On a local time formula of D. Williams, via filtering theory.

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1 Introduction.

$(\Omega, \mathcal{F}, P)$ is the reference probability space.

The following result is well known to Brownian motion 'students'.

Theorem 1 *Let $B = (B_t)_{t \geq 0}$ be a real-valued Brownian motion, starting at $x = 0$ (for simplicity).*

Define $A_t = \int_0^t 1_{(B_s > 0)}\, ds$, *and* $\tau_t = \inf\{ s \,/\, A_s > t\}$ $(t \geq 0)$.

Then, $\tilde{B} \underset{\text{déf}}{=} (B_{\tau_t})_{t \geq 0}$ *is a reflecting Brownian motion.*

Using the previous notation, define $(\mathcal{B}_t)$ (resp: $(\tilde{\mathcal{B}}_t)$) to be the natural filtration of B (resp: $\tilde{B}$), and if $(L_t)_{t \geq 0}$ is the local time at 0 of B, note $L'_t = \frac{1}{2} L_{\tau_t}$. A complement to Theorem 1 is:

(1) (L'_t) *is the local time at* 0 of $(\tilde{B}_t)$,

which is probably better expressed, to avoid any confusion concerning multiplicative constants, as:

(1') $(\tilde{B}_t - L'_t)$ *is a* $(\tilde{\mathcal{B}}_t)$ *martingale*

More information about the couple $(\tilde{B}_t, \tau_t)_{t \geq 0}$ is contained in the following formula, due to D. Williams [ ]:

for all $t \geq 0$, and $\alpha > 0$, one has

$$(2) \qquad E\left(\exp\left(-\frac{\alpha^2}{2}\tau_t\right) \Big| \tilde{\mathcal{B}}_\infty\right) = \exp\left\{-\frac{\alpha^2}{2}t - \alpha L'_t\right\}$$

In Vol. 1 of his book [ ], D. Williams writes (p. 154) that formula (2) is one way of introducing local time from global considerations. We regard the following theorem as a translation of the previous sentence (we note: $\Delta\tau_t = \tau_t - \tau_{t-}$)

Theorem 2: *For any $\alpha > 0$, the process:* $\sum_{(s \leq t)} \left\{1 - e^{-\frac{\alpha^2}{2}(\Delta\tau_s)}\right\} - \alpha L'_t$ *is a* $(\mathcal{B}_{\tau_t})$ *martingale.*

<u>Remark</u> The statement of theorem 2 is easily extended, using density arguments, as: <u>for any</u> $f \in L^1((0,\infty); n(dx))$, <u>the process</u>: $\sum_{(s\le t)} f(\Delta\tau_s) - L'_t\, n(f)$ <u>is a</u> $(\mathcal{B}_{\tau_t})$ <u>martingale</u>, where $n(dx)$ is the positive measure on $((0,\infty);\mathcal{B}(0,\infty))$ defined by $n(dx) = (2\pi x^3)^{-1/2}\,dx$, once one remarks ~~that~~: $\int n(dx)\left[1-e^{-\frac{\alpha^2 x}{2}}\right] = \alpha$, for ~~all~~ any $\alpha>0$ □

After proving Theorem 2 in §3, we show, in §4, that formula (2), and theorem 2 can be deduced from each other, making only use of simple 'facts'. To prove that Theorem 2 implies formula (2), we shall use a very particular case of a (semi-martingale) projection formula from (non-linear) filtering theory.

2). <u>Some precisions about theorem 1.</u>

N. El Karoui and M. Maurel [ ] have given a very ~~short~~ nice proof of Theorem 1, which is recalled here:

Tanaka's formula writes (3) $B_t^+ = \int_0^t 1_{(B_s>0)}\,dB_s + \frac{1}{2}L_t$

Note $X_t = \int_0^t 1_{(B_s>0)}\,dB_s$. It is, of course, a $(\mathcal{B}_t)$ continuous martingale, with $\langle X,X\rangle_t = \int_0^t 1_{(B_s>0)}\,ds = A_t$.

Dubins and Schwarz [ ] tell us that $X_{\tau_t} \overset{def}{=} \beta_t$ is a $(\mathcal{B}_{\tau_t})$ Brownian motion. Moreover, one gets, from formula (3):

$(\tilde{3})$ $\tilde{B}_t = B^+_{\tau_t} = \beta_t + L'_t$.

Now, $(\tilde{B}_t)$ is a non-negative, continuous process, and (L'_t) is a continuous increasing process such that dL'_s is carried by $\{s / \tilde{B}_s = 0\}$.

N. El Karoui and M. Maurel remark that these properties combined with $(\tilde{3})$ imply: (4) $\tilde{B}_t = \beta_t + \sup_{(s\le t)}(-\beta_s)$.

On the other hand, Tanaka's formula also gives:

(5) $$|B_t| = \gamma_t + L_t,$$
where $\gamma_t = \int_0^t \mathrm{sgn}(B_s)\,dB_s$ is a $(\mathcal{B}_t)$ Brownian motion.
The arguments used previously for $(\tilde{B}_t)$ imply:
$$|B_t| = \gamma_t + \sup_{(s\le t)}(-\gamma_s).$$
Consequently, $(\tilde{B}_t)$ and $(|B_t|)$ have the same distribution □.
We now add more useful complements to Theorem 1; these are well-known properties of reflecting Brownian motion.

__Proposition.__ (i) $(\tilde{\mathcal{B}}_t)$ __is the natural filtration of the__ $(\mathcal{B}_{\tau_t})$ __Brownian motion__ $\beta = (\beta_t)$.

(ii) __The space of square integrable__ $(\tilde{\mathcal{B}}_t)$ __martingales is identical to:__ $c + \int_0^{\cdot} \varphi(s)\,d\beta_s$,
__where__ $c \in \mathbb{R}$, __and__ φ __is a__ $(\tilde{\mathcal{B}}_t)$ __predictable process such that__ $E\left(\int_0^\infty \varphi^2(s)\,ds\right) < \infty$.

__Consequently, it consists of__ $(\mathcal{B}_{\tau_t})$ __martingales.__

(iii) __for every__ t, $\mathcal{B}_{\tau_t}$ __and__ $\tilde{\mathcal{B}}_\infty$ __are conditionally independent, given__ $\tilde{\mathcal{B}}_t$.

3 Proof of Theorem 2.

Let $\alpha > 0$. Using Tanaka's formula (5), and Ito's formula, we deduce that:

(6) $$M_t \overset{def}{=} \exp\left\{-\alpha|B_t| - \frac{\alpha^2 t}{2}\right\} + \alpha\int_0^t \exp\left(-\frac{\alpha^2 s}{2}\right) dL_s$$

__is a__ $(\mathcal{B}_t)$ __martingale__.

Moreover, as (M_t) is the sum of a bounded process, and of an increasing process, it is uniformly integrable.

As a consequence of (6),

(7) $$M_{\tau_t} = \exp\left\{-\alpha\tilde{B}_t - \frac{\alpha^2}{2}\tau_t\right\} + (2\alpha)\int_0^t \exp\left(-\frac{\alpha^2\tau_s}{2}\right) dL'_s$$

is a uniformly integrable $(\mathcal{B}_{\tau_t})$ ~~integrable~~ martingale.
Developing the exponential term with the help of Ito's formula, and $(\tilde{3})$, one gets:

$$(8) \qquad M_{\tau_t} = \{(\mathcal{B}_{\tau_t}) - \text{local martingale}\} + \int_0^t \exp\{-\alpha \tilde{B}_s - \frac{\alpha^2}{2}\tau_{s-}\} dC_s,$$

with:

$$(9) \qquad C_t = \sum_{(s \leq t)} \{\exp(-\frac{\alpha^2}{2}\Delta\tau_s) - 1\} + \alpha L'_t$$

(use the fact that: $\tau_t = t + \sum_{s \leq t} (\Delta\tau_s)$).
As $\exp\{-\alpha \tilde{B}_s - \frac{\alpha^2}{2}\tau_{s-}\}$ is a strictly positive, $(\mathcal{B}_{\tau_t})$ predictable process, one deduces from (8), that (C_t) is a $(\mathcal{B}_{\tau_t})$ local martingale, thus proving Theorem 2.

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4 · Equivalence of formula (2) and theorem 2.

4.1) We first deduce formula (2) from Theorem 2.

Fix $\alpha > 0$. Again using formula ($\tilde{3}$), and Itô's formula, we obtain:

$$N_t \stackrel{\text{def}}{=} \exp\left\{\alpha \tilde{B}_t - \frac{\alpha^2}{2}\tau_t\right\}$$

(10)
$$= 1 + \int_0^t N_s \,\alpha\, d\beta_s + \int_0^t N_{s-}\, dC_s .$$

Consequently, (N_t) is a $(\mathcal{B}_{\tau_t})$ local martingale, since, from Theorem 2, (C_t) is a $(\mathcal{B}_{\tau_t})$ martingale. From Jensen's inequality, and the finiteness of $E(\exp 2\alpha \tilde{B}_t) = E(\exp 2\alpha |B_t|)$, it is easy to see that (N_t) is, indeed, a $(\mathcal{B}_{\tau_t})$ martingale, such that $E(N_t^2) < \infty$, for every t.

The following lemma, which is a simple consequence of the proposition, may be found under a more or less hidden form in any course on filtering theory.

<u>Lemma.</u> <u>Let (V_t) be a $(\mathcal{B}_{\tau_t})$ martingale such that for all t's, $E(V_t^2) < \infty$, and $(\hat{V}_t)$ its $(\tilde{\mathcal{B}}_t)$ optional projection.</u>

<u>Then, a) there exists a $(\mathcal{B}_{\tau_t})$ optional process (v_t) such that $d\langle V;\beta\rangle_t = v_t\, dt$, and for any t,</u> $E\left(\int_0^t v_s^2\, ds\right) < \infty$.

b) <u>If $\hat{v}$ is the conditional expectation of v with respect to the measure $(ds\, dP)$, given the $(\tilde{\mathcal{B}}_t)$ optional σ-field, one has:</u>

(11)
$$\hat{V}_t = \hat{V}_0 + \int_0^t \hat{v}_s\, d\beta_s . \qquad \square$$

We apply the lemma with: $V \equiv N$.

Then, formula (10) yields: $v = \alpha N$, and therefore, formula (11)

reads: $$(12)\qquad \hat{N}_t = 1 + \int_0^t \hat{N}_s\, \alpha\, d\beta_s ,$$
which implies, as is well-known from the theory of <u>linear</u> stochastic differential equations:
$$(13)\qquad \hat{N}_t = \exp\left\{ \alpha\beta_t - \frac{\alpha^2}{2} t \right\}$$
On the other hand, from the definition of (N_t) (formula (10)), and part (iii) of the proposition, one has:
$$(14)\qquad \hat{N}_t = \exp\{\alpha \tilde{B}_t\}\, E\left(\exp\left(-\frac{\alpha^2}{2}\tau_t\right) \Big| \tilde{\mathcal{B}}_\infty \right)$$
Formula (2) now appears as a mere consequence of the comparison of (13) and (14), using again $(\tilde{3})$.

4.2) Conversely, we now <u>admit</u> formula (2).
By (N_t), we still mean the process $\left(\exp\left\{ \alpha \tilde{B}_t - \frac{\alpha^2}{2}\tau_t \right\} \right)_{t \geq 0}$.
Formula (2) tells us that the equality (13) holds. Therefore, $(\hat{N}_t)$ is a $(\tilde{\mathcal{B}}_t)$ martingale; in particular, for all t's, $E[N_t] = E[\hat{N}_t] = 1$.
Moreover, since (N_t) is the (τ_t)-time changed $(\mathcal{B}_t)$ <u>positive</u> martingale $\exp\left\{ \alpha B_t - \frac{\alpha^2}{2} t \right\}$, it is a $(\mathcal{B}_{\tau_t})$ supermartingale.
Its expectation being constantly equal to 1, it is a $(\mathcal{B}_{\tau_t})$ martingale.
Formula (10) now implies that (C_t), defined by (9), is a $(\mathcal{B}_{\tau_t})$ local martingale, implying in turn Theorem 2 $\square$.

References.

[1] <u>N. El Karoui et M. Maurel</u>: Un problème de réflexion et ses applications au temps local et aux équations différentielles stochastiques sur $\mathbb{R}$. Cas continu. Soc. Math. France. Astérisque 52-53 (1978), p. 117-144.

[2] <u>L. Dubins et G. Schwarz</u>:

[3] <u>D. Williams</u>: <u>Diffusions, Markov processes and ~~[illegible]~~ Martingales</u> <u>Vol 1: Foundations</u>. J. Wiley (1979).

[4] <u>D. Williams</u>: Markov properties of Brownian local time. Bull. Amer. Math. Soc. <u>75</u>, 1035-6 (1969).