# A unified treatment of asymptotic laws for planar BM.

(0.1) In recent years, a number of limit theorems for various functionals of complex BM, which we shall denote by $(Z_t)$, have been obtained: Kasahara-Kotani [ ] study functionals of the form $\int_0^t f(Z_s)\,ds$, with $f$ bounded with compact support, and such that $\int dx\, f(x)=0$; while Messulam-Yor [ ] are interested in $\int_0^t [u(Z_s)dX_s + v(Z_s)dY_s]$ with $u, v : \mathbb{C} \longrightarrow \mathbb{R}$ bounded, with compact support and Pitman-Yor ([ ], [ ]) give the asymptotic distribution of the winding numbers of $(Z_u, u \le t)$ around $(z_1, z_2, \cdots, z_n)$.

(0.2) Although the different techniques clearly show a common skeleton, we feel that at the present stage the whole picture is not as transparent as it could be. It is the first aim of this work to present a unified view of these different results and to show how all these convergences in distribution hold jointly.

(0.3) It also seems that there are still further interesting limit theorems which deserve to be exhibited, and this kind of exposition is the second aim of this paper. In particular, ----

# 1. Description of some known results.

(1.1) The following notation shall be used throughout this ~~A~~ paper:

- $Z = (Z_t, t \geq 0)$ is a complex Brownian motion starting from $z_0$ (which we shall often abbreviate as, $Z$ is a $BM_{z_0}(\mathbb{C})$),
- $(z_0, z_1, \ldots, z_n)$ are $(n+1)$ distinct points in $\mathbb{C}$,
- for each $j$, $1 \leq j \leq n$, $(\Phi^j_t, t \geq 0)$ is the continuous determination of the argument of $(Z_u, u \leq t)$ around $z_j$; moreover, for $r_j > 0$, we consider:

$$\Phi^j_\pm(t) = \int_0^t 1_{(|Z_s - z_j| \in I^j_\pm)} \, d\Phi^j_s,$$

where $I^j_- = (0, r_j)$ and $I^j_+ = (r_j, \infty)$.

- $L = (L(t), t \geq 0)$ is an increasing additive functional of $Z$, whose associated measure has total mass $2\pi$.
- $L^p_b(\mathbb{C})$ $(p \geq 1)$ is the set of locally bounded functions $\varphi : \mathbb{C} \to \mathbb{R}$ such that $\int dx dy \, |\varphi(z)|^p < \infty$.
- If $u, v \in L^2_b(\mathbb{C})$, we note $M^{u,v}_t = \int_0^t (u(Z_s) dX_s + v(Z_s) dY_s)$
- $h(t) = \frac{1}{2}(\log t)$.

(1.2) The asymptotic distribution of the winding numbers $(\Phi^j)$ is described in the following:

**Theorem 1** (Pitman-Yor [ ]): As $t \to \infty$,

$$\frac{1}{h(t)} \left[ \Phi^j_+(t), \Phi^j_-(t) \; (1 \leq j \leq n), L(t) \right] \xrightarrow{(d)} \left[ W_+, W^j_- \; (1 \leq j \leq n), \Lambda \right]$$

where:
- for each $j$, $(W_+, W_-^j, \Lambda)$ is distributed as

$$\left(\int_0^\sigma 1_{(\beta_s \geq 0)} d\gamma_s, \int_0^\sigma 1_{(\beta_s \leq 0)} d\gamma_s, \ell_\sigma\right)$$

with $(\beta, \gamma)$ a $BM_0(\mathbb{R}^2)$, $\sigma = \inf\{t : \beta_t = 1\}$,
and $(\ell_t, t \geq 0)$ is the local time of $\beta$ at $0$.
- the variables $W_+$ and $(W_-^j, 1 \leq j \leq n)$ are conditionally independent given $\Lambda$.

(1.3) The following theorem describes the asymptotic distribution of a class of additive functionals of bounded variation, such that the mass of their associated measure is $0$.

Theorem 2 (Kasahara-Kotani [ ]):
Let $f : \mathbb{C} \to \mathbb{R}$ be a bounded Borel function such that $\int |f(x)| \, |x|^\alpha dx < \infty$ for some $\alpha > 2$, and $\int f(x) dx = 0$.
Then:

$$\frac{1}{h(t)^{1/2}} \int_0^t f(Z_s) ds \xrightarrow{(d)} \sqrt{\langle f \rangle}\, U, \quad \text{where} \ldots$$

The work of Kasahara-Kotani inspired the following result

Theorem 2' (Messulam-Yor [ ]):
Let $u, v \in L_b^2(\mathbb{C})$. Then, as $t \to \infty$,

$$\frac{1}{h(t)^{1/2}} M_t^{u,v} \xrightarrow[t\to\infty]{(d)} \frac{1}{\sqrt{2\pi}} \left\| (u^2+v^2)^{1/2} \right\|_{L^2(\mathbb{C})} \cdot U.$$

In turn, theorem 2' implies theorem 2. Indeed, recall that if $\ell(x) \equiv \frac{1}{\pi} \log|x|$, then: $\frac{1}{2} \Delta \ell(x) = \delta_0(x)$, in the sense of Schwartz's distributions.
Therefore, if $F = f * \ell$, we obtain: $\frac{1}{2} \Delta F = f$,

and Ito's formula implies:

$$(1\text{-}a) \qquad F(Z_t) = F(Z_0) + \int_0^t (\nabla F(Z_s), dZ_s) + \int_0^t f(Z_s)\,ds.$$

Now, remark that we may as well assume $Z_0 = 0$, so that, using the hypothesis $\int dx\, f(x) = 0$, we obtain:

$$F(Z_t) \overset{(d)}{=} \frac{1}{\pi}\int dx\; f(x)\, \log\left|Z_1 - \frac{x}{\sqrt{t}}\right| \xrightarrow[(t\to\infty)]{(P)} 0$$

Hence, we have, from (1-a):

$$\frac{1}{h(t)^{1/2}}\int_0^t f(Z_s)\,ds + \frac{1}{h(t)^{1/2}}\int_0^t (\nabla F(Z_s), dZ_s) \xrightarrow[t\to\infty]{(P)} 0,$$

and therefore, it follows from theorem 2' that:

$$\frac{1}{h(t)^{1/2}}\int_0^t f(Z_s)\,ds \xrightarrow{(d)} \frac{1}{\sqrt{2\pi}} \|\nabla F\|_{L^2(\mathbb{C})} \cdot U$$

Integration by parts shows that: $\|\nabla F\|_{L^2(\mathbb{C})} = \sqrt{\langle f \rangle}$.

## 2. A general asymptotic theorem.

(2.1) For each $j$, we denote by $\zeta^j = \beta^j + i\theta^j$ the $BM_0(\mathbb{C})$ obtained by time-changing the conformal martingale:

$$\left( \log \frac{|Z_t - z_j|}{|Z_0 - z_j|} + i\,\Phi^j(t) \quad , \quad t \geq 0 \right)$$

via the common increasing process $U^j(t) = \int_0^t \frac{ds}{|Z_s - z_j|^2}$ of its real and imaginary parts.

Next, for $h > 0$, let $\zeta^{j,h}$ be the $BM_0(\mathbb{C})$ derived from $\zeta^j$ by the Brownian scaling operation:

$$\zeta^{j,h}(u) = \frac{1}{h}\,\zeta^j(h^2 u) \quad , \quad u \geq 0.$$

Theorem 1 is a consequence of the following

**Theorem 3:** As $h \to \infty$,

$$(\zeta^{j,h} \; ; \; j = 1, 2, \cdots, n) \xrightarrow{(d)} (\zeta^{j,\infty} \; ; \; j = 1, 2, \cdots, n)$$

where the limit is a family of $n$ complex BM's whose excursion processes ---

The definition of the excursion processes has been dealt with at length in our paper [ ] and in order to keep this ~~wat~~ paper relatively short, we refer to [ ] for more details.

(2.2) For $u, v : \mathbb{C} \longrightarrow \mathbb{R}$, belonging to $L^2_b(\mathbb{C})$, we call $\mu^{u,v}$ the $BM_0(\mathbb{R})$ which is associated with the martingale

$$M_t^{u,v} \equiv \int_0^t \{ u(Z_s)dX_s + v(Z_s)dY_s \}$$

Theorem 3 can be reinforced into

Theorem 3': As $h \to \infty$,

$$\left(\zeta^{j,h}, 1 \leq j \leq n ; \mu^{u,v;h^{1/2}}\right) \xrightarrow{(d)} \left(\zeta^{j,\infty}, 1 \leq j \leq n ; \nu\right)$$

where $(\zeta^{j,\infty} ; 1 \leq j \leq n)$ is distributed as indicated in theorem 3, $\nu$ is a $BM_0(\mathbb{R})$ and $(\zeta^{j,\infty}, 1 \leq j \leq n)$ and $\nu$ are independent.

Our main tool to prove theorem 3' will be, as in [ ], the following asymptotic version of Knight's theorem about continuous orthogonal martingales.

Theorem 4: Let $(M_j^{(h)}, 1 \leq j \leq n)$ be a family of $n$-tuples of continuous local martingales such that $\langle M_j^{(h)} \rangle_\infty = \infty$ for every $j$ and $h$, and such that:

for every pair of distinct indices $i$ and $j$, for every $u$,

$$(2.a) \qquad \int_0^{\langle M_*^{(h)} \rangle_u^{-1}} |d\langle M_i^{(h)}, M_j^{(h)} \rangle_s| \xrightarrow[h \to \infty]{(P)} 0$$

and $*$ may be taken, at convenience, to be either $i$ or $j$.

Then, as $h \to \infty$, the Brownian motions $\beta_j^{(h)}$ associated with $M_j^{(h)}$ are asymptotically independent.

We are now able to give a:

Proof of theorem 3':

a) We shall first show that condition (2.a) is satisfied for the families of martingales: $\frac{1}{h}\Phi^j$ and $\frac{1}{h^{1/2}}M^{u,v}$,

when $u$ and $v$ are assumed to be bounded, with compact support. Since, during the following argument, we shall fix $j$, and $(u,v)$, we may drop the superscripts $j, u, v$ for a moment.

Thus, condition (2.a) may be written, in our present case, as:

for all $u>0$, $\frac{1}{h^{3/2}}\int_0^{\langle\Phi\rangle^{-1}_{h^2 u}} |d\langle M,\Phi\rangle_s| \xrightarrow[h\to\infty]{(P)} 0$

which is easily seen to be equivalent to:

(2.b) $$\frac{1}{\langle\Phi\rangle_t^{3/4}}\int_0^t |d\langle M,\Phi\rangle_s| \xrightarrow[t\to\infty]{(P)} 0$$

For simplicity, we may assume that $z_j = 0$, so that we obtain:

$$d\langle M,\Phi\rangle_s = \frac{u(Z_s)Y_s - X_s v(Z_s)}{X_s^2+Y_s^2}ds \equiv [u,v](Z_s)ds,$$

where we denote: $[u,v](z) \equiv \frac{u(z)y - v(z)x}{x^2+y^2}$.

Since $u, v$ are bounded with compact support, the function $[u,v]$ is obviously in $L^1(\mathbb{C})$, and from the ergodic theorem for $BM(\mathbb{C})$, $\frac{1}{\log t}\int_0^t ds\,[u,v](Z_s)$ converges in distribution, while, on the other hand, $\frac{1}{(\log t)^2}\langle\Phi\rangle_t$ converges in distribution. These two results are easily seen to imply (2.b).

b) In the general case where $u$ and $v$ are simply assumed to be in $L_b^2(\mathbb{P})$, we need, in order to show that (2-b) is still true to approximate $u$ and $v$ by $u^{(n)}$ and $v^{(n)}$ in a suitable manner. From now on, we shall simply write $M^{(n)}$ for $M^{u_n, v_n}$.

We first remark that:

$$\int_0^t |d\langle M, \Phi\rangle_s| \leq \int_0^t |d\langle M - M^{(n)}, \Phi\rangle_s| + \int_0^t |d\langle M^{(n)}, \Phi\rangle_s|$$

$$\leq \left(\langle M - M^{(n)}\rangle_t \langle \Phi\rangle_t\right)^{1/2} + \int_0^t |d\langle M^{(n)}, \Phi\rangle_s|$$

so that:

(2-c) $$\frac{1}{\langle \Phi\rangle_t^{3/4}} \int_0^t |d\langle M, \Phi\rangle_s| \leq \left(\frac{\langle M - M^{(n)}\rangle_t}{\langle \Phi\rangle_t^{1/2}}\right)^{1/2} + \frac{1}{\langle \Phi\rangle_t^{3/4}} \int_0^t |d\langle M^{(n)}, \Phi\rangle_s|$$

For clarity, we may assume again that $z_j = 0$, and we introduce the following notation:

$$A_t = \frac{1}{\langle \Phi\rangle_t^{3/4}} \int_0^t |d\langle M, \Phi\rangle_s| \; ; \quad B_t = \left(\frac{\langle \Phi\rangle_{T_{\sqrt{t}}}}{\langle \Phi\rangle_t}\right)^{1/2}, \text{ where } T_r = \inf\{t : |Z_t| = r\}$$

$$C_t^{(n)} = \left(\frac{1}{\langle \Phi\rangle_{T_{\sqrt{t}}}} \langle M - M^{(n)}\rangle_t\right)^{1/2} \; ; \quad D_t^{(n)} = \frac{1}{\langle \Phi\rangle_t^{3/4}} \int_0^t |d\langle M^{(n)}, \Phi\rangle_s|$$

For any $\varepsilon > 0$, we have, from (2-c):

$$P(A_t > \varepsilon) \leq P\left(B_t C_t^{(n)} > \frac{\varepsilon}{2}\right) + P\left(D_t^{(n)} > \frac{\varepsilon}{2}\right)$$

$$\leq P(B_t > 2) + P\left(C_t^{(n)} > \frac{\varepsilon}{4}\right) + P\left(D_t^{(n)} > \frac{\varepsilon}{2}\right)$$

(2-d) $$\leq P(B_t > 2) + \frac{4}{\varepsilon} E\left[C_t^{(n)}\right] + P\left(D_t^{(n)} > \frac{\varepsilon}{2}\right).$$

Now, we have:

$$m_n(t) = \int dx\,dy\, f_n(z)\,\frac{1}{\log t}\, g_t(z),$$

with $g_t(z) = \int_0^t ds\, \exp\left(-\frac{|z-z_0|^2}{2s}\right)\frac{1}{2\pi s} \equiv \tilde{g}\left(\frac{2t}{|z-z_0|^2}\right)$,

where $$\tilde{g}(s) = \frac{1}{2\pi}\int_0^s \frac{du}{u}\exp\left(-\frac{1}{u}\right)$$

and $f_n(z) = \left(u(z)-u^n(z)\right)^2 + \left(v(z)-v^n(z)\right)^2.$

Consequently, we have:

$$m_n(t) \leq \int_{|z-z_0|\geq 1} dx\,dy\, f_n(z)\left(\frac{\tilde{g}(2t)}{\log t}\right) + \int_{|z-z_0|\leq 1} dx\,dy\, f_n(z)\,\frac{1}{\log t}\,\tilde{g}\left(\frac{2t}{|z-z_0|^2}\right)$$

Let $\|\varphi\|_p^p = \int_{|z-z_0|\leq 1} dx\,dy\, |\varphi(z)|^p$. We now remark:

(i) $\tilde{g}(t) = O(\log t) \qquad (t \to \infty)$

(ii) for any $q > 1$, $\left\|\tilde{g}\left(\frac{2t}{|\cdot - z_0|^2}\right)\right\|_q = O(\log t)$,

so that, for $p>1$, and $t$ sufficiently large, there exists a constant $C_p$ such that:

$$(2\text{-}f) \qquad m_n(t) \leq C_p \left\{ \int_{|z-z_0|\geq 1} dx\,dy\, f_n(z) + \|f_n\|_p \right\}$$

But, since $u$ and $v$ belong to $L^2_b(\mathbb{C})$, it is possible to find a sequence $(u^n, v^n)$ such that the right hand side of $(2\text{-}f)$ converges to $0$ as $n \to \infty$.

Now, we recall that, from ([ ], — ), we have:

$$(2\text{-}e) \qquad B_t \xrightarrow[t\to\infty]{(P)} 1 ,$$

so that $P(B_t > 2)$ may be chosen smaller than $\varepsilon$ for $t$ sufficiently large and, from the inequality (2-d) and part a) of this proof, it is now sufficient, in order to ~~prove~~ finish our proof of (2-b) in this general case to show:

$$E[C_t^{(n)}] \xrightarrow[n\to\infty]{} 0 ,$$

uniformly in $t$, for $t \geq T > 0$.

However, one has:

$$E[C_t^{(n)}] = E\left[\left(\frac{\log t}{\langle \Phi \rangle^{1/2}_{T_{\sqrt{t}}}}\right)^{1/2}\left(\frac{\langle M - M^{(n)}\rangle_t}{\log t}\right)^{1/2}\right]$$

$$\leq E\left[\frac{\log t}{\langle \Phi \rangle^{1/2}_{T_{\sqrt{t}}}}\right]^{1/2} E\left[\frac{\langle M-M^{(n)}\rangle_t}{\log t}\right]^{1/2}$$

But, one has the following distributional identities:

$$\frac{4}{(\log t)^2}\langle \Phi \rangle_{T_{\sqrt{t}}} \overset{(d)}{=} \sigma_1 \overset{(d)}{=} \frac{1}{\beta_1^2} ,$$

where $\sigma_1 = \inf\{t : \beta_t = 1\}$, and $\beta$ is a one dim. $BM_0$.

Therefore:

$$E[C_t^{(n)}] \leq E(|\beta_1|)^{1/2} E\left(\frac{\langle M-M^{(n)}\rangle_t}{\log t}\right)^{1/2} ,$$

and finally it remains to show that:

$$m_n(t) \equiv \frac{1}{\log t} E[\langle M-M^{(n)}\rangle_t] \xrightarrow[(n\to\infty)]{} 0 ,$$

uniformly in $t \geq 2$.

c) Strictly speaking, our proof so far only shows that, ~~for a fixed~~ $j$, we have:

$$(2\cdot 9)\qquad \left(\zeta^{j,h};\ \mu^{u,v;\,h^{1/2}}\right) \xrightarrow[h\to\infty]{(d)} \left(\zeta^{j,\infty};\ \nu\right)$$

However, we have in fact done almost all the work:
indeed, as is shown in detail in [ ], the $\sigma$-field generated by $(\zeta^j,\ j\le n)$ is that of the Brownian motion associated with

$$\Theta^j_\pm(t) + i\,\Phi^j_\pm(t) \equiv \int_0^t 1(|Z_s - z_j| \in I^j_\pm)\,\frac{dZ_s}{(Z_s - z_j)}$$

with $I^j_- = (0, |z_0 - z_j|)$, and $I^j_+ = (|z_0 - z_j|, \infty)$

these different Brownian motions $(\beta^j_\pm, \theta^j_\pm)$ are either asymptotically independent, or asymptotically identical, and our previous computations in a) and b) suffice to show that globally $(\beta^{j,h}_\pm, \theta^{j,h}_\pm;\ j\le n)$ is asymptotically independent of $\mu^{u,v;\,h^{1/2}}$, as $h\to\infty$

## 3. Applications.

(3.1) The following theorem states how, with the help of theorem 3', the convergences in distribution presented in theorems 1, 2, and 2' hold jointly.
We use the same notation as in theorems 1 and 2'.

Theorem 5: As $t\to\infty$,

$$\left[\frac{1}{h(t)}\left(\Phi^j_+(t), \Phi^j_-(t)\ (1\le j\le n), L(t)\right); \frac{1}{h(t)^{1/2}} M^{u_k,v_k}_{(t)}\ (1\le k\le m)\right]$$

$$\Big\downarrow (d)$$

$$\left[W_+, W^j_-\ (1\le j\le n);\ \Lambda;\ \sqrt{\frac{\Lambda}{2\pi}}\,\{\eta(u_k) + \chi(v_k)\}(k\le m)\right]$$

where $[W_+, W_-^j\ (1\le j\le n), \Lambda]$ has the distribution described in theorem 1, and $\eta$ and $\chi$ are two gaussian measures on $\mathbb{R}^2$, with intensity $dx\,dy$, independent of each other and of $\Lambda$.

Proof: a) We start in the case $m=1$, and we simply write $M$ for $M^{u_1,v_1}$. We have:

$$(3\cdot a)\qquad \frac{1}{h(t)^{1/2}} M_t = \mu^{h(t)^{1/2}}\left(\frac{1}{h(t)}\langle M\rangle_t\right)$$

As is explained in detail in [ ], theorem 3' implies theorem 1, and, in the same manner, we can deduce from theorem 3' that:

$$(3\cdot b)\qquad \left(\frac{1}{h(t)}\mathbb{F}(t);\ \frac{1}{h(t)}\langle M\rangle_t;\ \mu^{h(t)^{1/2}}\right) \Big\downarrow{\scriptstyle (d)} \left(\mathbb{F};\ \frac{1}{2\pi}\left\|(u^2+v^2)^{1/2}\right\|_{L^2(\mathbb{R}^2)}\cdot\Lambda;\ \gamma\right)$$

where, for simplicity, we have written $\mathbb{F}(t)$, resp: $\mathbb{F}$, for:

$$\frac{1}{h(t)}\left(\Phi_+^j(t), \Phi_-^j(t)\ (1\le j\le n), L(t)\right),$$

resp:

$$\left(W_+, W_-^j\ (1\le j\le n), \Lambda\right).$$

$\gamma$ denotes a $BM_0(\mathbb{R})$ which is independent of $\mathbb{F}$.

Next, we deduce from (3-a) and (3-b) that:

$$(3\cdot c)\qquad \left(\frac{1}{h(t)}\mathbb{F}(t),\ \frac{1}{h(t)^{1/2}}M_t\right)\xrightarrow[t\to\infty]{(d)}\left(\mathbb{F},\ \left\|(u^2+v^2)^{1/2}\right\|_{L^2(\mathbb{R}^2)}\sqrt{\frac{\Lambda}{2\pi}}\,N\right)$$

where $N$ is a gaussian random variable, with mean 0 and variance 1.

b) It is then immediate by looking at the Fourier transform of the left hand side of (3-c) that (3-c) implies in fact the full statement of theorem 5.

(3.2) From theorem 5, we immediately deduce a unified theorem about the joint convergence in distribution for the small and large windings $\Phi^j_\pm(t)$ about the $n$ points $z_j$ $(1 \le j \le n)$, and also the windings in annuli $\{z : r < |z - z_j| < 1\}$, as $r$ varies in $(0,1)$, the latter windings being denoted as:

$$\Phi^j_{(r,1)}(t) = \int_0^t 1_{(r < |Z_s - z_j| < 1)} \, d\Phi^j_s .$$

**Theorem 6:** As $t \to \infty$,

$$\left[ \frac{1}{h(t)} \left( \Phi^j_+(t), \Phi^j_-(t) \right) ; \frac{1}{h(t)^{1/2}} \Phi^j_{(r,1)}(t) \quad (1 \le j \le n) ; L(t) \right]$$

$$\Big\downarrow (f.d)$$

$$\left[ W_+, W^j_- , \sqrt{\Lambda}\, B^j_{-(\log r)} \quad (1 \le j \le n), \Lambda \right]$$

where $[W_+, W^j_-, \Lambda]$ are distributed as indicated in theorem 1, $[B^j_t, 1 \le j \le n, t \ge 0]$ is a $n$-dimensional gaussian process, independent of $F \equiv [W_+, W^j_- (j \le n), \Lambda]$, and whose covariance is given by:

(3d) $$E\left[ B^i_{-(\log \varepsilon)} B^j_{-(\log \varepsilon')} \right] = \frac{1}{2\pi} \int_{\{z : \varepsilon \le |z - z_i| \le 1\} \cap \{z : \varepsilon' \le |z - z_j| \le 1\}} dx\,dy \, \frac{(z - z_i)\cdot(z - z_j)}{|z - z_i|^2 |z - z_j|^2}$$

In particular, for any $i$, $(B^i_t, t \ge 0)$ is a $BM_0(\mathbb{R})$, and in addition, if $|z_j - z_i| \ge 2$, $B^i$ and $B^j$ are independent.

Finally, $(f.d)$ refers to the convergence of finite dimensional marginals, as $r$ varies in finite sets.

The part of theorem 6 concerning the windings in annuli is already found in theorem 11 of Messulam-Yor [ ]. However, in theorem 11 of [ ], ---