B- Further remarks on Wiener integrals with respect to a BES(3) process 1).

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$(R_t, t \geq 0)$ denotes a BES(3) process starting from 0, and

$$\hat{R}_t = R_t - E(R_t) \equiv R_t - \sqrt{t}\, E(R_1).$$

We have established that: there exists a universal constant C such that:

$$(1) \qquad E\left[\left(\int_0^\infty f(s)\, d\hat{R}_s\right)^2\right] \leq C \int_0^\infty f^2(s)\, ds,$$

for f with compact support on $(0,\infty)$, f bounded, which allows to extend the definition of $\int_0^\infty f(s)\, d\hat{R}_s$ to any $f \in L^2(\mathbb{R}_+, ds)$.

A main ingredient used in our previous proof of (1) is Hardy's L^2 inequality; we shall now see how Hardy's L^2 inequality allows to derive ~~three~~ several other analogues of (1); precisely:

Proposition: The inequality (1) also holds when R is replaced by:

a) $S_t = \sup_{s \leq t} B_s$, for B a 1-dim BM;

b) L_t : the local time of BM.

c) $(\rho_t, t \geq 0)$: Reflecting BM.

d) $A_t = \int_0^t ds \left(\frac{R_s}{s}\right)$.

e) $(r(t), t \leq 1)$: a standard BES(3) bridge, or standard reflecting BM bridge.

f) $(\ell_t, t \leq 1)$ the local time of reflecting Brownian bridge; g) the Brownian meander.

Proof of the Proposition:

a) We use Pitman's representation of BES(3) as:

$$R_t = 2S_t - B_t ;$$

hence, we have: $2\hat{S}_t = \hat{R}_t + \hat{B}_t$,

so that: $\int_0^\infty f(s)\, 2(d\hat{S}_s) = \int_0^\infty f(s)\, d\hat{R}_s + \int_0^\infty f(s)\, d\hat{B}_s$,

and the result follows immediately.

b) We can use: $(S_t, t \geq 0) \overset{(law)}{=} (L_t, t \geq 0)$.

c) We use another representation of BES(3), equivalent to the one recalled in a), thanks to Lévy's equivalence theorem: $R_t = \cancel{2}L_t + |B_t|$.

Hence, we have: $\widehat{|B_t|} = \hat{R}_t - \hat{L}_t$,

and the result follows.

d) We need to consider: $\int_0^t f(s) \dfrac{\hat{R}_s}{s}\, ds$.

We write simply: $\hat{R}_s = \int_0^s d\hat{R}_u$, and the previous integral becomes:

$$\int_0^t d\hat{R}_u \left(\int_u^t ds \frac{f(s)}{s} \right).$$

For simplicity, we may fix $t = 1$; then the transformation:

$$f \ (\in L^2[0,1]) \longrightarrow \int_\cdot^1 ds \frac{f(s)}{s} \equiv \tilde{H}f(\cdot)$$

is the dual of Hardy's transform: $f \longrightarrow Hf(\cdot) : x \longrightarrow \frac{1}{x}\int_0^x dy\, f(y)$.

Now, the result follows again from $\|\tilde{H}f\|_2 \leq 2\|f\|_2$.

e) to obtain the result for $(r(t), t \leq 1)$ a standard BES(3) bridge, I use the representation:

$$r(t) = (1-t)\, R_{t/(1-t)},$$

where R denotes a standard BES(3).

Then, we have:

$$\int_0^1 f(t)\, d\left((1-t)\hat{R}_{(t/1-t)}\right) = -\int_0^1 f(t)\, \hat{R}_{(t/1-t)}\, dt + \int_0^1 f(t)(1-t)\, d\left(\hat{R}_{(t/1-t)}\right).$$

$$= \int_0^1 F(t)\, d\hat{R}_{(t/1-t)}, \quad \text{where: } F(t) = f(t)(1-t) - \int_t^1 du\, f(u).$$

$$= \int_0^\infty F\left(\frac{h}{h+1}\right) d\hat{R}_h,$$

whose L^2 norm (squared) is smaller than a multiple of:

$$\int_0^\infty F^2\left(\frac{h}{h-1}\right) dh = \int_0^1 F^2(t)\, \frac{dt}{(1-t)^2}$$

$$\equiv \int_0^1 dt \left\{ f(t) - \frac{1}{(1-t)} \int_t^1 du\, f(u) \right\}^2$$

$$\leq 2 \int_0^1 dt \left(f^2(t) + (Hf)^2(t) \right)$$

$$\leq 2 \int_0^1 dt \left(f^2(t) \right) + 8 \int_0^1 dt\, f^2(t)$$

$$\equiv 10 \int_0^1 dt \left(f^2(t) \right).$$

The result for $(r(t), t \leq 1)$ now follows; the result for the standard reflecting bridge now follows from that for the reflecting Brownian motion, exactly in the same way.

4).

f) We use the following representation of $(p(t), t\leq 1)$ a standard Brownian bridge:

$$p(t) = \beta_t - \int_0^t ds \frac{p_s}{(1-s)} \quad , \quad t \leq 1,$$

where $(\beta_t, t\leq 1)$ denotes a standard Brownian motion.

Then, we have, from Tanaka's formula: $|p(t)| = \tilde{\beta}_t - \int_0^t ds \frac{|p_s|}{(1-s)} + \ell_t$.

Hence:
$$\int_0^1 f(t)\, d|p(t)|$$
$$= \int_0^1 f(t)\, d\tilde{\beta}_t - \int_0^1 ds\, f(s) \frac{|p(s)|}{(1-s)} + \int_0^1 f(s)\, d\ell_s \, ,$$

and the result we are seeking follows, because we know it holds for $|p|$, $\tilde{\beta}$, $\int_0^{\cdot} ds \frac{|p(s)|}{(1-s)}$ (once centered).

g). Finally, the result for the Brownian meander may be obtained from the known representation: (Pitman type)

$$m(t) = |p(t)| + \ell_t, \quad t \leq 1.$$

I shall continue thinking of other aspects -

- FIN DU MESSAGE -

1)

C - Looking for some adequate terminology. January 17th / (5 pages).

Consider a process $(R_t, t \geq 0)$, with second moments;
we say that it is an <u>o.c L^2-integrator</u>, which is an abbreviation for: <u>once centered L^2 integrator</u> if there exists a universal constant C such that, denoting: $\hat{R}_t = R_t - E(R_t)$,
there is the a priori inequality:

$$(1)_a \qquad E\left[\left(\int_0^\infty f(s)\, d\hat{R}_s\right)^2\right] \leq C \int_0^\infty f^2(s)\, ds,$$

Where f is a deterministic, <u>simple function</u>, say, so that the integral $\int_0^\infty f(s)\, d\hat{R}_s$ is well defined.

We have established recently that a number of "fundamental" stochastic processes are <u>o.c. L^2-integrators</u>; these processes are also semimartingales so that a priori one might consider more general integrators,
i.e: b) $f(s, R_s)$, where f is defined on $\mathbb{R}_+ \times \mathbb{R}$, and takes values in $\mathbb{R}$
c) more generally, $\phi(s, R)$, where ϕ is a predictable process (with respect to the natural filtration of R).
Thus, we look for semimartingales such that either:

$$(1)_b \qquad \mathrm{Var}\left(\int_0^\infty f(s, R_s)\, dR_s\right) \leq C\, E\left[\int_0^\infty ds\, f^2(s, R_s)\right]$$

$$(1)_c \qquad \mathrm{Var}\left(\int_0^\infty \phi(s, R)\, dR_s\right) \leq C\, E\left[\int_0^\infty ds\, \phi^2(s, R)\right].$$

again starting from bounded processes $\{f(s, R_s)\}$, $\sim \{\phi(s, R)\}$

To be more precise, we now use the terminology:

$$(2)\quad \begin{cases} \underline{\text{o.c } L^2_d\text{-integrator}} & \text{if } R \text{ satisfies } (1)_a \\ \underline{\text{o.c. } L^2_h\text{-integrator}} & \text{if } R \text{ satisfies } (1)_b \\ \underline{\text{o.c } L^2_p\text{-integrator}} & \text{if } R \text{ satisfies } (1)_c \end{cases}$$

Here, d, resp: h, resp: p, stands for: deterministic, resp: homogeneous, resp: predictable.

As a test case, let us consider: $(b(t), t\le 1) \overset{(law)}{=} (B_t - tB_1, t\le 1)$ the standard Brownian bridge.

We note that, for any $f \in L^2([0,1])$, there is the identity in law:

$$\int_0^1 f(s)\,db(s) \overset{(law)}{=} \int_0^1 f(s)\,dB_s - \left(\int_0^1 f(s)ds\right) B_1$$

$$\overset{(law)}{=} \int_0^1 (f(s) - \bar{f})\,dB_s,$$

where $\bar{f} = \int_0^1 ds\, f(s)$.

Hence:

$$E\left[\left(\int_0^1 f(s)\,db(s)\right)^2\right] = \int_0^1 (f(s) - \bar{f})^2 ds$$

$$= \int_0^1 f^2(s)\,ds - (\bar{f})^2.$$

and, clearly, $(b(t), t\le 1)$ is a (o.c) L^2_d-integrator.

Let us consider whether it is a o.c L^2_p-integrator.

In order to develop such a discussion, we write the canonical decomposition of $(b(t), t\le 1)$ as a semimartingale, i.e: there exists a Brownian motion β such that:

$$(3)\qquad b(t) = \beta(t) - \int_0^t ds\, \frac{b(s)}{(1-s)}$$

and we integrate a bounded predictable process: $\Phi(s,b)$

Thus, we obtain:

$$(4) \qquad \int_0^t \Phi(s,b)\, db(s) = \int_0^t \Phi(s,b)\, d\beta_s - \int_0^t ds\, \Phi(s,b) \frac{b_s}{(1-s)}$$

Since $b(1)=0$, we may write: $-\int_0^t ds\, \Phi(s,b) \frac{b_s}{(1-s)}$

$$= \int_0^t ds\, \Phi(s,b) \frac{\int_s^1 db(u)}{(1-s)}$$

$$= \int_0^1 db(u) \int_0^{u\wedge t} \frac{ds\, \Phi(s,b)}{(1-s)},$$

so that bringing this expression on the LHS of (4), we obtain:

$$(5) \qquad \int_0^1 db(u) \left\{ \Phi(u,b)\, 1_{(u\le t)} - \int_0^{u\wedge t} \frac{ds\, \Phi(s,b)}{(1-s)} \right\} = \int_0^t \Phi(s,b)\, d\beta_s .$$

We would now like to use (5) in order to discuss some L^2_p-integration property of b. Let us take $t=1$ in (5), and consider the integrand

$$(6) \qquad \boxed{\Psi(u,b) \stackrel{\text{def}}{=} \Phi(u,b) - \int_0^u ds \frac{\Phi(s)}{(1-s)}}$$

as given. We then consider the identity (6) as an equation with the given data $\Psi(u)$, and unknown function $\Phi(u)$.

Denote: $f(u) = \Phi(u) - \Psi(u)$; then:

$$(7). \qquad f(u) = \int_0^u ds \frac{\Psi(s)}{(1-s)} + \int_0^u ds \frac{f(s)}{(1-s)}.$$

This 1st order linear equation is solved as: (8): $f(u) = \frac{1}{(1-u)} \int_0^u ds\, \Psi(s).$

so that: (9) $\quad \Phi(u) = \psi(u) + \frac{1}{(1-u)} \int_0^u ds\, \psi(s).$ $\quad$ 4)

We now plug this result into (5) to obtain:

(10) $$\int_0^1 d\beta(u)\, \psi(u) = \int_0^1 d\beta_u \left(\psi(u) + \frac{1}{(1-u)} \int_0^u ds\, \psi(s) \right)$$

- Before going any further, we note from (6) that ψ is not "any" predictable process, as it satisfies:

(11) $$\int_0^1 ds\, \psi(s) = 0$$

Thus, $$\psi(u) + \frac{1}{(1-u)} \int_0^u ds\, \psi(s) = \psi(u) - \frac{1}{(1-u)} \int_u^1 ds\, \psi(s).$$

Denote: $$K_0\psi(u) = \frac{1}{(1-u)} \int_u^1 ds\, \psi(s).$$

Then, we note: $K_0\psi(1-u) = \frac{1}{u} \int_0^u dv\, \psi(1-v)$: this is the Hardy transform of $\psi(1-\cdot)$

and $K_1\psi(u) = \psi(u) + K_0\psi(u).$

Thus, we have: $$\int_0^1 du\, (K_1\psi(u))^2 \leq 2 \int_0^1 du\, \left(\psi^2(u) + (K_0\psi)^2(u)\right)$$

(12). $$\leq \overset{10}{10} \int_0^1 du\, \psi^2(u).$$

Thus, we may go back to (#) and write: (13) $$\int_0^1 db(u)\, \psi(u) = \int_0^1 d\beta_u\, (K_1\psi)(u)$$

for every predictable process ψ such that:

$$E\left[\int_0^1 \psi^2(u)\, du\right] < \infty, \text{ and } \int_0^1 du\, \psi(u) = 0.$$

Moreover, from (12), we deduce:

$$E\left[\left(\int_0^1 db(u)\, \psi(u)\right)^2\right] \leq 10\, E\left[\int_0^1 du\, \psi^2(u)\right].$$

Thus, we have obtained the following

Theorem: If ψ denotes any predictable process for the natural filtration of b such that: $E\left[\int_0^1 du\, \psi^2(u)\right] < \infty$ and $\int_0^1 du\, \psi(u) = 0$,

~~and~~ then $\int_0^1 db(u)\,\psi(u)$ is well defined, and centered, and it satisfies:

$$E\left[\left(\int_0^1 db(u)\,\psi(u)\right)^2\right] \leq 10\, E\left[\int_0^1 du\, \psi^2(u)\right]$$

Remark: In the case when: $\int_0^1 \psi(s)\,ds \neq 0$, it may be of some interest to find /present/ a formula which "replaces" the above formula:

(10) $$\int_0^1 \psi(s)\,db(s) = \int_0^1 d\beta_s \left(\psi(s) + \int_0^s \frac{dh\, \psi(h)}{\cancel{(1-h)}\ 1-s}\right)$$, valid for ψ such that: $\int_0^1 du\, \psi(u) = 0$.

Indeed, starting from: $$b(t) = \beta(t) - \int_0^t ds\, \frac{b(s)}{(1-s)},$$

there is the well-known representation of b in terms of β:

(3) $$b(t) = (1-t)\int_0^t \frac{d\beta_s}{(1-s)},$$

from which we may now deduce in full generality:

(14) $$\int_0^1 \psi(s)\,db(s) = -\int_0^1 \psi(s)\,ds \int_0^s \frac{d\beta_u}{(1-u)} + \int_0^1 d\beta_s\, \psi(s).$$

In the case when: $\int_0^1 ds\, \psi(s) = 0$, we may write, at least formally, the double integral on the RHS of (14) as:

$$\int_0^1 \frac{d\beta_u}{(1-u)}\left(-\int_u^1 ds\, \psi(s)\right) = \int_0^1 \frac{d\beta_u}{(1-u)}\left(\int_0^u ds\, \psi(s)\right),$$

thus formally recovering (10) from (14) $\square$