A criterion for the convergence in law of Brownian functionals to time changed Brownian motion.

Let $(B_t, t \geq 0)$ be a Brownian motion, and consider $(X_n(t), t \geq 0)$ a sequence of processes with continuous paths.

I give a criterion, which seems to be applicable in many cases, for the following kind of convergence in law:

$$(1) \qquad (B_t, X_n(t); t \geq 0) \xrightarrow[(law)]{(n \to \infty)} (B_t; \beta_{\ell_t}; t \geq 0)$$

when $(\ell_t, t \geq 0)$ is an increasing process, measurable with respect to $(B_t, t \geq 0)$ and $(\beta_u, u \geq 0)$ is a Brownian motion which is independent of $(B_t, t \geq 0)$.

The assumptions about $(X_n(t), t \geq 0)_{n \in \mathbb{N}}$ are:

i) $X_n(t) = Y_n(t) + Z_n(t)$,

i') the family (P_n) of the laws of the (X_n)'s is tight.

ii) for every t, $Y_n(t) \xrightarrow[n \to \infty]{(P)} 0$;

iii) $Z_n(t) = \int_0^t \varphi_n(s, \cdot)\, dB_s$, where $\varphi_n(s, \cdot)$ is a previsible process; iv) for every t, $\int_0^t \varphi_n(s, \cdot)\, ds \xrightarrow[n \to \infty]{(P)} 0$

v) there exists a continuous process $(\ell_t, t \geq 0)$ such that:

for every $\lambda \geq 0$, for every $t > 0$, $E\left[\exp \lambda \left| \int_0^t ds\, \varphi_n^2(s) - \ell_t \right| \right] \xrightarrow[(n \to \infty)]{} 0$

<u>Theorem:</u> <u>Under the preceding assumptions</u>, (1) <u>holds.</u>

<u>Proof:</u> a) First, consider $F(B) = \exp\left(\int_0^\infty f(u)\,dB_u - \frac{1}{2}\int_0^\infty f^2(u)\,du\right)$

for some simple deterministic function f.
We first show:

$$(2) \qquad E\left[F \quad \exp\left\{ i\int_0^\infty g(u)\,dX_n(u) + \frac{1}{2}\int_0^\infty g^2(u)\,\varphi_n^2(u)\,du \right\}\right] \xrightarrow[n\to\infty]{} 1$$

for every simple deterministic function g.
Indeed, we may replace, on the left-hand side, X_n by Z_n, thanks to ii).
Then, we use the fact that:

$$\left(\exp\left\{\left(\int_0^t f(u)\,dB_u + i\int_0^t g(u)\,dZ_n(u)\right) - \frac{1}{2}\int_0^t \left(f(u) + ig(u)\varphi_n(u)\right)^2 du\right\}, t$$

is a martingale, with mean 1, and we obtain (2) with the help of iv).

b) (2) may now be extended to any $F \in L^2$, if we replace on the right-hand side 1 by $E(F)$, thanks to the density in L^2 of the vector space generated by $\left(\exp\left(\int_0^\infty f(u)\,dB_u\right), f \in L^2(\mathbb{R}_+)\right)$

We denote by (2') this extended convergence result.

c) On the left-hand side of (2'), we may now replace

$\int_0^\infty g^2(u)\,\varphi_n^2(u)\,du$ by $\int_0^\infty g^2(u)\,dl_u$, thanks to (v).

Thus, we have:

$$(3) \qquad E\left[F \exp\left(\frac{1}{2}\int_0^\infty g^2(u)\,d\ell_u\right) \exp\left(i\int_0^\infty g(u)\,dX_n(u)\right)\right] \xrightarrow[n\to\infty]{} E(F$$

d) Finally, we can replace in (3) F by $\Phi \equiv F \exp\left(-\frac{1}{2}\int_0^\infty g^2(u)\,d\ell_u\right.$

Thus, we obtain:

$$(4) \qquad E\left[\Phi \exp\left(i\int_0^\infty g(u)\,dX_n(u)\right)\right] \xrightarrow[(n\to\infty)]{} E\left[\Phi \exp\left(-\frac{1}{2}\int_0^\infty g^2(u)\,d\right.\right.$$

which proves the desired result. $\square$

From : Marc Yor, to : Professor Mihael PERMAN

Message length (including this page) : 3 pages /

Berkeley, 24th of Aug. 94.

Dear Mihael,

Here is a partial answer to your question.

I hope this may help you to solve the question completely. I will keep thinking about it. I am in Paris from Aug. 28th, onwards, where you can reach me at the usual Fax #.

We used your paper a lot this summer. I hope we have lots of (Jim & I) contacts when you are in Cambridge.

All the best, Marc.

Answer to Mihael's second question :

$$E^{\nu}\left[\exp\left(-\rho X_{t\wedge T_0}\right)\right] = E^{\nu}\left[\exp\left(-\rho X_t\right) 1(t<T_0)\right] + P^{\nu}(T_0 \leq t)$$

$$= E^{\nu}\left[\exp\left(-\rho X_t\right)\right] - E^{\nu}\left[(T_0 \leq t) E_0\left[\exp(-\rho X_u)\Big|_{u=(t-T_0)}\right]\right]$$

$$+ P^{\nu}(T_0 \leq t)$$

and now, we know all the quantities (in particular,

$E^{\nu}\left[\exp\left(-\rho X_t\right)\right]$ is explicit and simple) and :

$$P^{\nu}_x\left(T_0 \in dt\right) = \frac{x^{2\nu}\,dt}{\Gamma(\nu)\, t^{\nu+1}} \exp\left(-\frac{x^2}{2t}\right)$$