(4)

On ranked lengths of excursions, III: from Brownian motion to the Brownian bridge.

July, 27^{th}, 1994.

The main relation, which will allow us to pass to the Brownian bridge, is the following: in the sequel, let $c = \sqrt{\frac{\pi}{2}}$, and define: $\beta_u^{(1)} = \frac{1}{\sqrt{g_{H(1)}}} B_{u g_{H(1)}}$

Then, we have:

$$(1)_f \qquad c\, E\left[F\left(\beta_u^{(1)}, u \le 1\right) f\left(g_{H(1)}\right)\right] = E\left[\frac{1}{\sqrt{\tau_1}} F\left(\frac{B_{u\tau_1}}{\sqrt{\tau_1}}, u \le 1\right) \hat{f}\left(\frac{\tau_1}{V_1(\tau_1)}\right)\right]$$

$$\text{where: } \quad \hat{f}(x) = \int_0^x \frac{du}{2\sqrt{u}} f(u)$$

and we can go from the pseudo-bridge $B_u^{\#} = \frac{1}{\sqrt{\tau_1}} B_{u\tau_1}$ to the bridge $(b(u), u \le 1)$ thanks to BLY, which gives:

$$(2)_f \qquad c^2\, E\left[F\left(\beta_u^{(1)}, u \le 1\right) f\left(g_{H(1)}\right)\right] = E\left[F(b(u), u \le 1)\, \hat{f}\left(\frac{1}{V_1(b)}\right)\right]$$

We now take $f \equiv 1$, and we get:

$$(1')_{f=1} \qquad c\, E\left[F\left(\beta_u^{(1)}, u \le 1\right)\right] = E\left[F\left(B_u^{\#}, u \le 1\right) \frac{1}{\sqrt{V_1(\tau_1)}}\right]$$

and

$$(2')_{f=1} \qquad c^2\, E\left[F\left(\beta_u^{(1)}, u \le 1\right)\right] = E\left[F(b(u), u \le 1) \frac{1}{\sqrt{V_1(b)}}\right].$$

2)

Now, we apply $(2')_{f=1}$ to $F(X) = f\left(\frac{V_2(X)}{V_1(X)}, \frac{V_3(X)}{V_1(X)}, \dots\right)$

and we use two facts: a) $V_{n}(g_{H^{(1)}}) = V_{n+1}(H^{(1)})$;

b) $H^{(1)}$ is admissible.

Thus, we deduce from $(2')_{f=1}$ that:

$$(3) \quad \frac{\pi}{2} E\left[f\left(\frac{V_3(1)}{V_2(1)}, \frac{V_4(1)}{V_2(1)}, \text{———}\right)\right] = E\left[\frac{1}{\sqrt{V_1(b)}} f\left(\frac{V_2(b)}{V_1(b)}, \frac{V_3(b)}{V_1(b)}, \dots\right)\right]$$

Now, we remark that, if we denote: $R_1^b = \frac{V_2(b)}{V_1(b)}$, $R_n^b = \frac{V_{n+1}(b)}{V_n(b)}$, then:

$$\frac{1}{V_1(b)} = 1 + R_1^b + R_1^b R_2^b + R_1^b R_2^b R_3^b + \dots,$$

so that we can write (3) in the form:

$$(3') \quad \frac{\pi}{2} E\left[f(R_2, R_3, \dots)\right] = E\left[\left(1 + R_1^b + R_1^b R_2^b + \dots\right)^{1/2} f(R_{\cancel{2}\,1}^b, R_{\cancel{3}\,2}^b, \dots)\right]$$

or equivalently:

$$(3'') \quad \boxed{E\left[f(R_{\cancel{2}\,1}^b, R_{\cancel{3}\,2}^b, \dots)\right] = \frac{\pi}{2} E\left[\left(1 + R_2 + R_2 R_3 + \dots\right)^{-1/2} f(R_2, R_3, \dots)\right]}$$

We now complete this discussion with remarks about the sequence:

$$\left(V_1(z_1), V_2(z_1), \dots\dots\right).$$

Starting now from $(1')_{f=1}$, we obtain:

$$(4) \quad c\, E\left[f\left(\frac{V_3(1)}{V_2(1)}, \frac{V_4(1)}{V_3(1)}, \frac{V_5(1)}{V_4(1)}, \text{——}\right)\right] = E\left[\frac{1}{\sqrt{V_1(\tau_1)}} f\left(\frac{V_2(\tau_1)}{V_1(\tau_1)}, \frac{V_3(\tau_1)}{V_2(\tau_1)}, \frac{V_4(\tau_1)}{V_3(\tau_1)}, \right.\right.$$

(Note that, in order to deduce (4) from $(1')_{f=1}$, I have used again the facts a) and b) above).

Now, I introduce the notation: $R'_1 = \frac{V_2(\tau_1)}{V_1(\tau_1)}$, $R'_2 = \frac{V_3(\tau_1)}{V_2(\tau_1)}$, $R'_3 = \frac{V_4(\tau_1)}{V_3(\tau_1)}$, ...

We know, since τ_1 is admissible, that:

$$(5) \quad (R_1, R_2, \text{——}, R_n, \text{——}) \overset{(law)}{=} (R'_1, R'_2, \text{——}, R'_n, \text{——})$$

This being recalled, we see that formula (4) offers (... gives / yields ...) an expression for the Radon-Nikodym density of the distribution of:
$(R_{n+1}, n \geq 1)$ with respect to that of $(R_n, n \geq 1)$.

Since the R_k's are independent, it is easy to find an expression for this density; we shall do this now, and then compare with (4).

<u>Proposition</u>: 1. <u>For every</u> $n \geq 1$, <u>one has</u>:

$$E\left[f(R_2, R_3, \text{——}, R_{n+1})\right] = E\left[(n+1)(R_1 R_2 \text{—} R_n)^{1/2} f(R_1, R_2, \text{——}, R_n)\right]$$

2. $\left(\Delta_n \overset{def}{=} (n+1)(R_1 R_2 \cdots R_n)^{1/2}, n \geq 1\right)$ <u>is an</u> L^2 <u>convergent martingale</u>, and <u>we have</u>:
<u>for every measurable functional</u> f <u>defined on</u> $[0,1]^{\mathbb{N}}$, $f \geq 0$,

$$(6) \quad \boxed{E\left[f((R_{n+1}, n \geq 1))\right] = E\left[\Delta_\infty f((R_n, n \geq 1))\right]},$$

<u>where</u>: $\Delta_\infty = \lim_{n \to \infty} n (R_1 R_2 \cdots R_n)^{1/2}$.

Proof: Recall that: $R_k \overset{(law)}{=} Z_{\frac{k}{2},1}$; hence, we have:

$$E\left[f(R_{k+1})\right] = \left(\frac{k+1}{2}\right)\int_0^1 dx\, x^{\frac{k+1}{2}-1} f(x) = \left(\frac{k+1}{k}\right)\left(\frac{k}{2}\right)\int_0^1 dx \left(x^{1/2}\right) x^{\frac{k}{2}-1} f(x).$$

$$= E\left[\left(\frac{k+1}{k}\right)(R_k)^{1/2} f(R_k)\right],$$

and the assertions in the proposition follow from the independance of the R_k's $\square$

We now compare (4) and (6): from the expression of Δ_∞ given in the Proposition, we deduce that:

$$\frac{1}{c\sqrt{V_1(\tau_1)}} = \lim_{n\to\infty} n\left(\frac{V_2(\tau_1)}{V_1(\tau_1)}\cdot\frac{V_3(\tau_1)}{V_2(\tau_1)}\cdot \text{———} \cdot\frac{V_{n+1}(\tau_1)}{V_n(\tau_1)}\right)$$

which simplifies, to give: $\lim_{n\to\infty} c\,n\left(V_n(\tau_1)\right)^{1/2} = 1$,

or equivalently: (7) $\boxed{\lim_{n\to\infty}\left(\frac{\pi}{2}\, n^2 V_n(\tau_1)\right) = 1}$

This agrees with our previous limit theorem which I presented partly in my Fax of June 28th, [and which you amplified, correcting my completely wrong conjecture!!] i.e:

(8) $\boxed{\lim_{n\to\infty}\left(\frac{\pi}{2}\, n^2 V_n(t)\right) = L_t^2}$

For clarity, I have now written 4 "Berkeley notes":

(1) On ranked lengths of excursions, I | July 25th | 9 pages

(2) ———————, II | July 26th | 5 pages

(3) Random scaling and admissibility | 26th | 4 pages

(4) On ranked lengths of excursions, III: from Brownian motion to the Brownian bridge | July 27th | 4 pages