A solution to Skorokhod's problem for the age process of Brownian excursions

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1. Motivation and Introduction.

(1.1) Let $(B_t, t \geq 0)$ denote a 1-dim BM starting from 0, and $S_t = \sup_{u \leq t} B_u$.

Azéma-Yor [·] gave the following explicit solution to Skorokhod's problem of embedding the probability μ (with: $\int d\mu(x)|x| < \infty$; $\int d\mu(x)\, x = 0$) into Brownian motion, precisely:

if $T_\mu = \inf\{t:\ S_t \geq \psi_\mu(B_t)\}$, with $\psi_\mu(x) = \frac{1}{\mu([x,\infty))} \int_{[x,\infty)} t\, d\mu(t)$,

then: $B_{T_\mu} \sim \mu$.

(1.2). In this paper, we would like to do the same kind of construction for Azéma's martingale $\mu_t \stackrel{def}{=} c\, sgn(B_t)\sqrt{(t - g_t)}$, where c is a universal constant (e.g., take $c = 1$), and $g_t = \sup\{s \leq t:\ B_s = 0\}$.

For $c =$, $(\mu_t, t \geq 0)$ is the projection of $(B_t, t \geq 0)$ on the filtration generated by the ~~zero set~~ (signs) of B, i.e.: $\mathcal{Z}_t \equiv \sigma\{sgn(B_s), s \leq t\}$.

Hence, it is a somewhat natural object, which might even be considered as some kind of Brownian motion —

2. A computation using excursion theory.

(2.1) Let $T'_\varphi = \inf\{t : \sqrt{t - g_t} \geq \varphi(\ell_t)\}$

Then, one has:

$$P\left(\ell_{T'_\varphi} \geq \lambda\right) = P\left(T'_\varphi \geq \tau(\lambda)\right)$$

$$= P\left\{\begin{array}{l}\text{on the time interval } [0, \tau(\lambda)], \text{ no excursion had} \\ \text{a lifetime exceeding } \underbrace{\varphi^2(\cdot)}_{\text{denoting } \varphi^2 \text{ composed with the l.t. for this excursion}}\end{array}\right\}$$

$$= P\left\{\sum_{\mu \leq \lambda} 1_{(V(e_\mu) \geq \varphi^2(\mu))} = 0\right\}$$

But the random variable $\sum_{\mu \leq \lambda} 1_{(V(e_\mu) \geq \varphi^2(\mu))}$ is a Poisson variable, with parameter:

$$\int_0^\lambda d\mu \; n\left(V \geq \varphi^2(\mu)\right) = \int_0^\lambda d\mu \left(\int_{\varphi^2(\mu)}^\infty \frac{dv}{\sqrt{2\pi v^3}}\right)$$

$$= \int_0^\lambda d\mu \frac{1}{\sqrt{2\pi}} (2) \frac{1}{\varphi(\mu)}.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\lambda \frac{d\mu}{\varphi(\mu)}.$$

Thus, we have obtained:

$$P\left(\ell_{T'_\varphi} \geq \lambda\right) = \exp\left(-\sqrt{\frac{2}{\pi}} \int_0^\lambda \frac{d\mu}{\varphi(\mu)}\right).$$

(2.2). From the previous result, we can now deduce the law of $\sqrt{A_{T'_\varphi}} \equiv \varphi(\ell_{T'_\varphi})$

Indeed, if we denote by ν the law of $\sqrt{A_{T'_\varphi}}$, we have:

$$\bar{\nu}(x) \stackrel{\text{def}}{=} P\left(\sqrt{A_{T'_\varphi}} \geq x\right) = P\left(\varphi(\ell_{T'_\varphi}) \geq x\right)$$

So that: $\underline{\bar{\nu}(\varphi(x))} = P\left(\ell_{T'_\varphi} \geq x\right) = \underline{\exp\left(-\sqrt{\frac{2}{\pi}} \int_0^x \frac{d\lambda}{\varphi(\lambda)}\right)}$

(2.3) It seems quite natural to relate the above result with the computations of Azéma–Yor (Sem. Prob. XIII), who establish that if we define:

$$T_\phi = \inf\{t : \phi(S_t) \geq B_t\},$$

then:

$$P(S_{T_\phi} \geq x) = \exp - \int^x \frac{d\lambda}{(\lambda - \phi(\lambda))}$$

Hence, there is a correspondance between the 2 solutions, which may be obtained via:

$$\sqrt{\frac{2}{\pi}} \frac{1}{\varphi(\lambda)} = \frac{1}{\lambda - \phi(\lambda)}.$$

(2.4) We now <u>develop some explicit examples</u>.