Enlarging the filtration of the BES(3) process, with its total local times.

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1. Let $(R_t, t\geq 0)$ be a 3-dimensional Bessel process, starting from 0.

I would like to know the distribution of $(R_t, t\geq 0)$, given $\sigma\{\ell^r_\infty; r\geq 0\}$, where $(\ell^r_t; t\geq 0)$ denotes the local time of R at level r.

In order to do this, we may consider the filtration $(\mathcal{R}_t \equiv \sigma\{R_s; s\leq t\})$ enlarged with $\sigma\{\ell^r_\infty; r\geq 0\}$; let $(\mathcal{R}'_t, t\geq 0)$ be the new filtration.

We would like to know the canonical decomposition of $(R_t, t\geq 0)$ as a semimartingale (if it is still one!) in $(\mathcal{R}'_t, t\geq 0)$.

2. <u>Some general formulae</u>.

To realize the above program, I hope to be able to use the formulae in Théorème 1, p. 7, of my paper [ ] in Lecture Notes 1118. Here are some details:

let $(\beta_t; t\geq 0)$ be the (Brownian) martingale part of $(R_t, t\geq 0)$ in $(\mathcal{R}_t, t\geq 0)$, i.e.:

$$R_t = \beta_t + \int_0^t \frac{ds}{R_s} \qquad (t\geq 0);$$

let $L: \Omega \longrightarrow \mathbb{R}^m$ be a random variable which is $\mathcal{R}_\infty$-measurable and such that we have (with my notation in [ ]):

$$(1)\quad \boxed{\lambda_t(g) = \lambda_0(g) + \int_0^t \dot{\lambda}_s(g)\, d\beta_s, \text{ with: } \dot{\lambda}_s(dx) = \lambda_s(dx)\, \ell(x,s)}.$$

<u>Nota bene</u>:

$(\lambda_t(g), t\geq 0)$ is simply a "good" version of $(E[g(L)\,|\,\mathcal{R}_t], t\geq 0)$.

Then, under some integrability condition on $\lambda, \dot{\lambda}, \ldots,$ we obtain:

$$(2)\quad \boxed{\beta_t = \tilde{\beta}_t + \int_0^t \ell(L;s)\, ds.}$$

which is the canonical decomposition of $(\beta_t, t \geq 0)$ in the enlarged filtration $(\mathcal{R}_t^{\sigma(L)}; t \geq 0)$.

In particular, $(\tilde{\beta}_t, t \geq 0)$ is a BM with respect to this enlarged filtration.

3. <u>Application to some additive functionals.</u>

Let $A_t \equiv (A_t^1, A_t^2, \dots, A_t^n)$ be an $\mathbb{R}^n$-valued additive functional, i.e.: for $s \leq t$, $A_t = A_s + A_{t-s} \circ \theta_s$ $(s < t)$.

We are interested in fact in additive functionals $(A_t, t \geq 0)$ which converge as $t \to \infty$. Hence, we have:

$$A_\infty = A_t + A_\infty \circ \theta_t \quad , \text{ for every } t \geq 0.$$

We may now compute $(\lambda_t(g), t \geq 0)$ and $(\dot{\lambda}_t(g), t \geq 0)$, for $L = A_\infty$.

We have:

$$\begin{aligned} \lambda_t(g) &= E[g(A_\infty) | \mathcal{R}_t] \\ &= E_{R_t}[g(A_t(\omega) + A_\infty)] \\ &= E[g(A_\infty)] + \int_0^t d\beta_s \left. \frac{\partial}{\partial r} \right|_{r=R_s} E_r[g(A_s(\omega) + A_\infty)]. \end{aligned}$$

Hence, we have:

$$\boxed{\lambda_t(g) = G(R_t, A_t), \text{ and } \dot{\lambda}_t(g) = \frac{\partial G}{\partial r}(R_t, A_t), \quad \text{where: } G(r,a) = E_r[g(a + A_\infty)].}$$