A remarkable subordinator with Generalized Gamma Convolutions marginals.

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This Note is devoted to the presentation of a number of properties of a subordinator, which we shall denote here as $(\Delta_t, t \geq 0)$, whose Lévy-Khintchine representation is simply:

$$(1) \qquad E\left[\exp(-\lambda \Delta_t)\right] = \left(\sqrt{1+\lambda} - \sqrt{\lambda}\right)^{2t}, \quad t, \lambda \geq 0.$$

$$(2) \qquad \equiv \left(1 + 2\lambda - 2\sqrt{(1+\lambda)\lambda}\right)^{t}$$

Despite the simplicity - or, perhaps, thanks to the simplicity! - of this formula, the process $(\Delta_t, t \geq 0)$ occurs in several instances in probability theory, and enjoys many interesting properties.

a) Consider a real-valued Brownian motion $(B_u, u \geq 0)$, and denote:

$$d_s = \inf\{u > s : B_u = 0\}; \quad g_s = \sup\{u < s : B_u = 0\}$$

Then, for $\mathbb{e}$ an independent r.v., with standard exponential distribution, one has: $E\left[\exp(-\lambda(d_{\mathbb{e}} - g_{\mathbb{e}}))\right] = \sqrt{1+\lambda} - \sqrt{\lambda}$;

hence: $d_{\mathbb{e}} - g_{\mathbb{e}} \sim \Delta_{1/2}$ (3).

The identity (3) is a very particular case of the identification of the distribution of $d_{\mathbb{e}} - g_{\mathbb{e}}$, when $(B_u, u \geq 0)$ is replaced by a general

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diffusion $(X_u, u \geq 0)$; see BFRY, when X is a $BES(d)$, for $0<d<2$, and Winkel [ ] for a more general study.

b) As formula (3) suggests, the process $(\Delta_{t/2}, t \geq 0)$ may occur more naturally than $(\Delta_t, t \geq 0)$; throughout this Note, the factor 2 will appear repeatedly, and formulae need to be carefully checked (because of this feature in particular!!) -

c). The process $(\Delta_t, t \geq 0)$ is a very particular example among the subordinators indexed by $\mathbb{R}_+$-valued r.v.'s G, which we shall denote by $(\Delta_t(G), t \geq 0)$, and whose Lévy-Khintchine representation is:

$$(4) \quad E\left[\exp\left(-\lambda \Delta_t(G)\right)\right] = \exp\left(-t\int_0^\infty dx\,(1-e^{-\lambda x})\,\frac{E(e^{-xG})}{x}\right)$$

Indeed, the process $(\Delta_t, t \geq 0)$ is $(\Delta_t(G_{1/2}), t \geq 0)$, where $G_{1/2} \sim \beta(\frac{1}{2},\frac{1}{2})$, a beta $(\frac{1}{2},\frac{1}{2})$ r.v., also known as arc-sine variable, i.e.:

$$P\left(\beta(\tfrac{1}{2},\tfrac{1}{2}) \in du\right) = \frac{du}{\pi\sqrt{u(1-u)}} \qquad (0<u<1)$$

Later in the Note, more general beta(a,b) variables will play some role; recall

$$P\left(\beta(a,b) \in du\right) = \frac{u^{a-1}(1-u)^{b-1}\,du}{B(a,b)} \qquad (0<u<1).$$

We also indicate that in BFRY, the subordinators $(\Delta_t(G_\alpha), t \geq 0)$ occur, for every $\alpha \in (0,1)$, with G_α defined in terms of two independent stable(α) variables T_α and T'_α; for $\alpha = 1/2$, ~~it is well known that:~~ $(G_{1/2} \sim)\ \beta_{1/2,1/2} \sim T_{1/2}/(T_{1/2}+T'_{1/2})$

d) As a first occurrence of the care to take with factors 2, or 1/2, in our discussion, we note that the Laplace transform which ~~occurs~~ is featured in the Lévy-Khintchine formula (4) for $(\Delta_t \equiv \Delta_t(G_{1/2}), t \geq 0)$ is:

$$(5) \quad \underline{E\left[\exp\left(-x G_{1/2}\right)\right]} = \underline{E\left[\exp\left(-x \beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)\right]} = \underline{e^{-x/2} I_0\left(x/2\right)}.$$

This leads us naturally to the following statement found in [MNY, Theorem 1.1, p. 214.]

<u>Theorem</u>: a) The <u>modified Bessel functions</u> $(I_\nu, \nu \geq 0)$ <u>satisfy the Lipschitz-Hankel formula:</u>

$$(6) \quad \nu \int_0^\infty e^{-ax} I_\nu(x) \frac{dx}{x} = \left(a + \sqrt{a^2-1}\right)^{-\nu}, \quad a \geq 1;$$

b) There exists a <u>subordinator</u>, <u>denoted by</u> $(\mathcal{J}_\nu, \nu \geq 0)$, <u>such that its Lévy measure is:</u>

$$(7) \quad \ell_{\mathcal{J}}(dx) = I_0(x) e^{-x} \frac{dx}{x}$$

c) For every $\nu > 0$, the law of $\mathcal{J}_\nu$ is given by:

$$(8) \quad \underline{P(\mathcal{J}_\nu \in dx)} = \underline{\nu I_\nu(x) e^{-x} \frac{dx}{x}}.$$

and its Laplace transform is:

$$(9) \quad E\left[\exp(-\lambda \mathcal{J}_\nu)\right] = \left((1+\lambda) + \sqrt{(1+\lambda)^2 - 1}\right)^{-\nu}$$

We now compare formula (9) with formulae (1) and (2), or rather with the obviously equivalent formulae:

$$(1') \quad E\left[\exp(-\lambda \Delta_t)\right] = \left(\sqrt{1+\lambda} + \sqrt{\lambda}\right)^{-2t}$$

$$(2') \quad \qquad = \left(1 + 2\lambda + 2\sqrt{(1+\lambda)\lambda}\right)^{-t}$$

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Thus, we obtain, by replacing λ by $(2J)$ in formula (9):

$$(10) \quad \left(\Delta_t \equiv \Delta_t(\Theta_{1/2}),\ t \geq 0 \right) \sim \left(2J_t,\ t \geq 0 \right)$$

by comparison with formula (2').

e) Furthermore, the identification of the density of J_t, as given in (8) now leads to:

Proposition: For fixed $t>0$, the distribution of $\Delta_t \sim 2J_t$ is that of: $\gamma_t / \beta(\frac{1}{2}, \frac{1}{2}+t)$, where γ_t and $\beta(\frac{1}{2}, \frac{1}{2}+t)$ are independent, and γ_t indicates a gamma (t) variable.

We shall give two proofs of this result: the first one simply consists in identifying the Laplace transform, in general, of $\gamma_a / \beta(b,c)$, for 3 general values of the parameters a, b, c, and to identify this Laplace transform, for $a=t$, $b=1/2$, $c=\frac{1}{2}+t$, with that of Δ_t, as given by (1).

Undoubtedly, the following lemma is well-known:

Lemma: The Laplace transform $\gamma_a / \beta(b,c)$ is given by:

$$E\left[\exp\left(-\lambda \frac{\gamma_a}{\beta(b,c)}\right)\right] = \frac{\Gamma(a+b)\,\Gamma(b+c)}{\Gamma(b)\,\Gamma(a+b+c)} \frac{1}{\lambda^a} F\left(a, a+b, a+b+c, -\frac{1}{\lambda}\right)$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function with 3 parameters (α, β, γ), which may be defined via the integral representation:

$$F(\alpha, \beta, \gamma, z) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 dt\, t^{\beta-1} (1-t)^{\gamma-\beta-1} \frac{1}{(1-tz)^\alpha}$$

$$\equiv E\left[\frac{1}{(1-\beta(\beta, \gamma-\beta)z)^\alpha}\right]$$

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f) As the reader may have guessed, we did not arrive to the result of the Proposition, i.e. the identification of the law of Δ_t as that of a beta-gamma ratio, with the proof we just gave. In fact, we first obtained this result as the consequence of the following identification of $(\Delta_t, t \geq 0)$ as a subordinator $(\hat{T}_u, u \geq 0)$ being time changed with an independent gamma process $(\gamma_t, t \geq 0)$.
Precisely, there is the following

<u>Proposition</u>: (i) The <u>subordinator</u> $(\Delta_{t/2}, t \geq 0)$ <u>may be represented</u> <u>as</u>: $(\hat{T}_{\gamma_t}, t \geq 0)$ <u>where</u> $(\hat{T}_u, u \geq 0)$ <u>is a subordinator, independent of the standard gamma process</u> $(\gamma_t, t \geq 0)$, <u>and</u> <u>(***)</u>

(ii) <u>As a consequence, the density of the variable Δ_t is given by</u>:

$$P(\Delta_t \in dx)/dx = \frac{2^{2t}\,\Gamma(1+t)}{(2\pi)\,\Gamma(2t)}\, x^{t-1} \int_0^1 dm\, e^{-mx} \left(m(1-m)\right)^{\left(t-\frac{1}{2}\right)} \quad (11)$$

(iii) Hence,

$$\Delta_t \sim \gamma_t \big/ \beta\left(\tfrac{1}{2}, \tfrac{1}{2}+t\right) \quad . \quad (12)$$

<u>Proof</u>: (i) This follows from formula (1'), which we rewrite as follows:

$$E\left[\exp\left(-\lambda \Delta_{t/2}\right)\right] = \frac{1}{\left(1+\left(\sqrt{1+\lambda}-1\right)+\sqrt{\lambda}\right)^t}$$

$$= \frac{1}{\Gamma(t)} \int_0^\infty du\, u^{t-1} e^{-u\left[1+\left(\sqrt{1+\lambda}-1\right)+\sqrt{\lambda}\right]}$$

$$= \frac{1}{\Gamma(t)} \int_0^\infty du\, u^{t-1} e^{-u}\, E\left[e^{-\lambda \tilde{T}_u}\right] E\left(e^{-\lambda T_u}\right)$$

$$= E\left[\exp\left(-\lambda \hat{T}_{\gamma_t}\right)\right]$$

(***) $\hat{T}_u = \tilde{T}_u + T_u$, $u \geq 0$, with $(\tilde{T}_u)$ and (T_u) two independent subordinators, which may be characterized through their Laplace transforms:

$$\begin{cases} E\left[e^{-\lambda \tilde{T}_u}\right] = \exp\left(-u\left(\sqrt{1+\lambda}-1\right)\right) \\ E\left[e^{-\lambda T_u}\right] = \exp\left(-u\sqrt{\lambda}\right) \end{cases}$$

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(iii) ~~(ii)~~/ This point will now follow easily from (i), and the fact that the marginal distributions of $(T_u, u \geq 0)$ and $(T_u, u \geq 0)$ are well-known. We leave the details to the interested reader.
(→ cf: my Fax to Lancelot on Aug. 23rd)
[Proposition 1 and 2].

γ) Occurrences of the subordinator $(\Delta_t, t \geq 0)$.
(a discussion from the references below).

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