we then deduce that for all $i \in \mathbb{I}_d$, $\lambda \in (0,1)$, w_i^λ is a supersolution to

$$\min\Big\{ rw_i^\lambda - \mathcal{L}_i w_i^\lambda - f_i, w_i^\lambda - \max_{j\neq i}(w_j^\lambda - g_{ij}) \Big\} \;\; \geq \;\; \lambda\delta, \quad \text{ on } \; (0,\infty), \tag{A.4}$$

where $\delta = \underline{\nu} \wedge 1 > 0$.

Step 2. In order to prove the comparison principle, it suffices to show that for all $\lambda \in (0,1)$:

$$\max_{j\in\mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;\; \leq \;\; 0$$

since the required result is obtained by letting λ to 0. We argue by contradiction and suppose that there exists some $\lambda \in (0,1)$ and $i \in \mathbb{I}_d$ s.t.

$$\theta \; := \; \max_{j\in\mathbb{I}_d} \sup_{(0,+\infty)} (u_j - w_j^\lambda) \;\; = \;\; \sup_{(0,+\infty)} (u_i - w_i^\lambda) \; > \; 0. \tag{A.5}$$

From the linear growth condition (A.1), and since $p > 1$, we observe that $u_i(x) - w_i^\lambda(x)$ goes to $-\infty$ when x goes to infinity. By choosing also $C_1' \geq \max_i w_i(0^+)$, we then have $u_i(0^+) - w_i^\lambda(0^+) = u_i(0^+) - w_i(0^+) + \lambda(w_i(0^+) - C_1') \leq 0$. Hence, by continuity of the functions u_i and w_i^λ, there exists $x_0 \in (0,\infty)$ s.t.

$$\theta \;\; = \;\; u_i(x_0) - w_i^\lambda(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned} \Phi_\varepsilon(x,y) &= u_i(x) - w_i^\lambda(y) - \phi_\varepsilon(x,y), \\ \phi_\varepsilon(x,y) &= \frac{1}{4}|x - x_0|^4 + \frac{1}{2\varepsilon}|x-y|^2, \end{aligned}$$

for all $x, y \in (0, \infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (0,\infty)^2$, which converges (up to a subsequence) to (x_0, x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} \quad = \quad 0. \tag{A.6}$$

Applying Theorem 3.2 in [3], we get the existence of M_ε, $N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned} (p_\varepsilon, M_\varepsilon) &\in J^{2,+} u_i(x_\varepsilon), \\ (q_\varepsilon, N_\varepsilon) &\in J^{2,-} w_i^\lambda(y_\varepsilon) \end{aligned}$$

where

$$\begin{aligned} p_\varepsilon &= D_x \phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_{_0})^3 \\ q_\varepsilon &= -D_y \phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) \end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & -N_\varepsilon \end{pmatrix} \le D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon \left(D^2 \phi_\varepsilon(x_\varepsilon, y_\varepsilon) \right)^2 \tag{A.7}$$

with

$$D^2 \phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \begin{pmatrix} 3(x_\varepsilon - x_{_0})^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix},$$

By writing the viscosity subsolution property (3.9) of u_i and the viscosity strict supersolution property (A.4) of w_i^λ, we have the following inequalities:

$$\min \Big\{ r u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_{_0})^3 \right) b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon)\,, \quad u_i(x_\varepsilon) - \max_{j \neq i}(u_j - g_{ij})(x_\varepsilon) \Big\} \le 0 \tag{A.8}$$

$$\min \Big\{ r w_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon)\,, \quad w_i^\lambda(y_\varepsilon) - \max_{j \neq i}(w_j^\lambda - g_{ij})(y_\varepsilon) \Big\} \ge \lambda\delta \tag{A.9}$$

We then distinguish the following two cases :

(1) $u_i(x_\varepsilon) - \max_{j \neq i}(u_j - g_{ij})(x_\varepsilon) \le 0$ in (A.8).
By sending $\varepsilon \to 0$, this implies

$$u_i(x_0) - \max_{j \neq i}(u_j - g_{ij})(x_0) \quad \le \quad 0. \tag{A.10}$$

On the other hand, we have by (A.9) :

$$w_i^\lambda(y_\varepsilon) - \max_{j \neq i}(w_j^\lambda - g_{ij})(y_\varepsilon) \quad \ge \quad \lambda\delta,$$

so that by sending ε to zero :

$$w_i^\lambda(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \quad \geq \quad \lambda\delta. \tag{A.11}$$

Combining (A.10) and (A.11), we obtain :

$$\begin{aligned} \theta \;=\; u_i(x_0) - w_i^\lambda(x_0) \quad & \leq \quad -\lambda\delta + \max_{j\neq i}(u_j - g_{ij})(x_0) - \max_{j\neq i}(w_j^\lambda - g_{ij})(x_0) \\ & \leq \quad -\lambda\delta + \max_{j\neq i}(u_j - w_j^\lambda)(x_0) \\ & \leq \quad -\lambda\delta + \theta, \end{aligned}$$

which is a contradiction.

(2) $ru_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) b_i x_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - f_i(x_\varepsilon) \leq 0$ in (A.8).
Since by (A.9), we also have :

$$rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) b_i y_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - f_i(y_\varepsilon) \quad \geq \quad \lambda\delta,$$

this yields by combining the above two inequalities :

$$\begin{aligned} & ru_i(x_\varepsilon) - rw_i^\lambda(y_\varepsilon) - \frac{1}{\varepsilon} b_i (x_\varepsilon - y_\varepsilon)^2 - (x_\varepsilon - x_0)^3 b_i x_\varepsilon \\ & \qquad + \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon - \frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon + f_i(y_\varepsilon) - f_i(x_\varepsilon) \quad \leq \quad -\lambda\delta. \end{aligned} \tag{A.12}$$

Now, from (A.7), we have :

$$\frac{1}{2}\sigma_i^2 x_\varepsilon^2 M_\varepsilon - \frac{1}{2}\sigma_i^2 y_\varepsilon^2 N_\varepsilon \leq \frac{3}{2\varepsilon}\sigma_i^2 (x_\varepsilon - y_\varepsilon)^2 + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right),$$

so that by plugging into (A.12) :

$$\begin{aligned} r\left(u_i(x_\varepsilon) - w_i^\lambda(y_\varepsilon)\right) \quad \leq \quad & \frac{1}{\varepsilon} b_i (x_\varepsilon - y_\varepsilon)^2 + (x_\varepsilon - x_0)^3 b_i x_\varepsilon + \frac{3}{2\varepsilon}\sigma_i^2 (x_\varepsilon - y_\varepsilon)^2 \\ & + \frac{3}{2}\sigma_i^2 x_\varepsilon^2 (x_\varepsilon - x_0)^2 \left(3\varepsilon(x_\varepsilon - x_0)^2 + 2\right) + f_i(y_\varepsilon) - f_i(x_\varepsilon) - \lambda\delta \end{aligned}$$

By sending ε to zero, and using (A.6), continuity of f_i, we obtain the required contradiction: $r\theta \leq -\lambda\delta < 0$. This ends the proof of Theorem 4.3. □

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