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A list of formulae on the distributions of extremes of the Brownian bridge and other processes.

<u>Introduction</u>: In this work, we present formulae for the extremes of a number of random processes, such as the Brownian bridge $(b(t), t \leq 1)$, the brownian meander $(m(t), t \leq 1)$, free (i.e: unconditioned) Brownian motion $(B_t, t \leq 1)$, perturbed Brownian bridges $(b(t) + \mu \ell_t^{(b)}, t \leq 1)$, where $(\ell_t^{(b)}, t \leq 1)$ denotes the local time at 0 of $(b(t), t \leq 1)$, Walsh's brownian motions taking values in n rays in the plane, and finally Bessel processes and Bessel bridges.

Many papers have already been published about such distributions, among which:

However, our presentation differs somewhat from the existing ones, in that our formulae exhibit the laws of $|N| X$, where X is one of the extreme variables listed above, and N is a $N(0,1)$ variable, which is independent of X. Without the factor $|N|$, the formulae found in the above list of papers involve theta functions and seem in general hard to use, whereas our formulae are written in terms of the hyperbolic functions sinh and cosh. (see, e.g., the formulae for the Brownian bridge in A).

In fact, the simplicity of these formulae makes it easy to compare some of them, and, thus, obtain some striking identities in law. Thus, comparing formulae (1.a") and (4.a") in the list below, we arrived to the identity in law:

$$\left(\sup_{t\leq 1} b(t),\ \inf_{t\leq 1} b(t)\right) \overset{(law)}{=} \left(\sup_{t\leq 1}\left(|b(t)|-\frac{1}{2}\ell_t^{(b)}\right),\ \inf_{t\leq 1}\left(|b(t)|-\frac{1}{2}\ell_t^{(b)}\right)\right)$$

(note that, on the right hand side, the infimum is precisely $-\frac{1}{2}\ell_1^{(b)}$, but it is more suggestive to write ~~to~~ ~~the~~ the identity in this form).

This led us to ~~the~~

Theorem 1: Let $X_t = |b(t)| - \frac{1}{2}\ell_t^{(b)}$, $t\leq 1$.

The processes of local times $(\ell_1^a(b);\ a\in\mathbb{R})$ and $(\ell_1^a(X);\ a\in\mathbb{R})$ have the same distribution.

Theorem 2: Let $(r(t), t\leq 1)$ be the 3-dimensional standard Bessel bridge, or standard Brownian excursion. Let $Y_t = |b(t)| + \frac{1}{2}\ell_t^{(b)}$, $t\leq 1$.
The processes of local times $(\ell_1^a(r);\ a\geq 0)$ and $(\ell_1^a(Y);\ a\geq 0)$ have the same distribution.

These theorems are the main really new results about the Brownian bridge, or the 3-dimensional Bessel process, to be found in our paper. They are discussed and proved in section D, where some recent extensions by Carmona-Petit-Yor [ ] are also stated.

For the moment, we should like to explain how the random factor $|N|$ arises in most of the computations presented in this paper.

It is well known that the computation of the distribution of a Brownian functional $(H_u, u \geq 0)$ at a fixed time t is more complicated than when t is replaced by T, an exponential random variable, which is independent of the Brownian motion; the latter computation may be done with the help of Feynman-Kac formula; since the seventies, this formula has been "disintegrated" and interpreted in terms of the theory of Brownian excursions (see, e.g., Biane-Yor [ ], or Revuz-Yor [ ], Chap. $\overline{\text{XII}}$, for some examples).

When $(H_u, u \geq 0)$ is an homogeneous functional of the Brownian motion $(B_u, u \geq 0)$, by which we mean

$$(1) \qquad \text{for } c > 0, \quad (B_{ct}; H_{ct}; t \geq 0) \overset{(law)}{=} (\sqrt{c}B_t; \sqrt{c}H_t; t \geq 0),$$

we obtain, if T is independent of B:

$$(1') \qquad (B_T; H_T) \overset{(law)}{=} \sqrt{T}\,(B_1; H_1),$$

and the law of the couple (B_1, H_1) may be recovered from the knowledge of the law of the left-hand side of $(1')$.

In this paper, we shall exploit the homogeneity property of H in order to obtain, with the help of a relation which is similar to $(1')$ the law of $H^{(b)}$, which denotes the value of H at time 1, when the Brownian motion B is replaced by the Brownian bridge b, from the law of H_{g_T}, where we have denoted $g_t = \sup\{s < t : B_s = 0\}$.

To this effect we recall that, for $t > 0$, the process $\left(b(u) = \frac{1}{\sqrt{g_t}} B_{u g_t}; u \leq 1\right)$ is a standard Brownian bridge, which is independent of the σ-field

generated by the variable g_t and the process $(B_{g_t+u};\, u \geq 0)$.
Consequently, we have:

(2) $$H_{g_T} \overset{(law)}{=} \sqrt{g_T}\, H^{(b)}.$$

In the sequel, $\tilde{T}$ will denote an exponential random variable with parameter $\left(\frac{1}{2}\right)$, which is independent of B. It is then easy to show that:

$$\sqrt{g_{\tilde{T}}} \overset{(law)}{=} |N|,$$

where N denotes a $N(0,1)$ random variable. Hence, the identity (2) may be written in the equivalent form:

(2') $$|N|\, H^{(b)} \overset{(law)}{=} H_{g_{\tilde{T}}},$$

and the homogeneous functional H may be ~~vector~~ valued in $\mathbb{R}^n$.

In order to compute the law of the right-hand side of (2'), we shall use the following relations:

if $(\ell_t, t \geq 0)$ denotes the local time of B at 0, and if $\tau_s = \inf\{t : \ell_t > s\}$ then:

$$P(\ell_{\tilde{T}} \in d\ell) = \exp(-\ell)\, d\ell,$$

and, for every Brownian functional F, taking values in $\mathbb{R}_+$, we have:

(3) $$E\left[F(B_u;\, u \leq g_{\tilde{T}}) \,\middle|\, \ell_{\tilde{T}} = \ell\right] = e^{\ell}\, E\left[\exp\left(-\frac{\tau_\ell}{2}\right) F(B_u;\, u \leq \tau_\ell)\right]$$

consequently:

(3') $$E\left[F(B_u;\, u \leq g_{\tilde{T}})\right] = \int_0^\infty d\ell\; E\left[F(B_u;\, u \leq \tau_\ell) \exp\left(-\frac{\tau_\ell}{2}\right)\right]$$

Bringing the identities (2') and (3') together, we obtain the key identity:

(4) $$P\left(|N| H^{(b)} \in dh;\, |N| \ell^{(b)} \in d\ell\right) = E\left[H_{\tau_\ell} \in dh;\, \exp\left(-\frac{\tau_\ell}{2}\right)\right] d\ell$$

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A. List of formulae.

1. The Brownian bridge.

$(b(t), t \leq 1)$ denotes a standard Brownian bridge, and $(\ell_t^{(b)}, t \leq 1)$ is its local time at 0.

Let $S^{(b)} = \sup_{t \leq 1} b(t)$, $I^{(b)} = -\inf_{t \leq 1} b(t)$, $M^{(b)} = \sup_{t \leq 1} |b(t)|$

Then:

(1.a) $P\left(|N| S^{(b)} \leq x, |N| I^{(b)} \leq y, |N| \ell^{(b)} \in ds\right) = \exp\left(-\frac{s}{2}\left(\coth x + \coth y\right)\right) ds$

Consequently:

(1.a') $P\left(|N| S^{(b)} \leq x ; |N| I^{(b)} \leq y\right) = 2 \Big/ (\coth x) + (\coth y)$.

(1.a'') $P\left(|N| S^{(b)} \in dx ; |N| I^{(b)} \in dy\right) = \dfrac{4 \sinh(x) \sinh(y)\, dx\, dy}{\left(\sinh(x+y)\right)^3}$

Taking $x = y$ in (1.a), we obtain:

(1.b) $P\left(|N| M^{(b)} \leq x ; |N| \ell^{(b)} \in ds\right) = \exp\left(-s(\coth x)\right) ds$

Consequently:

(1.b') $P\left(|N| M^{(b)} \leq x\right) = \tanh(x)$; (1.b'') $P\left(|N| M^{(b)} \in dx\right) = \dfrac{dx}{(\cosh^2 x)}$.

2. The Brownian meander.

$(m(t), t \leq 1)$ denotes a standard Brownian meander. Let $S^{(m)} = \sup_{t \leq 1} m(t)$.

Then:

(2.a) $P\left(|N| S^{(m)} \leq x\right) = \cancel{\tanh(2x)}$.

Ceci serait: $th(x/2)$??

Verif. à la [illegible]!!

$P\left(|N| \left(\frac{1}{2} S^{(m)}\right) \leq x\right) = P\left(|N| S^{(m)} \leq 2x\right) = th(x)$. /

Comparing (2.a) and (1.b'), we obtain Kennedy's identity:

$$(2.b) \qquad M^{(b)} \overset{(law)}{=} \frac{1}{2} S^{(m)}.$$

[See complement on opposite page]

3. The (free) Brownian motion.

$(B_t, t \geq 0)$ is a 1-dimensional Brownian motion starting from 0, $S_t = \sup_{s \leq t} B_s$, $I_t = -\inf_{s \leq t} B$ and $(\ell_t, t \geq 0)$ is the local time of B at 0. $\tilde{T}$ denotes an exponential random variable, with parameter $\left(\frac{1}{2}\right)$, which is independent of B. We have:

$$(3.a) \qquad P\left(S_{\tilde{T}} \leq x ; I_{\tilde{T}} \leq y\right) = 1 - \frac{(\sinh x) + (\sinh y)}{\sinh(x+y)}$$

(Note that, thanks to scaling, the left hand side of (3.a) is equal to:

$$P\left(\sqrt{\tilde{T}}\, S_1 \leq x ; \sqrt{\tilde{T}}\, I_1 \leq y\right))$$

Formula (3.a) may be generalized as a 4-variate formula:

$$(3.b) \qquad P\left(\sqrt{\tilde{T}}\, S_1 \leq x ; \sqrt{\tilde{T}}\, I_1 \leq y ; \sqrt{\tilde{T}}\, \ell_1 \in d\ell ; \sqrt{\tilde{T}}\, |B_1| \in dz\right)$$

$$= d\ell \left\{ \exp\left(-\frac{\ell}{2}(\coth x + \coth y)\right)\right\} \frac{1}{2} (\sinh z)\, dz\, 1_{(z \leq x \wedge y)} \left[\frac{1}{\sinh^2(x)} + \frac{1}{\sinh^2(y)}\right]$$

||| il y a une erreur ds cette formule !! et il faut établir, ce que l'on peut faire assez aisément !! ??

Consequently, we have:

$$P\left(\sqrt{\tilde{T}}\, S_1 \leq x ; \sqrt{\tilde{T}}\, I_1 \leq y ; \sqrt{\tilde{T}}\, |B_1| \in dz\right)$$

$$(3.c) \qquad = \frac{\left(\sinh^2(x) + \sinh^2(y)\right) (\sinh z)\, dz}{\sinh(x+y)\, \sinh(x) \sinh(y)} 1_{(z \leq x \wedge y)}$$

4. Perturbed Brownian bridges.

We use the notation introduced above for the standard Brownian bridge. Moreover, for $\mu > 0$, and $\nu > 0$, we define:

$$S_\mu^{(b)} = \sup_{t \leq 1} \left(b(t) - \mu \ell_t^{(b)} \right) \quad , \quad I_\nu^{(b)} = \sup_{t \leq 1} \left(-b(t) - \nu \ell_t^{(b)} \right).$$

Then:

$$(4.a) \quad P\left(|N| S_\mu^{(b)} \leq x ; |N| I_\nu^{(b)} \leq y ; |N| \ell^{(b)} \in ds \right) = \left(\frac{\sinh(x)}{\sinh(x+\mu s)} \right)^{\frac{1}{2\mu}} \left(\frac{\sinh(y)}{\sinh(y+\nu s)} \right)^{\frac{1}{2\nu}} ds$$

In particular, if we define: $M_\mu^{(b)} = \sup_{t \leq 1} \left(|b(t)| - \mu \ell_t^{(b)} \right)$, we obtain:

$$(4.a') \quad P\left(|N| M_\mu^{(b)} \leq x ; |N| \ell^{(b)} \in ds \right) = \left(\frac{\sinh(x)}{\sinh(x+\mu s)} \right)^{\frac{1}{\mu}} ds,$$

and, consequently:

$$(4.a'') \quad P\left(|N| M_\mu^{(b)} \in dx ; |N| \ell^{(b)} \in ds \right) = \frac{\sinh(\mu s)}{(\sinh(x+\mu s))^2} \frac{1}{\mu} \left(\frac{\sinh(x)}{\sinh(x+\mu s)} \right)^{\frac{1}{\mu}-1} dx\, ds.$$

If we define: ${}^+M_\mu^{(b)} = \sup_{t \leq 1} \left(|b(t)| + \mu \ell_t^{(b)} \right)$, then:

$$(4.b') \quad P\left(|N| \, {}^+M_\mu^{(b)} \leq x ; |N| \ell^{(b)} \in ds \right) = \left(\frac{\sinh(x-\mu s)}{\sinh(x)} \right)^{\frac{1}{\mu}} ds \qquad (x > \mu s)$$

and, consequently:

$$(4.b'') \quad P\left(|N| \, {}^+M_\mu^{(b)} \in dx ; |N| \ell^{(b)} \in ds \right) = \frac{\sinh(\mu s)}{(\sinh(x))^2} \frac{1}{\mu} \left(\frac{\sinh(x-\mu s)}{\sinh(x)} \right)^{\frac{1}{\mu}-1} dx\, ds\, 1_{(x>\mu}$$

5. Walsh's Brownian motions.

Let $(B_k(t);t\geq 0)$ denote a Walsh Brownian motion taking values in k rays $I_1, I_2, \dots, I_k$, such that when B_k is at the meeting point of the rays, it chooses, informally speaking, either of the k rays with probability $\frac{1}{k}$.

Let $(b_k(t);t\leq 1)$ be the associated bridge, i.e: $(B_k(t), t\leq 1)$, conditioned by $B_k(1)=0$. We note:

$S_{(j)}^{(b_k)} = \sup_{t\leq 1}\{|b_k(t)|;\, b_k(t)\in I_j\}$, and simply $S_{(j)} = \sup_{t\leq 1}\{|B_k(t)|;\, B_k(t)\in I_j\}$

$\tilde{T}$ denotes an exponential variable, with parameter $\left(\frac{1}{2}\right)$, which is independent of B_k.
Then, we have:

$$(5.a)\quad P\left(|N|S_{(1)}^{(b_k)}\leq x_1;\,\dots\dots;|N|S_{(k)}^{(b_k)}\leq x_k\right) = k\Big/\left(\sum_{j=1}^{k}\coth(x_j)\right).$$

$$(5.b)\quad P\left(\sqrt{\tilde{T}}\,S_{(1)}\leq x_1;\,\dots\dots;\sqrt{\tilde{T}}\,S_{(k)}\leq x_k\right) = \frac{\sum_{j=1}^{k}\tanh\left(\frac{x_j}{2}\right)}{\sum_{j=1}^{k}\coth\left(x_j\right)}$$

(we indicate here the elementary formula: $\tanh\left(\frac{x}{2}\right) = (\coth x) - \frac{1}{\sinh(x)}$ which helps to simplify further computations; see C'.4, below).

6. Bessel processes and Bessel bridges.

(i) $(R_t, t\geq 0)$ denotes a Bessel process starting at 0, with dimension $\delta = 2(1+\mu) \in (0,2)$.
μ is the index of R; let $\nu = -\mu \in (0,1)$. $P^{(\mu)}$ is the distribution of R.
We choose the local time of R at level 0 such that its inverse process
$(\tau(s); s\geq 0)$ satisfies:

$$E^{(\mu)}\left(\exp -\frac{\lambda}{2}\tau(s)\right) = \exp\left(-\lambda^{\nu} s\right)$$

Then, if $M_t = \sup_{s\leq t} R_s$, we have:

$$(6.a)_\mu \quad E^{(\mu)}\left[M_{\tau(1)} \leq x ; \exp\left(-\frac{\lambda}{2}\tau(1)\right)\right] = \exp\left(-\lambda^{\nu} h_\mu(x\sqrt{\lambda})\right), \text{ where } h_\mu(x) = \frac{I_\mu(x)}{I_\nu(x)},$$

and I_θ denotes the usual modified Bessel function, with parameter θ.

(ii) Let $(r(t), t\leq 1)$ be the corresponding Bessel bridge, and let $M^{(r)} = \sup_{t\leq 1} r(t)$.
Then, we have:

$$(6.b)_\mu \quad P^{(\mu)}\left(\sqrt{2Z_\nu}\, M^{(r)} \leq x\right) = \frac{1}{h_\mu(x)} = \frac{I_\nu(x)}{I_\mu(x)},$$

where Z_ν denotes a gamma variable with parameter ν, independent of the process r.

(iii) Let $(m(t); t\leq 1)$ be the Bessel meander, and let: $M^{(m)} = \sup_{t\leq 1} m(t)$.
Then, we have:

$$(6.c)_\mu \quad P^{(\mu)}\left(\sqrt{2Z_{1-\nu}}\, M^{(m)} \leq x\right) = \frac{c_\mu I_\mu(x) - x^{\mu}}{c_\mu I_\nu(x)},$$

where $Z_{1-\nu}$ denotes a gamma variable with parameter $(1-\nu)$, independent of the process m.

(iv) Let $(\tilde{R}_t, t\geq 0)$ be the symmetrized Bessel process with index μ, and $(\tilde{r}(t), t\leq 1)$ be the corresponding bridge. We write $\tilde{P}^{(\mu)}$ for the law of $\tilde{R}$, and we note:

$$S^{\tilde{r}} = \sup_{t\leq 1} \tilde{r}(t), \quad S_1 = \sup_{t\leq 1} \tilde{R}_t, \quad I_1 = -\inf_{t\leq 1} \tilde{R}_t .$$

Then, we obtain, with the same conventions as above:

$$(6.d)_\mu \quad \tilde{P}^{(\mu)}\left(\sqrt{2Z_\nu}\, S^{\tilde{r}} \leq x ; \sqrt{2Z_\nu}\, I^{\tilde{r}} \leq y\right) = \frac{2}{h_\mu(x)+h_\mu(y)}$$

$$(6.e)_\mu \quad \tilde{P}^{(\mu)}\left(\sqrt{\tilde{T}}\, S_1 \leq x ; \sqrt{\tilde{T}}\, I_1 \leq y\right) = \frac{\varphi_\mu(x)h_\mu(x)+\varphi_\mu(y)h_\mu(y)}{h_\mu(x)+h_\mu(y)} .$$

where $\varphi_\mu(x) = 1 - \dfrac{x^\mu}{c_\mu I_\mu(x)}$, and $c_\mu = 2^\mu \Gamma(\mu+1)$.