A discussion on pure and extremal continuous martingales.

(exemplaire incomplet)

Introduction $(\Omega, \mathcal{F}, P)$ denotes a probability space, which varies throughout the paper. If $X = (X_t)_{t \geq 0}$ is a (real-valued) process defined on $(\Omega, \mathcal{F}, P)$, $\mathcal{F}(X) = \{\mathcal{F}(X)_t, t \geq 0\}$ is the smallest right-continuous filtration, composed of $(\mathcal{F}, P)$ complete σ-fields, with respect to (in short, w.r.t.) which the process X is adapted.

An important theorem due to K. Ito states that any local martingale $M = (M_t)_{t \geq 0}$ w.r.t. the natural filtration $\mathcal{F}(B)$ of a real valued Brownian motion $B = (B_t)_{t \geq 0}$, with $B_0 = 0$, may be represented as:

$$M_t = c + \int_0^t \phi(s) dB_s , \qquad (t \geq 0) ,$$

with $c \in \mathbb{R}$, and ϕ a predictable process w.r.t $\mathcal{F}(B)$, such that: $\forall t, \int_0^t \phi^2(s) ds < \infty$, a.e.

The Brownian motion is said to have the (predictable) representation property (w.r.t. $\mathcal{F}(B)$).

With the developments of the theories

- on one hand: of stochastic integrals w.r.t general local martingales
- on the other hand: of non-linear filtering,

the question of knowing which (continuous) local martingales X have the representation property (w.r.t. $\mathcal{F}(X)$) has been open for some time; the answer to this question is now known (Yan-Yoeurp [18], Jacod-Yor [10], Kussmaul [13], Liptser [15]), and goes as follows:

a local martingale $(X_t)_{t \geq 0}$ (w.r.t. $\mathcal{F}(X)$) has the representation property iff its distribution (defined on the canonical space of cadlag functions) is extremal among all the distributions of local martingales (we also say that X is extremal, making thus a slight abuse of language).

Another important theorem in the theory of continuous martingales shows that any continuous local martingale $(X_t)_{t\geq 0}$, with $X_0 = 0$, and $\langle X, X\rangle_\infty = \infty$ i.e may be expressed as a Brownian motion $(\beta_t)_{t\geq 0}$ run with the new clock $A_t = \langle X, X\rangle_t$, e.g: $X_t = \beta_{A_t}$. This result has been obtained independently by K. Dambis [3], and L. Dubins and G. Schwarz [5]. We refer to it as the Dambis-Dubins-Schwarz (in short, D.D.S) theorem.

In another paper [6], L. Dubins and G. Schwarz call a continuous local martingale $(X_t)_{t\geq 0}$ (with $X_0 = 0$, and $\langle X, X\rangle_\infty = \infty$, a.e) a <u>pure martingale</u> if:

$$(1) \qquad \mathcal{F}(X)_\infty = \mathcal{F}(\beta)_\infty,$$

where β is the D.D.S Brownian motion attached to X. The same authors also remark that a pure martingale is extremal, but that the converse statement is false. This was also realized much later in [20], using a very different ~~argument~~ counter-example.

The present paper is wholly devoted to obtaining a deeper understanding of the distinction between extremal and pure continuous martingales.

Some preliminaries on changes of time are developed in chapter one; the results improve slightly those of N. Kazamaki [11]. Then, we get a first characterization of pure continuous martingales, which may be considered as a suitable extension of the D.D.S theorem, when one replaces the Brownian motion by a pure martingale.

In chapter two, as a generalization of the example built in [20], we construct a general setting for which it is possible to discuss exactly the extremality and/or the purity of the martingales defined there.

In chapter three, it is remarked that a continuous martingale with respect to each of the filtrations $(\mathcal{G}_t)_{t\geq 0}$, and $(\mathcal{F}_t)_{t\geq 0}$, with: $\mathcal{G}_t \subseteq \mathcal{F}_t$, for every t, may have the representation property w.r.t. $(\mathcal{F}_t)$, but not w.r.t $(\mathcal{G}_t)$. This enables us to obtain a second characterization of pure martingales among extremal ones.

In chapter four, the behavior of pure martingales under absolute continuous changes of probabilities is investigated.
Firstly, B. S. Tsirel'son's well-known example [16] is used as a new means to obtain extremal, but non-pure martingales. The same construction shows that there exists a pure martingale X (under a probability P) and an other probability Q, with $Q \simeq P$ on $\mathcal{F}(X)_\infty$, such that the martingale part $\tilde{X}$ of the canonical decomposition of X - considered as a $(\mathcal{F}(X), Q)$ semi-martingale - is extremal but not pure. Changing slightly this example, it is even possible to show that there exists a pure martingale X such that $\tilde{X}$ is not extremal.

In chapter five, we raise some concluding questions, in order to understand better the links between extremal continuous martingales and Brownian filtrations. Some equivalences or implications between these questions are obtained; in particular, the answers to the two questions asked at the end of [20] are simultaneously positive or negative -

1. Changes of time and pure martingales.

$(\Omega, \mathcal{F}, P)$ is the basic probability space, endowed with a right-continuous, $(\mathcal{F}, P)$ complete ~~family~~ filtration $(\mathcal{F}_t)_{t \geq 0}$. $\mathcal{O}_{(\mathcal{F}_t)}$, resp: $\mathcal{P}_{(\mathcal{F}_t)}$, denotes the optional, resp: predictable, σ-field on $\Omega \times \mathbb{R}_+$, associated to $(\mathcal{F}_t)$.

In agreement with N. Kazamaki [11], a $(\mathcal{F}_t)$-change of time denotes here a family $T = (\tau_t)_{t \geq 0}$ of finite valued, $(\mathcal{F}_t)$ stopping times, such that, for almost all ω, the trajectory $\tau_\cdot(\omega)$ is increasing and right-continuous.

Given a $(\mathcal{F}_t)$ change of time $T = (\tau_t)$, and a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$ measurable process $(X_t)_{t \geq 0}$, we denote by $T(\mathcal{F}_\cdot)$, resp: $T(X)$, the filtration $(\mathcal{F}_{\tau_t})$, resp: the process (X_{τ_t}).

If X is a real valued process, a $\mathcal{F}(X)$-change of time is called a X-adapted change of time. A $(\mathcal{F}_t)$ change of time $T = (\tau_t)$ is said to be X-continuous if, outside an evanescent set, X is constant on each interval $[\tau_{t-}, \tau_t]$, and on $[0, \tau_0]$.

In the following, X is always a continuous $(\mathcal{F}_t)$ local martingale, with $X_0 = 0$. The interest of X-continuous changes of time appears in the next

(1.1) Proposition. Let $T = (\tau_t)$ be a $(\mathcal{F}_t)$ change of time. The following assertions are equivalent:

(i) T is X-continuous.

(ii) $T(X)$ is a $T(\mathcal{F}_t)$ local continuous martingale, with increasing process $T(\langle X \rangle)$.

Moreover, if Y is another continuous $(\mathcal{F}_t)$ local martingale, such that T is X- and Y- continuous, the only $T(\mathcal{F}_t)$-adapted, and continuous, process, with bounded variation, associated to the product $T(X)T(Y)$ is given by:

$$(2) \qquad \langle T(X), T(Y) \rangle = T(\langle X, Y \rangle).$$

The main ingredient in the proof of proposition (1.1) is the following

(1.2) Lemma (Getoor and Sharpe, [7])

X and $\langle X \rangle$ have the same intervals of constancy, almost surely.

Proof of proposition (1.1).

(i) $\Rightarrow$ (ii): Kazamaki proved in Proposition 1 of [11] that if T is X-continuous, then $T(X)$ is a local martingale w.r.t $T(\mathcal{F}_t)$.

From lemma (1.2), T is also $\langle X \rangle$-continuous; therefore, T is Y-continuous, where: $Y = X^2 - \langle X \rangle$, so that, using again Kazamaki's result, $T(Y) = T(X)^2 - T(\langle X \rangle)$ is a $T(\mathcal{F}_t)$ local martingale. As $T(\langle X \rangle)$ is a continuous, $T(\mathcal{F}_t)$ adapted, increasing process, this proves that:

$$\langle T(X) \rangle = T(\langle X \rangle).$$

(ii) $\Rightarrow$ (i) In particular, $T(\langle X \rangle)$ is continuous, e.g: T is $\langle X \rangle$-continuous, therefore, from lemma (1.2), T is X-continuous.

To show (2), we only need to remark that if T is X- and Y- continuous, it is $(X+\lambda Y)$-continuous for any $\lambda \in \mathbb{R}$. Thus, from our previous results, we get: $\langle T(X+\lambda Y) \rangle = T(\langle X+\lambda Y \rangle)$. Developing w.r.t λ, we obtain (2).

~~We no~~ Now, we investigate the effect of X-continuous changes of time on stochastic integrals w.r.t. X.

(1.3) Proposition. Let X be a continuous $(\mathcal{F}_t)$ local martingale, with $X_0 = 0$, and $T = (\tau_t)$ a X-continuous change of time.

If C is a $(\mathcal{F}_t)$ optional process such that: $\forall t$, $\int_0^t C_s^2 \, d\langle X \rangle_s < \infty$ a.e., we denote by $C \cdot X$ the stochastic integral $\int_0^{\cdot} C_s \, dX_s$.

Then, the process $T(C)$ is $T(\mathcal{F}_t)$ optional, $(TC)\cdot(TX)$ is well defined, and

$$T(C \cdot X) = (TC)\cdot(TX) \tag{3}$$

Proof: 1) If C is right-continuous and $(\mathcal{F}_t)$ adapted, $T(C)$ is right-continuous, $T(\mathcal{F}_t)$ adapted, and consequently $T(\mathcal{F}_t)$ optional.

Thus, by the monotone class theorem, if C is $(\mathcal{F}_t)$ optional, $T(C)$ is $T(\mathcal{F}_t)$ optional.

2) Suppose $(\gamma_t)_{t\geq 0}$ is a continuous increasing process, not necessarily $(\mathcal{F}_t)$ adapted, but such that T is γ- continuous. Then, for any positive Borel function $u : [0,\infty[\longrightarrow \mathbb{R}_+$, one has:

$$\forall t, \quad \int_0^{\tau_t} u_s \, d\gamma_s = \int_0^t u_{\tau_s} \, d(\gamma_{\tau_s}),$$

which, in short, may be written as: $\quad (3') \quad T(u\cdot\gamma) = (Tu)\cdot(T\gamma)$

3) As a consequence of 2), and of Proposition (1.1), (ii), the finiteness of $\int_0^t C_s^2 \, d\langle X\rangle_s$, for every t, implies that of $\int_0^t (TC)_s^2 \, d\langle TX\rangle_s$, for every t, and so, $(TC)\cdot(TX)$ is well defined.

4) To prove (3), we only have to show that:

$$I \overset{\text{def}}{=} \langle T(C\cdot X) - (TC)\cdot(TX)\rangle = 0.$$

Developing the left-hand side, one gets:

$$I = \langle T(C\cdot X)\rangle - 2\,(TC)\cdot\langle TX, T(C\cdot X)\rangle + (TC)^2\cdot T(\langle X\rangle)$$

Now, remark that, from lemma (1.2), T is $\langle C\cdot X\rangle$-, and thus $(C\cdot X)$- continuous, and from formula (2), we deduce:

$$I = T(\langle C\cdot X\rangle) - 2\,(TC)\cdot T(\langle X, C\cdot X\rangle) + (TC)^2\cdot(T\langle X\rangle)$$
$$= 2\,(TC)^2\cdot T(\langle X\rangle) - 2\,(TC)^2\cdot T(\langle X\rangle) = 0.$$

X

The above preliminaries on changes of time will play an important part in the sequel. But, even now, they are helpful to sketch a proof of the Dambis - Dubins - Schwarz (D.D.S in the following) result already stated in the introduction, and to draw several remarks from it.

So, let X be a $(\mathcal{F}_t)$ continuous local martingale such that $X_0 = 0$, and $\langle X\rangle_\infty = \infty$ a.e. Let $\tau_t = \inf\{s / \langle X\rangle_s > t\}$. $T = (\tau_t)$ is a $(\mathcal{F}_t)$ change of time, which is X-adapted (as the process $\langle X\rangle$ is adapted to $\mathcal{F}(X)$), and even X-continuous, as $T(\langle X\rangle)_t = t$ (then, use lemma (1.2))

Thus, from Proposition (1.1), $T(X)$ is a $T(\mathcal{F}_t)$ local continuous martingale, with increasing process t, e.g., a $T(\mathcal{F}_t)$-Brownian motion, from Paul Lévy's theorem. We shall call $\beta \stackrel{def}{=} T(X)$, the D.D.S Brownian motion attached to X, and we note $\beta = \beta(X)$. Moreover, as T is X-continuous, we also have: (4) $X_t = \beta_{\langle X\rangle_t}$.

The effects of changes of time on the D.D.S Brownian motion are studied in the next

(1.4) Lemma: Let X be a continuous $(\mathcal{F}_t)$ local martingale, with $X_0 = 0$, and $\langle X, X\rangle_\infty = \infty$ a.e. Let $R = (\rho_t)_{t\geq 0}$ be a $(\mathcal{F}_t)$ change of time, which is X-continuous, and such that $\rho_\infty = \infty$ a.e.

Then, one has: (5) $\beta(R(X)) = \beta(X)$

Proof: Note that, from Proposition (1.1), $Y = R(X)$ is a $R(\mathcal{F}_t)$ continuous local martingale, with $\langle Y\rangle = R(\langle X\rangle)$, and so, from formula (4),

$$Y_t = \beta_{R(\langle X\rangle)_t} = \beta_{\langle Y\rangle_t},$$

where $\beta = \beta(X)$. From this last formula, we finally deduce: $\beta(Y) = \beta$.

In the D.D.S result, we may regard $(\langle X\rangle_t)$ as a $(\mathcal{F}_{\tau_t})$ change of time, as: $\forall t$, $\langle X\rangle_t = \inf\{s / \tau_s > t\}$. Thus, in lemma (1.4), we have composed changes of time. We shall need the following easy result concerning this situation.

(1.5) Lemma Let $T = (\tau_t)$ be a $(\mathcal{F}_t)$ change of time, and $S = (\sigma_t)$ be a $(\mathcal{F}_{\tau_t})$ change of time.

Then, $ST = (\tau_{\sigma_t})$ is a $(\mathcal{F}_t)$ change of time, and

$$(6) \quad \forall t, \quad (\mathcal{F}_{\tau_\cdot})_{\sigma_t} = \mathcal{F}_{\tau_{\sigma_t}}.$$

The first part of the lemma has been proved by Kazamaki ([11], lemma 2), and here is a sketch of the proof of (6): recall that if $(\mathcal{G}_t)_{t\geq 0}$ is a usual filtration, and u a finite $(\mathcal{G}_t)$ stopping time, any $(\mathcal{G}_u)$ measurable

r.v.'s may be expressed as : Z_u , where Z is a $(\mathcal{G}_t)$ optional process.
Now, suppose Z is a $(\mathcal{F}_t)$ optional process. Then, $Z_{\tau_{\sigma_t}} = (Z_{\tau_\cdot})_{\sigma_t}$.
Moreover, $(Z_{\tau_t})_{t \geq 0}$ is a $(\mathcal{F}_{\tau_t})$ optional process, and this proves:

$$(6') \qquad \mathcal{F}_{\tau_{\sigma_t}} \subseteq (\mathcal{F}_{\tau_\cdot})_{\sigma_t} .$$

Conversely, a monotone class argument shows that any $(\mathcal{F}_{\tau_t})$ optional process may be written as : $f(t; \tau_t, \omega)$, where f is a $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{O}_{(\mathcal{F}_t)}$ measurable function. This easily implies the equality :

$$(6'') \qquad (\mathcal{F}_{\tau_\cdot})_{\sigma_t} = \mathcal{F}_{\tau_{\sigma_t}} \vee \sigma\{\sigma_t\} .$$

Thus, to prove (6), one needs only show that σ_t is $\mathcal{F}_{\tau_{\sigma_t}}$-measurable.
This last result is easily obtained, using the dyadic approximations from above of σ_t , and the right-continuity of (τ_t).

We are now able to study some properties of pure continuous martingales, the definition of which was given in the introduction.

Let (X_t) be a continuous $(\mathcal{F}_t)$ local martingale, with $X_0 = 0$, and $A = \langle X \rangle$ its increasing process, such that $A_\infty = \infty$ a.e. Note β the D.D.S Brownian motion attached to X, and $\tau_t = \inf\{s / A_s > t\}$ $(t \geq 0)$.
It is easy to show ([19], for example) that X is pure , ie , verifies:

$$(1) \qquad \mathcal{F}(X)_\infty = \mathcal{F}(\beta)_\infty \qquad \text{iff}$$

$$(1') \qquad \forall t, \ \mathcal{F}(X)_{\tau_t} = \mathcal{F}(\beta)_t \qquad \text{is verified, or equivalently}$$

$$(1'') \qquad A = (A_t) \text{ is a } \mathcal{F}(\beta)\text{-change of time} .$$

Also, remark that, if $\mathcal{B} \overset{def}{=} \mathcal{F}(\beta)$, and X is pure , then:

$$(7) \qquad \forall t, \quad \mathcal{F}(X)_t = \mathcal{B}_{A_t} .$$

In the following proposition, we show that pure martingales are left stable under "nice" changes of time.

(1.6) Proposition. Let $M = (M_t)_{t \geq 0}$ be a pure continuous $(\mathcal{F}_t)$ local martingale, with $M_0 = 0$, and $\langle M, M \rangle_\infty = \infty$ a.e.
Then, if $T = (\tau_t)_{t \geq 0}$ is a M-adapted, and M-continuous change of time increasing to ∞, as $t \uparrow +\infty$, $N = T(M)$ is a pure local martingale.

Proof: From proposition (1.1), N is a $T(\mathcal{F}_t)$ local continuous martingale, with increasing process $T(\langle M \rangle)$.

From lemma (1.4), the D.D.S Brownian motions attached to M and N are equal. Denote this process by β, and $\mathcal{B} = \mathcal{F}(\beta)$.

So, to prove that N is pure, we need only show, from (1''), that $T(\langle M \rangle)$ is a $\mathcal{B}$-change of time. But, as ~~MM~~ M is pure, $\langle M \rangle$ is a $\mathcal{B}$-change of time; from (7), T is a $(\mathcal{B}_{\langle M \rangle})$ change of time, and, from lemma (1.5), $T(\langle M \rangle)$ is a $\mathcal{B}$-change of time $\blacksquare$.

Remark For an improvement of Proposition (1.6), the reader is invited to look already at theorem (3.3), which gives a new characterization of pure martingales.

The following statement, which is also a characterization of pure martingales, can be considered as an extension of the D.D.S theorem, when one replaces the Wiener measure (i.e: the Brownian distribution) by pure distributions (i.e: distributions of pure martingales).

(1.7) Theorem Let P be the distribution of a continuous local martingale X, such that $X_0 = 0$, and $\langle X \rangle_\infty = \infty$ a.e.

Then, P is pure iff, for any local continuous martingale M, defined on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P')$, with $M_0 = 0$, and $\langle M \rangle_\infty = \infty$ a.e, there exists a M-adapted, and M-con-

-tinuous change of time $L=(\lambda_t)_{t\geq 0}$ such that the distribution of $L(M)$ is P (we note: $P=\mathcal{L}((M_{\lambda_t})_{t\geq 0})$)

Proof 1) Suppose the condition holds. It holds in particular when M is the real valued Brownian motion $(B_t)_{t\geq 0}$, with $B_0=0$. Let $R=(\rho_t)_{t\geq 0}$ be a B-adapted, and B-continuous (this amounts here to be continuous, from Lemma (1.2), and the fact that $\langle B\rangle_t=t$) change of time such that $P=\mathcal{L}((B_{\rho_t})_{t\geq 0})$. Then, from Proposition (1.1), $Y=B_{\rho_\cdot}$ is a continuous local martingale w.r.t $\mathcal{F}(Y)$, and $\langle Y\rangle=\rho_\cdot$. The D.D.S Brownian motion attached to Y is obviously B, and from (1''), Y is pure.

2) Conversely, suppose P is a pure distribution. By definition, $P=\mathcal{L}(Y)$, where Y is a continuous local martingale, which may be written as $Y=B_{\rho_\cdot}$, with B a Brownian motion, and $R=(\rho_t)_{t\geq 0}$ a B-adapted, and continuous, change of time.

Now, let M be a continuous local martingale, defined on a filtered probability space $(\Omega',\mathcal{F}',(\mathcal{F}'_t),P')$, with $M_0=0$, and $\langle M\rangle_\infty=\infty$, a.s. P'. Note $\Lambda_t=\inf\{s/\langle M\rangle_s>t\}$ $(t\geq 0)$. Then, from the D.D.S theorem, $(\beta_t=M_{\Lambda_t},\ t\geq 0)$ is a real valued Brownian motion.

X By "transport" of the B-adapted, and continuous, change of time $R=(\rho_t)_{t\geq 0}$ on the probability space where M is defined, there exists a β-adapted change of time $R'=(\rho'_t)_{t\geq 0}$ such that $P=\mathcal{L}((\beta_{\rho'_t})_{t\geq 0})$. Now, we may write:

$$\forall t,\quad M_{\Lambda_{\rho'_t}}=\beta_{\rho'_t},$$

and finally, we only need to show that $(\Lambda_{\rho'_t})_{t\geq 0}$ is a M-adapted, and M-continuous change of time: it is M-adapted, as a consequence of lemma (1.5), because (Λ_t) is a $(\mathcal{F}(M)_t)$-change of time, and $R'=(\rho'_t)_{t\geq 0}$ a $(\mathcal{F}(\beta)_t)$-, and therefore $(\mathcal{F}(M)_{\Lambda_t})$-change of time; it is M-continuous, as (Λ_t) is $\langle M\rangle$-, and thus M-continuous, and

ρ' is continuous.

2. A general discussion.

As announced in the Introduction, the main aim of this chapter is to obtain a general method for the construction of extremal, but not pure continuous martingales.

For this, we rely mainly upon the characterization of extremal martingales given in the next theorem; there, X is a continuous local martingale, with $X_0 = 0$, $\langle X, X\rangle_\infty = \infty$ a.e, β is the D.D.S Brownian motion attached to X, and $\tau_t = \inf\{s / \langle X\rangle_s > t\}$ $(t \geq 0)$.

(2.1) Theorem. ([20], theorem 2). X is extremal iff $\beta = X_{\tau.}$ has the (predictable) representation property w.r.t the filtration $\{\mathcal{F}(X)_{\tau_t}\}$, e.g: every $\{\mathcal{F}(X)_{\tau_t}\}$ (local) martingale M may be written as

$$M_t = c + \int_0^t \phi(s)\,d\beta_s ,$$

where $c \in \mathbb{R}$, and ϕ is a $(\mathcal{F}(X)_{\tau_t})$ predictable process such that $\forall t$, $\int_0^t \phi^2(s)\,ds < \infty$ a.e.

For completedness, we give a sketch of the proof of this result: it was recalled in the Introduction that X is extremal iff it has the predictable representation property w.r.t $(\mathcal{F}(X)_t)$. This is equivalent to the following: for every $Y \in L^2(\mathcal{F}(X)_\infty)$ there exist $c \in \mathbb{R}$, and ϕ a predictable process w.r.t $\mathcal{F}(X)$, such that $E\left(\int_0^\infty \phi^2(s)\,d\langle X\rangle_s\right) < \infty$, and $Y = c + \int_0^\infty \phi(s)\,dX_s$. But, if this property is true, we have, from Proposition (1.3), as $\tau_\infty = \infty$,

$$\int_0^\infty \phi(s,\omega)\,dX_s = \int_0^\infty \phi(\tau_s\omega,\omega)\,d\beta_s = \int_0^\infty \psi(s,\omega)\,d\beta_s ,$$

where ψ is the $L^2(\Pi, ds\,dP)$-projection of $\phi(\tau_\cdot, \cdot)$, and Π is the predictable σ-field w.r.t $(\mathcal{F}(X)_{\tau_t})$.

The converse is also easily obtained from Proposition (1.3), when one remarks that:

(i) $(\langle X\rangle_t)$ is a continuous $(\mathcal{F}(X)_{\tau_t})$ change of time.

and

(ii) if ψ is a Π-measurable process, then: $(t,\omega) \longrightarrow \psi(\langle X\rangle_t, \omega)$ is predictable w.r.t $\{\mathcal{F}(X)_t\}$.

We now develop the promised "constructive" method. Let $(\Omega, \mathcal{G}, (\mathcal{G}_t), P)$ be a given "usual" filtered probability space such that:

(A1) $L^1(\Omega, \mathcal{G}_\infty, P)$ is separable.

(A2) there exists a $(\mathcal{G}_t)$ Brownian motion $(\beta_t)_{t\geq 0}$ (with $\beta_0 = 0$).

Dellacherie and Stricker [4] have remarked that the assumption (A1) is equivalent to the existence of a (far from unique!) increasing process $(\tau_t)_{t\geq 0}$ such that $\mathcal{G}_t = \mathcal{F}(\tau_\cdot)_t$, for every t. Of course, (τ_t) can be chosen to be strictly increasing, continuous, and $\tau_t \geq t$. We will always assume that these properties are verified.

Now, define $A_t = \inf\{s \,/\, \tau_s > t\}$ $(t \geq 0)$. From the inequality: $A_t \leq t$, and Doob's optional sampling theorem, we deduce that $X_t = \beta_{A_t}$ is a $(\mathcal{G}_{A_t})$ continuous martingale, with increasing process $\langle X\rangle = A$, and associated D.D.S Brownian motion β.

The following theorem is devoted to the discussion of the extremality and/or the purity of this martingale X in terms of β. Although quite easy to obtain, these results will be very helpful in the sequel of the paper.

(2.2) Theorem X, defined by the previous construction, verifies one, and only one of the following properties:

(i) X is pure iff: $\mathcal{F}(\beta)_\infty = \mathcal{G}_\infty$

(ii) X is extremal but not pure iff:
β has the predictable representation property with respect to $(\mathcal{G}_t)$, but $\mathcal{F}(\beta)_\infty \subsetneq \mathcal{G}_\infty$.

(iii) X is not extremal iff:
β does not have the predictable representation property w.r.t $(\mathcal{G}_t)$.

Theorem (2·2) is a simple consequence of theorem (2·1), once the following result is known.

(2·3) Lemma. (8) $\forall t, \quad \mathcal{G}_t = \mathcal{F}(X)_{\tau_t}$.

Remark Notice that: a) (8) implies (8_∞) : $\mathcal{G}_\infty = \mathcal{F}(X)_\infty$, because $\tau_t \uparrow \infty$, as $t \uparrow \infty$.

b) From lemma (1·5), (8) is equivalent to

$(8')$: $\forall t, \quad \mathcal{G}_{A_t} = \mathcal{F}(X)_t$.

Proof of the lemma: Indeed, we shall show that $(8')$ is true.

i) For every t, $A_t = \inf\{s \,/\, \tau_s > t\}$; so, A_t is a $(\mathcal{G}_s)_{s \geq 0}$ stopping time, and $X_t = \beta_{A_t}$ is $\mathcal{G}_{A_t}$-measurable.

Thus, $\mathcal{F}(X)_t \subseteq \mathcal{G}_{A_t}$, for every t.

ii) Conversely, for every t, one has $\mathcal{G}_{(A_t)^-} = \sigma\{\tau_{(s \wedge A_t)}, s \geq 0 ; A_t\}$, up to negligible sets, as $\mathcal{G}_t = \mathcal{F}(\tau_\cdot)_t$.

As A_t is $\mathcal{F}(X)_t$-measurable, and $\mathcal{G}_{(A_t)} = \bigcap_n \left(\mathcal{G}_{(A_{t+\frac{1}{n}})^-}\right)$, we need only show, to prove the inclusion: $\mathcal{G}_{A_t} \subseteq \mathcal{F}(X)_t$, that:

(9) $\forall s \quad \tau_{(s \wedge A_t)}$ is $\mathcal{F}(X)_t$-measurable.

But, $\tau_{(s \wedge A_t)} = \tau_s 1_{(\tau_s < t)} + t 1_{(A_t \leq s)}$.

As τ_s is a $(\mathcal{F}(X)_t)$ stopping time, $\tau_s 1_{(\tau_s < t)}$ is $\mathcal{F}(X)_t$-measurable; Finally, as A_t is $\mathcal{F}(X)_t$ measurable, (9) is proved, and so is (8).

The assertion (ii) of theorem (2·2) provides an easy tool to construct extremal but not pure, continuous martingales. Indeed, theorem (2·2) shows that all is needed is to exhibit filtrations $(\mathcal{G}_t)$ supporting a $(\mathcal{G}_t)$ Brownian motion (β_t), which possesses the representation property w.r.t $(\mathcal{G}_t)$, but such that $\mathcal{F}(\beta)_\infty \subsetneq \mathcal{G}_\infty$.

In this paper, we develop two methods to produce such couples $(\mathcal{G}, \beta)$: the end of this chapter is devoted to the first one, and the second one is dealt with in the fourth chapter.

Our first method is to take for $(\mathcal{G}_t)$ the natural filtration of a real valued Brownian motion $(B_t)_{t\geq 0}$, with $B_0 = 0$. Then, from Itô's theorem, B has the predictable representation property w.r.t. $(\mathcal{G}_t)$; moreover, as any $(\mathcal{G}_t)$ Brownian motion (β_t), with $\beta_0 = 0$, may be written as: $\beta_t = \int_0^t \zeta_s \, dB_s$, with $|\zeta| = 1$ $ds\,dP(\omega)$ a.e., any such β has the representation property w.r.t $(\mathcal{G}_t)$. So, all is needed is to construct a $(\mathcal{G}_t)$ Brownian motion (β_t) such that $\mathcal{G}_\infty \supsetneq \mathcal{F}(\beta)_\infty$.

The easiest example is probably the one given in [20], e.g.:

$\beta_t = \int_0^t \mathrm{sgn}(B_s)\, dB_s$, as $\mathcal{F}(\beta) = \mathcal{F}(|B|)$, and $\mathcal{F}(\beta)_\infty \subsetneq \mathcal{F}(B)_\infty$.