A Gaussian martingale which is the sum of two gaussian independent non-semimartingales.

1. Certain mixed Fractional Brownian motions are semimartingales

In his thesis, P. Cheridito [1] obtained the following remarkable result:
if $(B_t, t \geq 0)$ and $(B_t^{(H)}, t \geq 0)$ denote two independent Gaussian processes, the first one being a Brownian motion, and the second one a fractional Brownian motion with Hurst parameter $H \in]\frac{3}{4}, 1]$, ie:

$$E\left[\left(B_t^{(H)} - B_s^{(H)}\right)^2\right] = |t-s|^{2H}, \quad s, t \geq 0,$$

then, for every $\alpha \in \mathbb{R}$, the sum:

$$\Sigma_t^{(H)} = B_t + \alpha B_t^{(H)}, \quad t \geq 0,$$

is a semimartingale with respect to its own natural filtration.
Notice that, for $H = 1$, one has: $B_t^{(1)} = t\xi$, where ξ is a standard Gaussian variable, and, consequently,

$(\Sigma_t^{(1)}, t \geq 0)$ is a semimartingale in the filtration

$$\mathcal{B}_t^{(\xi)} \stackrel{def}{=} \sigma\{B_s, s \leq t; \xi\}, \text{ made right contin[uous]}$$

hence, a fortiori, with respect to its own filtration —

However for $H \in]\frac{3}{4}, 1[$, $(B^{(H)}_t, t \geq 0)$ has zero quadratic variation, but infinite variation on every time interval, hence it is not a semimartingale with respect to its own filtration, which makes Cheridito's result remarkable —

Note: Throughout the rest of this paper, when we say that a process $(\Pi_t, t \geq 0)$ is a semimartingale, with no further qualification, we mean: semimartingale with respect to its own filtration, made right continuous, and P-complete —

2. Some related questions.

(2.1) In the light of Cheridito's result, one may ask the following question:

(*)
$$\boxed{\text{to give a "simpler" example of a pair of independent, centered, Gaussian processes, } (X_t, t \geq 0) \text{ and } (Y_t, t \geq 0), \text{ one of which at least is not a semimartingale, but such that the sum is a semimartingale}}$$

In Section 3, we shall give an example where $(X_t, t \geq 0)$ is constructed from a Brownian bridge and is not a semimartingale whereas $(Y_t, t \geq 0)$ has bounded variation —

In Section 4, pushing the construction of Section 3 one step further, we shall give another example of (*) where neither (X_t), nor (Y_t), are semimartingales —

(2.2) For the moment, we simply note that, in order to ~~give~~ obtain some positive answer to (*), at least ~~of~~ one of the gaussian processes (X_t), or (Y_t), must have some non-zero quadratic variation, ie:

$$\sum_{\tau_n} (\Delta X_{t_i})^2 \quad , \text{ where } \tau_n = (0 = t_0 < t_1 < \dots < t_{p_n} = 1)$$

~~and~~ as $\phi(\tau_n) \equiv \sup_{\tau_n} (t_{i+1} - t_i) \xrightarrow[(n \to \infty)]{} 0$

does not converge to 0. This assertion follows from the

Lemma: i) Assume that X and Y are two independent centered, Gaussian processes, and τ is a subdivision of $[0,1]$.
Then:

$$\max \left(E\left[\sum_\tau |\Delta X_{t_i}|\right] ; E\left[\sum_\tau |\Delta Y_{t_i}|\right] \right)$$

$$\leq E\left[\sum_\tau |\Delta (X+Y)_{t_i}|\right] \leq E\left[\sum_\tau |\Delta X_{t_i}| + \sum_\tau |\Delta Y_{t_i}|\right]$$

ii) If both X and Y have 0-quadratic variation, and if one of them has infinite variation (on a set of >0 probability), then ~~X+~~ X+Y also enjoys these two properties –

Proof: i) Only the LHS inequality needs to be proven; but this follows from:

$$E\left[\cancel{\Sigma} |\Delta (X+Y)_{t_i}|\right] = \sqrt{\frac{2}{\pi}} \, \| (\Delta X_{t_i}) + (\Delta Y_{t_i}) \|_2 \geq \sqrt{\frac{2}{\pi}} \, \| (\Delta X_{t_i}) \|_2$$

$$= E\left[|\Delta X_{t_i}| \right].$$

ii) It follows from i), and our hypothesis in ii) that:

$$E\left[\int_0^1 |d(X_s+Y_s)|\right] = \infty ;$$

but, from Fernique's integrability result for the norms of Gaussian vectors, it follows that $\int_0^1 |d_s(X_s+Y_s)|$ cannot be finite a.s.

3. Brownian bridges and a first solution to (*).

(3.1) Let $u>0$, and denote by $(p_u(t), t\leq u)$ a Brownian bridge of length u, ie: $(B_t, t\leq u)$, conditioned to be equal to 0 at time u.

Recall that the canonical decomposition of p_u is:

$$p_u(t) = \beta_t - \int_0^t ds\, \frac{p_u(s)}{(u-s)}, \quad t\leq u,$$

where $(\beta_t, t\leq u)$ is a Brownian motion in the filtration $(\mathcal{P}_t^{(u)}; t\leq u)$ of p_u.

Furthermore, there is the following

Proposition: Let $f\in L^2([0,u])$. Then:

a) the process: $\int_0^t f(s)\, dp_u(s) \equiv \int_0^t f(s)\, d\beta_s - \int_0^t ds\, f(s)\frac{p_u(s)}{(u-s)}$

is well defined, for every $t\leq u$, with:

$$\int_0^u f(s)\, dp_u(s) = (L^2 \text{ and a.s}) \lim_{t\uparrow u} \int_0^t f(s)\, dp_u(s).$$

b) $\left\{ \int_0^t f(s)\, d\mu_u(s) \,,\ t \le u \right\}$ is a semimartingale with respect to $\left(\mathcal{P}_t^{(u)} = \sigma\{ \mu_u(h), h \le t \} ,\ t \le u \right)$,

iff: $$\int^u ds\, |f(s)| \frac{1}{\sqrt{u-s}} < \infty .$$

Proof: a) The L^2 and a.s. convergence results are easily obtained from the representation of μ_u as:

$$\mu_u(t) = B_t - \frac{t}{u} B_u .$$

b) The semimartingale property of $\left(\int_0^t f(s)\, d\mu_u(s) ,\ t \le u \right)$ is clearly equivalent to: $\displaystyle\int_0^u |f(s)| \frac{|\mu_u(s)|\, ds}{(u-s)} < \infty$

The arguments developed in Jeulin-Yor [2] show that this is equivalent to:

$$\int^u ds\, |f(s)| \frac{1}{\sqrt{u-s}} < \infty .$$

In order to give explicit examples for $(*)$ in the sequel of this paper, let us point out that the function:

$$\varphi(t) = \frac{1}{\sqrt{u-t}} \left(-\log(u-t) \right)^{-\alpha} 1_{\left(\frac{u}{2} < t < u\right)}$$

satisfies: $\displaystyle\int_0^u dt\, \varphi^2(t) < \infty$, but: $\displaystyle\int^u dt\, \varphi(t) \frac{1}{\sqrt{u-t}} = \infty .$

(3.2) To obtain a solution to $(*)$, we decompose, as above, a Brownian motion $(B_t, t \leq u)$ as:

$$B_t = p_u(t) + \frac{t}{u} B_u \quad , \quad t \leq u ,$$

and we consider $f_* \in L^2([0,u])$ such that:

$$\int_0^u ds \, |f_*(s)| \frac{1}{\sqrt{u-s}} = \infty \quad , \quad f_*(s) \neq 0 \text{, for every } s.$$

Then, taking: $\begin{cases} X_t = \int_0^t f_*(s) \, d_s(p_u(s)) \\ Y_t = \left(\frac{B_u}{u}\right) \int_0^t f_*(s) \, ds , \end{cases}$

we obtain a solution to $(*)$, since: $X_t + Y_t = \int_0^t f_*(s) dB_s$, is a martingale.

4. A "full" solution to $(*)$.

Let $u \in]0,1[$. We shall use the same idea as in Section 3, but twice instead of once, by decomposing first $(B_t, t \le u)$ ~~into~~

into: $p_u(t) + \frac{t}{u} B_u$, then:

$(\hat{B}_t \equiv B_{t+u} - B_u,\ t \le 1-u)$ into: $\hat{p}_{1-u}(t) + \frac{t}{(1-u)} \hat{B}_{(1-u)}$.

Next, for $f \in L^2([0,1])$, we write:

$$\int_0^t f(s)\,dB_s = \int_0^t f(s) 1_{(s \le u)}\,dB_s + \int_u^t f(s)\,dB_s$$

$$= \int_0^t f(s) 1_{(s \le u)}\, d_s(p_u(s)) + \left(\frac{B_u}{u}\right) \int_0^t ds\, f(s) 1_{(s \le u)}$$

$$+ \int_u^t f(k)\, d_k(\hat{p}_{1-u}(k-u)) + \left(\frac{B_1 - B_u}{1-u}\right) \int_u^t f(k)\,dk$$

We then choose f_* such that: $\int^u ds\, |f_*(s)| \frac{1}{\sqrt{u-s}} = \infty$,

and $\int^1 |f_*(v)| \frac{dv}{\sqrt{1-v}} = \infty$, $\quad f_*(t) \ne 0,\ \forall t < 1$.

Using the semimartingale characterization property in part b) of the above Proposition, it is then easily shown that:

$$\begin{cases} X_t = \int_0^t f_*(s) 1_{(s \le u)}\, d_s(p_u(s)) + \left(\frac{B_1 - B_u}{1-u}\right) \int_u^t dk\, f_*(k) \\ Y_t = \int_u^t f_*(k)\, d_k(\hat{p}_{1-u}(k-u)) + \left(\frac{B_u}{u}\right) \int_0^t ds\, f_*(s) 1_{(s \le u)} \end{cases}$$

are not semimartingales —

However, we give a few details:

- Concerning (X_t), we see that: $X_t = \tilde{X}_t$ for $t<u$,

where $\tilde{X}_t = \int_0^t f_*(s) 1_{(s<u)} d_s(p_u(s))$, hence the

non-semimartingale property of X follows from that of $\tilde{X}$, discussed in Section 3.

- Concerning (Y_t), we have:

$Y_u = \left(\frac{B_u}{u}\right) \int_0^u ds\, f_*(s)$, and $Y_t - Y_u = \int_u^t f_*(k) d_k(\hat{p}_{1-u}(k-u))$, $t \geq u$

Now, Y, being a gaussian process, could only be a semimartingale if it were a quasimartingale — (see, e.g., Stricker)

Since, with obvious notation, we have:

$$\mathcal{Y}_{u+h} = \sigma\{B_u ;\ \hat{p}_{1-u}(\mu),\ \mu \leq h\},$$

we find that, for $h<k$:

$$E[Y_{u+k} - Y_{u+h} \mid \mathcal{Y}_{u+h}] = E[Y_{u+k} - Y_{u+h} \mid \hat{\mathcal{P}}_h^{(1-u)}]$$

~~and~~ which now yields the non semimartingale property of Y.

References: [1] P. Cheridito: Ph.D Thesis, ETH Zürich (April 2001).

[2] T. Jeulin, M. Yor: Inégalités de Hardy, semimartingales et faux-amis — Sem. Probas. XIII, Lect. Notes in Maths., vol. 721 (1979). p. 332-359. Springer