Berkeley, July 25th.

On ranked lengths of excursions, I

Since June 28th, we started again to exchange a number of results on the sequence $(V_1(1), \dots, V_n(1), \dots)$ with several motivations in mind, the main one being that the distribution of this sequence is complicated, and that some very natural questions about it are still unanswered.

In my various Faxes, I have come up, roughly, with 8 questions, some of which are the following ones:

Question a): Compute $E[V_n(1)]$, even for $n=1$.

Compute positive and negative moments of $V_n(1)$, even for $n=1$.

Soln. for negative moments: If H_{2k} is the $(2k)^{\text{th}}$ Hermite polynomial (in two variables):

$$H_{2k}(x,t) = \sum_{j=0}^{k} \alpha_{k,j}\, x^{2j} t^{k-j},$$

we then project the martingale: $(H_{2k}(B_t, t), t \geq 0)$ on $(\mathcal{F}_{g_t})$; we get:

attention

$$P_k(y,t) \stackrel{\text{def}}{=} \sum_{j=0}^{k} \underbrace{\alpha_{k,j}\, E[m_1^{2j}]}_{\beta_{k,j}}\, y^j t^{k-j},$$

and $(P_k(t-g_t, t), t \geq 0)$ is a martingale.

These are some special kind of hypergeometric polynomials; see Azéma-Yor: Etude d'une mart. remarquable. Sém. XXIII?

Hence, by the optional sampling theorem:

$$0 = \sum_{j=0}^{k} \beta_{k,j}\, E[(H_1)^{k-j}].$$

From there, we deduce, by recurrence, a formula for: $E[(H_1)^n] = E\left[\frac{1}{(V_1(1))^n}\right]$

(It may be interesting to write down explicitly, say, the 1st #5 moments of H_1).

Question b): To give a large class of permissible stopping times.

Now, we know that $H_n = H^{(n)}$ is permissible, for every n.
What about: $T = \inf\{t : \Phi(\underline{V}(t)) \geq 1\}$

where: $\Phi(\lambda \underline{v}) = \lambda \Phi(\underline{v})$? (and more generally, I think that we should include the signs).

All the permissible times we have obtained up to now are of this type, i.e.: apart from the $H^{(n)}$'s, we have τ_1, the inverse local time, and
$$\alpha_1 = \inf\{u : A^+_u \geq 1\}.$$

I believe ($\sim$ I seem to remember) that, with what I ~~was~~ wrote last summer here on the scaling property, we should have a positive answer for many cases....

Question: What is the distr. $\Lambda^{(n)}_1$??

Question c): Motivated by the identity:

$$(1)\quad E\left[F(B_u, u \leq 1)\, 1_{(1-g_1 = V_n(1))}\right] = E\left[\frac{1}{H^{(n)}} F\left(\frac{1}{\sqrt{H^{(n)}}} B_{uH^{(n)}}, u \leq 1\right)\right],$$

I introduced the increasing processes: $\Lambda^{(n)}_t = \int_0^t ds\, 1_{(s-g_s = V_n(s))}$,

and their inverses: $\lambda^{(n)} \equiv \lambda^{(n)}_1 = \inf\{t : \Lambda^{(n)}_t \geq 1\}$.

Now, I know for sure (: this is a particular case of the discussion I have in my paper on Random Brownian scaling*, presented in March 1993 in Ascona), that:

$$(2)\quad E\left[F(B_u, u \leq 1)\, 1_{(1-g_1 = V_n(1))}\right] = E\left[\frac{1}{\lambda^{(n)}} F\left(\frac{1}{\sqrt{\lambda^{(n)}}} B_{u\lambda^{(n)}}, u \leq 1\right)\right].$$

* this is a predecessor of the version I am looking for, here...
(↑ manuscript)

Thus, the right-hand sides of (1) and (2) are equal, i.e:

$$(3) \qquad E\left[\frac{1}{\lambda^{(n)}} F\left(\frac{1}{\sqrt{\lambda^{(n)}}} B_{u\lambda^{(n)}}, u \leq 1\right)\right] = E\left[\frac{1}{H^{(n)}} F\left(\frac{1}{\sqrt{H^{(n)}}} B_{uH^{(n)}}, u \leq 1\right)\right]$$

but we should not (a priori!!) deduce from this that $\lambda^{(n)}$ and $H^{(n)}$ have the same distribution (or, equivalently, that $\Lambda_1^{(n)}$ and $V_n(1)$ have the same distribution; however, they have the same first moment, by PY 92 and scaling).

In fact, what we can really deduce from (3) is that:

$$(4) \qquad E\left[\frac{1}{\lambda^{(n)}} \varphi\left(\frac{V_n(\lambda^{(n)})}{\lambda^{(n)}}, \frac{1}{\lambda^{(n)}}\right)\right] = E\left[\frac{1}{H^{(n)}} \varphi\left(\frac{1}{H^{(n)}}, \frac{\Lambda^{(n)}_{H^{(n)}}}{H^{(n)}}\right)\right]$$

which, of course, is very far, a priori, from: $\lambda^{(n)} \overset{(law)}{=} H^{(n)}$ (!).

Sub Question: Can we now deduce from (3), and the fact that $H^{(n)}$ is admissible, that also $\lambda^{(n)}$ is admissible?

(I would not be surprised at all by this...).

Here is another interesting family of increasing processes:

$$L_n(t) = \int_0^t \frac{ds}{s} V_n(s)$$

(which has the scaling property of order 1).

Then, we have, as a consequence of the general scaling relationship:

$$(5) \qquad E\left[F(B_u, u \leq 1) V_n(1)\right] = E\left[\frac{1}{\ell^{(n)}} F\left(\frac{1}{\sqrt{\ell^{(n)}}} B_{u\ell^{(n)}}, u \leq 1\right)\right],$$

where: $\ell^{(n)} = \inf\{t : L_n(t) \geq 1\}$.

Consequently, the quantity: $E\left[F\left(V_k(1), k\geq 1\right) 1_{(1-g_1 = V_n(1))}\right]$ is equal to:

$$E\left[\frac{1}{\lambda^{(n)}} F\left(\frac{V_k(\lambda^{(n)})}{\lambda^{(n)}}, k\geq 1\right)\right], E\left[\frac{1}{H^{(n)}} F\left(\frac{V_k(H^{(n)})}{H^{(n)}}, k\geq 1\right)\right],$$

$$E\left[\frac{1}{\ell^{(n)}} F\left(\frac{V_k(\ell^{(n)})}{\ell^{(n)}}, k\geq 1\right)\right] \ldots\ldots$$

<u>Question d)</u> : [<u>A general question and motivation</u>].

It seems fruitful to ask a number of questions about the process $(t-g_t, t\geq 0)$, or rather $(\sqrt{t-g_t}, t\geq 0)$, by analogy with <u>reflecting Brownian motion</u>.

This is most natural if we think of : $(\mathrm{sgn}(B_t)\sqrt{t-g_t}, t\geq 0)$, Azéma's martingale, as an analogue of $(B_t, t\geq 0)$.
For instance:

- $(X_t \equiv \mathrm{sgn}(B_t)\sqrt{t-g_t}, t\geq 0)$ is a martingale, with increasing process $\left(\frac{t}{2}\right)$
- It has the <u>chaos representation property</u>
- then, from there, we can ask many questions, one of them being:
  (i) <u>what about the BDG inequalities</u>?

More precisely, is it true that, at least, as far as moments are concerned, $V_1(T)$ is equivalent to T, i.e.: the longest length is "roughly" equivalent to T; more precisely again: let $k>0$; does there exist a constant $C_k>0$ such that for every $(\mathcal{F}_t)$ stopping time T,

$$(6) \qquad C_k E((T)^k) \leq E\left[(V_1(T))^k\right] \leq E(T^k).$$

(obvious)

At the moment, I can prove (6) by restricting the class of stopping times T to $(\mathcal{F}_{g_t})$ stopping times, or, more generally $\mathcal{G}_t \overset{\text{def}}{=} (\mathcal{F}_{g_t}) \vee \sigma(\mathrm{sgn}(B_t))$ stopping times.

Indeed, (X_t) is a $(\mathcal{G}_t)$ martingale which satisfies:

$$(7) \qquad d[X,X]_t = -X_{t-}\,dX_t + dt$$

(that is: the structure equation for the Azéma martingale) / Proof: Check jumps: $\Delta X_t = -X_{t-} 1_{(\Delta X_t \neq 0)}$. /

Thus, for a $(\mathcal{G}_t)$ stopping time T, we have, from (7):

$$E((T)^k) \leq C_k \left\{ E\left[[X,X]_T^k \right] + E\left[\left| \int_0^T X_{t-}\,dX_t \right|^k \right] \right\}$$

$$\leq C_k \left\{ E\left[(X_T^*)^{2k} \right] + E\left[\left(\int_0^T X_{t-}^2\, d[X,X]_t \right)^{k/2} \right] \right\}$$

from the BDG inequalities for discontinuous martingales;

Finally, we get the result by using Cauchy-Schwarz on the right hand side, i.e:

$$E\left[\left(\int_0^T X_{t-}^2\, d[X,X]_t \right)^{k/2} \right] \leq E\left[(X_T^*)^k [X,X]_T^{k/2} \right]$$

$$\leq \left(E\left[(X_T^*)^{2k} \right] \right)^{1/2} E\left[[X,X]_T^k \right]^{1/2}$$

$$(\text{BDG again}) \leq C_k \; E\left[(X_T^*)^{2k} \right],$$

and, finally, we have obtained:

$$(6') \qquad E\left[T^k \right] \leq C_k \; E\left[(V_1(T))^k \right].$$

(obviously, above, the constant C_k changes from line to line) /.

So now, the question is: Can we extend (6') from stopping times relative to $(\mathcal{G}_t)$ to any general $(\mathcal{F}_t)$ stopping time?

I believe that the answer is yes, but I have to think again about the "good λ-inequalities" techniques (most of them are discussed in Revuz-Yor)

On the contrary, it is wrong that such inequalities are true with $V_2(t)$ replacing $V_1(t)$ ((this gives some more interest to V_1 again!))

i.e.: at least for $k \geq \frac{1}{2}$, there exists no constant C_k such that

$$(8) \qquad E(T^k) \leq C_k \, E((V_2(T))^k).$$

Indeed, take $T = H^{(2)}$; then,

$$E\left[(H^{(2)})^k\right] = \infty, \quad \text{since: } H^{(2)} \geq d_{H(1)} \geq d_{H(1)} - H^{(1)} \overset{(\text{law})}{=} \hat{T}_{(B_{H(1)})}$$

where $B_{H(1)}$ is independent from $\hat{T}$, so that (scaling):

$$\hat{T}_{B_{H(1)}} \overset{(\text{law})}{=} (B_{H(1)})^2 \, \hat{T}_1, \quad \text{hence the result.}$$

Question: Is (8) true for $k < \frac{1}{2}$? / Similar question for V_n ....)

(ii) Let us now study more closely the distribution of $V_1(t)$, for fixed t, by looking at exponential moments of $V_1^{H(1)}$.

Recall that: $\cos(\lambda B_t) \exp\left(\frac{\lambda^2}{2} t\right)$ is a martingale, and define:

$$\tilde{\Psi}(z) = E\left[\cos(z\, m_1)\right].$$

Call γ_* the first >0 zero of this function (I guess: $\gamma_* < \infty$).

We remark that: $\tilde{\varphi}(\lambda\sqrt{t-g_t})\, e^{\frac{\lambda^2 t}{2}}$ is a martingale?,
and it is not difficult to | remark / prove that:

$$E\left[\exp\left(\frac{\lambda^2}{2} H^{(1)}\right)\right] = \frac{1}{\tilde{\varphi}(\lambda)}, \quad \text{for } \lambda < \gamma_* .$$

What is precisely γ_*? /.

<u>Application</u>* : We look for asymptotics, as $u \to \infty$, of:

$$P\left(V_1(1) < \frac{1}{u}\right) = P\left(u < \frac{1}{V_1(1)}\right) = P\left(u < H^{(1)}\right) = P\left(\exp\left(\frac{\lambda^2 u}{2}\right) < \exp\left(\frac{\lambda^2}{2} H^{(1)}\right.\right.$$

$$\leq \exp\left(-\frac{\lambda^2 u}{2}\right) E\left[\exp\left(\frac{\lambda^2}{2} H^{(1)}\right.\right.$$

|| Can we be as precise about this, as one is classically for $\sup_{s \leq 1} |B_s|$?

(This is another example of the analogy I have in mind ...).

/ * We should probably look at some papers by Revesz, quoted in Mihael's paper .... /

A completely different discussion:

In Caracas, I thought that the identity (16) in my 2^{nd} message (July 12^{th}) is very interesting:

$$(16) \qquad c\, E\left[G\left(\frac{V_1(\tau_1)}{\tau_1} ; \left(\frac{V_{n+1}(\tau_1)}{\tau_1}, n \geq 1 \right) \right) \right] = E\left[\frac{1}{\sqrt{\tau_1}} \hat{G}\left(\frac{V_1(\tau_1)}{\tau_1} ; \left(\frac{V_n(\tau_1)}{\tau_1}, n \geq \ldots \right.\right.\right.$$

for the following reasons:

i) Suppose G depends only on n variables, and define:

$$q_{n-1}(v_1, \ldots, v_{n-1}) = E\left[\frac{1}{\sqrt{\tau_1}} \,\middle|\, \frac{V_k(\tau_1)}{\tau_1} = v_k,\ k \leq n-1 \right]$$

Then, we deduce the law of: $(V_1(1), \ldots, V_n(1))$ from that of $(V_1(1), \ldots, V_{n-1}(1))$ and q_{n-1},

or conversely, if we know the n marginals for every n, then, we deduce q_{n-1}, for $n \geq 2$.

We could probably get a nice charact. result from this....

$\left(\frac{1}{\sqrt{\tau_1}} \text{ is in the way !!} \right)$. [for the law of $\underline{V}(1)$]

ii) I would also like to exploit the ergodic theory character of (16). More precisely, suppose G does not depend on the 1^{st} variable, which plays a very particular rôle ...). I write such a function G_0. Then, we have:

$$E\left[G_0(\underline{V} \circ \underset{\substack{\uparrow \\ \text{(the shift operator)}}}{\mathcal{S}}) \right] = E\left[(\text{loc. time})\, \hat{G}_0(\underline{V}) \right]$$

What can we deduce from this? Is it true that the law of $\underline{V} \cdot S$ is abs. cont. with respect to the law of $\underline{V}$?
If yes, <u>is there a simple density</u> ??

<u>A puzzling question:</u>

In one of your faxes to me on July 11th, you thought that for some stopping time T, one had:

$$E\left[F\left(\frac{1}{\sqrt{T}} B_{uT}, u \leq 1\right)\right] = E\left[F(B_u, u \leq 1)\right].$$

This is not true, etc... <u>But</u>: Could we show that <u>the only such T's are the Cts</u>?

<u>A final question:</u>
We showed that the sequences: $\left(\frac{1}{V_1(1)}, \dots, \frac{1}{V_n(1)}, \dots\right)$ and $\left(H^{(1)}, \dots, H^{(n)}, \dots\dots\right)$ do not have the same distribution.
It would be nice to show that:

$$\left(\frac{1}{V_1(1)}, \frac{1}{V_2(1)}\right) \text{ differs from } \left(H^{(1)}, H^{(2)}\right),$$

and more generally:

$$\left(\frac{1}{V_n(1)}, \frac{1}{V_{n+1}(1)}\right) \text{ differs from } \left(H^{(n)}, H^{(n+1)}\right).$$

All for now. Marc.