On the asymptotic behavior of Bessel processes, as the dimension goes to ∞. (II)

Notation: $X_n(t) = \frac{1}{n}(\rho^{(n)}(t))^2$; $\rho^{(n)}(t)$: $BES(n) = \beta(t) + \frac{n-1}{2}\int_0^t \frac{ds}{\rho^{(n)}(s)}$; $L_{(n)} = \sup(t : X_n(t)=1)$; $T_{(n)} = \inf(t : X_n(t)=1)$

1\. My interest in the subject originates from Gallardo's paper [2], who ~~shows~~ gave a probabilistic proof and interpretation of Poincaré's lemma (cf: Mc Kean [1]) by showing: $T_{(n)} \xrightarrow[(n\to\infty)]{} 1$.

I then thought of replacing the sequence $T_{(n)}$, by $L_{(n)} = \sup(t : X_n(t)=1)$, which is as suitable as $T_{(n)}$, as far as Poincaré's lemma is concerned. Moreover, explicit formulae relative to $L_{(n)}$ are certainly far simpler than the $T_{(n)}$-ones (for those formulae with L_y, see Pitman-Yor [3] for instance).

2\. Some individual results about $L_{(n)}$.

Let $L_y^{(\nu)} = \sup\{t : \rho_t^{(\nu)} = y\}$. Then, from [3], we know that $1/L_y^{(\nu)}$ follows the law $\gamma(\nu; \frac{y^2}{2})$.

Therefore, writing $\nu = \frac{n}{2} - 1$; $y^2 = n$, we get, for any $p > 0$:

$$E[(L_{(n)})^p] = \int_0^\infty dt\ \Gamma(\nu)^{-1}\left(\frac{y^2}{2}\right)^\nu t^{\nu-p-1}\exp\left(-\frac{y^2}{2}t\right)$$

$$= \left(\frac{y^2}{2}\right)^p \frac{\Gamma(\nu-p)}{\Gamma(\nu)}$$

$$= \left(\frac{y^2}{2}\right)^p \frac{1}{(\nu-1)(\nu-2)\cdots(\nu-p)} = \frac{(\nu+1)^p}{(\nu-1)(\nu-2)\cdots(\nu-p)} \quad (*)$$

(when p is an integer)

Therefore, for any p integer, $E[(L_{(n)})^p] \xrightarrow[(n\to\infty)]{} 1$,

which implies: $E[(L_{(n)}-1)^p] \xrightarrow[(n\to\infty)]{} 0$ (develop: $(a-1)^p$)

Moreover, taking $p=4$, one finds: $E[(L_{(n)}-1)^4] = O\left(\frac{1}{n^2}\right)$,

and therefore: $L_{(n)} \xrightarrow[a.s.]{(n\to\infty)} 1$

2. On the asymptotic behavior of $X_n(t)$, up to $T_{(n)}$.

The main results there are:

(i) for any $p>0$, $$E\Big[\sup_{t\le T_{(n)}} |X_n(t)-t|^p\Big] \xrightarrow[(n\to\infty)]{} 0.$$

(ii) for any $p>0$, $$E\Big[\sup_{t\le T_{(n)}} \Big|\sqrt{n}(X_n(t)-t) - 2\int_0^t \sqrt{s}\, d\beta(s)\Big|^p\Big] \xrightarrow[(n\to\infty)]{} 0.$$

As a corollary, one gets:

(i') $$E\big[|T_{(n)}-1|^p\big] \xrightarrow[(n\to\infty)]{} 0 \; ; \text{ also: } T_{(n)} \xrightarrow[(n\to\infty)]{a.s} 1.$$

(i'') $$E\Big[\Big|\sqrt{n}\,(1-T_n) - 2\int_0^1 \sqrt{s}\, d\beta(s)\Big|^p\Big] \xrightarrow[(n\to\infty)]{} 0.$$

The proof of (i) and (ii) is obtained as a consequence of the Burkholder-Davis-Gundy inequalities. In fact, one obtains a more general type of result, inspired by Burkholder's work [4] and Davis [5].

Burkholder [4] showed that if $c_{p,n}$ & $C_{p,n}$ are the "best" constants such that, for any stopping time T:

$$c_{p,n}\, E[T^p] \le E\Big[\sup_{t\le T} (X_n(t))^p\Big] \le C_{p,n}\, E[T^p].$$

~~Davis [ ] improved on Burkholder~~

then, $$\lim_{n\to\infty} c_{p,n} = \lim_{n\to\infty} C_{p,n} = 1$$

Davis [5] improved on Burkholder's result by showing that:

$$E\Big[\sup_{t\le T} \big|\sqrt{X_n(t)} - \sqrt{t}\big|^p\Big] \le \frac{C_p}{n^{p/2}} E[T^{p/2}]$$

I find it simpler(*) to consider the expressions:

$$E\Big[\sup_{t\le T} |X_n(t)-t|^p\Big],$$ and the analogous expression related to (ii).

(*) and just as useful

3. On the asymptotic behavior of $X_n(t)$, up to $L_{(n)}$.

It is now natural to seek to prove the analogues of (i)-(ii); (i')-(ii'), when one replaces $T_{(n)}$, by $L_{(n)}$. The analogous results are true. To see this, we shall use a method ("B-D-G inequalities") similar to that used in §2, apart from the fact that we are looking at the processes X_n up to last hitting times (not stopping times). Therefore, we must first establish "right" analogues of B-D-G.

3.1) Note L for $L_y^{(\gamma)} \equiv \sup\{t : \rho_t^{(\gamma)} = y\}$ (but the following arguments apply in a far more general set-up (see Jeulin-Yor [6], for instance). Let $(\mathcal{F}_t^L)$ be the smallest filtration which makes L a stopping time, and contains $(\mathcal{F}_t)$. It is then known [6] that if (Y_t) is a $(\mathcal{F}_t)$ continuous local martingale, there exists $(\tilde{Y}_t)$ a $(\mathcal{F}_t^L)$ continuous local martingale such that:

$$(1) \qquad Y_{t\wedge L} = \tilde{Y}_t + \int_0^{t\wedge L} \frac{1}{Z_s^L} d\langle Y, M^L\rangle_s ,$$

where (Z_t) is the (right-cont.) supermart. $P(L > t / \mathcal{F}_t)$, and (M_t^L) is its martingale part. Then, one has the:

Lemma: Let $l, m \in]1,\infty[$ be such that: $\frac{1}{l} + \frac{1}{m} = 1$.

Note $U_L = \left(\int_0^L \frac{d\langle M^L\rangle_s}{(Z_s^L)^2}\right)^{1/2}$.

Then, one has, for all $k > 0$, and any cont. $(\mathcal{F}_t)$ martingale Y_t:

$$(**) \qquad E[(Y_L^*)^k] \le C_{k,l} E[\langle Y\rangle_L^{\frac{kl}{2}}]^{1/l} \left\{1 + E[U_L^{km}]^{\frac{1}{m}}\right\}$$

Proof: One has: $\langle \tilde{X}\rangle_t = \langle Y\rangle_{t\wedge L}$.

It now remains to use (1), the usual B-D-G inequalities,

the majorization:
$$\int_0^L \frac{d\langle Y, M^L\rangle_s}{Z_s^L} \leq \langle Y\rangle_L^{1/2}\, U_L\, ,$$
and finally Hölder, with ℓ and m.

Let's apply this now to $L = L_y^{(\nu)}$. The scale function of $BES^{(\nu)}$ is:
(denoted here by: R_t)
$s(x) = -1/x^{2\nu}$

then, one has: $Z_t^L \equiv P(L>t / \mathcal{F}_t) = \left(\frac{s(R_t)}{s(y)}\right)\wedge 1$,

and, from Itô's formula:
$$M_t^L = \frac{1}{s(y)}\int_0^t 1_{(R_u > y)}\, d(s(R_u)).$$

Therefore:
$$U_L^2 \equiv \int_0^L \frac{d\langle M^L\rangle_s}{(Z_s^L)^2} = \int_0^L \frac{1(R_u>y)}{s(R_u)^2}\, d\langle s(R_u)\rangle$$
$$= \int_0^L (4\nu^2)\, \frac{1(R_u>y)}{R_u^2}\, du\, .$$

In the following, the majorization:
$$U_L^2 \leq \left(\frac{4\nu^2}{y^2}\right)(L-T),\ \text{where } T \equiv T_y^{(\nu)} \equiv \inf\{t : R_t = y\}$$
will be crucial, since:
$$U_{L_{(n)}}^2 \leq \underset{\uparrow\,(\text{Universal Ct})}{c}\, n\,(L_{(n)} - T_{(n)}).$$

We shall certainly be able to prove that for all $p>0$, $\sup_n E\left[\{n(L_{(n)}-T_{(n)})\}^p\right] < \infty$

Therefore, we obtain from the lemma, that for any k, ℓ, m $(k>0;\ \ell, m>1)$,
there exists $C_{k,\ell,m}$ $\ni$:
for all n's, and all Cts. Martingales:
$$E\left[(Y^*_{L_{(n)}})^k\right] \leq C_{k,\ell,m}\, E\left[\langle Y\rangle_{L_{(n)}}^{\frac{k\ell}{2}}\right]^{1/\ell}.$$

3.2) Some explicit Computations.

It is then natural to try to Compute explicitly : $E\left[\sup_{t\le L_{(n)}} |X_n(t)-t|^p\right]$ to see precisely as this expression goes to 0, as $(n\to\infty)$.

The Computation relies on the following formula:

if $\begin{cases} H_t \uparrow, \\ \text{right-Cont} \\ (\mathcal{F}_t)\text{ adapted.} \end{cases}$ and $T_a^H = \inf\{t : H_t \ge a\}$, one has:

$$P\{H_L \ge a\} = P\{L > T_a^H\}$$
$$= E\left[Z^L_{T_a^H}\right]$$
$$= E\left[\left(\frac{s(R_{T_a^H})}{s(y)}\right)\wedge 1\right]$$

(on the diffusion case; with: $L = L_y$).

As a consequence, if $F(da)$ is a ~~cont~~ (diffuse) measure on $(0,\infty)$, one has:

$$E[F(H_L)] = \int_0^\infty dF(a)\, E\left[\frac{s(R_{T_a^H})}{s(y)}\wedge 1\right].$$

As an application, take: $H_t = \sup_{(s\le t)} (X_n(s)-s)$

$$\tilde{T}_a^{(n)} \equiv T_a^H = \inf\{t : \rho^{(n)}(t) > \sqrt{n}\,(a+t)^{1/2}\}$$

(Note that such stopping times have been considered by Shepp [7])

Therefore, One gets: $E\left[\left(\frac{s(R_{T_a^H})}{s(y)}\right)\wedge 1\right]$

$$= E\left[\frac{y^{2\nu}}{(\rho^{(n)}(T_a^H))^{2\nu}}\wedge 1\right] = E\left[\frac{y^{2\nu}}{y^{2\nu}(a+\tilde{T}_a^{(n)})^{\nu}}\wedge 1\right]$$

$$= E\left[\frac{1}{a^{\nu}(1+\tilde{T}_1^{(n)})^{\nu}}\wedge 1\right] \quad (\text{by scaling}).$$

Finally, one gets, with $X = H_{L_{(n)}}$; $Y = 1+\tilde{T}_1^{(n)}$

$$\boxed{E[X^p] = \left(\frac{\nu}{\nu-p}\right) E\left[\frac{1}{Y^p}\right]}$$

It now remains to Compute: $E\left[\frac{1}{(1+\tilde{T}_c^{(n)})^p}\right]$

$\tilde{T}_c^{(n)} = \inf\{t : \rho_t^{(n)} = c\sqrt{1+t}\}$

We use Shepp's method [7], mutatis mutandis –

$\rho_t^{-\nu} I_\nu(\theta\rho_t)\, e^{-\frac{\theta^2 t}{2}}$ is a martingale ($\theta>0$)

$$\Rightarrow \quad E\left[\underbrace{\left(\frac{\theta\rho_{\tilde{T}_c}}{2}\right)^{-\nu} I_\nu(\theta\rho_{\tilde{T}_c})}_{\overset{\text{def}}{=}\ \tilde{I}_\nu(\theta\rho_{\tilde{T}_c})}\, e^{-\frac{\theta^2}{2}\tilde{T}_c}\right] = \frac{1}{\Gamma(\nu+1)}$$

D'où :

$$E\left[\tilde{I}_\nu(\theta c\sqrt{1+\tilde{T}_c})\, e^{-\frac{\theta^2\tilde{T}_c}{2}}\right] = \frac{1}{\Gamma(\nu+1)}$$

$$\Rightarrow \forall\varphi\geq 0,\ E\left[\int e^{-\frac{\theta^2}{2}}\varphi(\theta)\,d\theta\ \tilde{I}_\nu(\theta c\sqrt{1+\tilde{T}_c})\, e^{-\frac{\theta^2}{2}\tilde{T}_c}\right] = \frac{1}{\Gamma(\nu+1)}\int e^{-\frac{\theta^2}{2}}\varphi(\theta)\,d\theta$$

Taking $u=\theta\cdot\sqrt{1+\tilde{T}_c}$, and $\varphi(\theta)=\theta^p$:

$$E\left[\frac{1}{(1+\tilde{T}_c)^{\frac{1+p}{2}}}\right]\cdot\int e^{-\frac{u^2}{2}}u^p\,\tilde{I}_\nu(uc) = \frac{1}{\Gamma(\nu+1)}\int e^{-\frac{\theta^2}{2}}\theta^p\,d\theta \cdot \quad \underline{\text{QED}}.$$

§ Taking $X = \sup_{t\leq L_{(n)}}|\beta(t)|$, and $Y=\rho^{(n)}(T_1^H)$, one gets similarly:

$$E[X^p] = \frac{2\nu}{2\nu-p}\,E\left[\left(\frac{y}{Y}\right)^p\right],$$

but I have not been able to Compute $E\left[\frac{1}{Y^p}\right]$.

References:

[1] Mc Kean: Geometry of Differential Space – Ann. Proba (73).

[2] Gallardo: Au sujet du contenu probabiliste d'un lemme de Poincaré. Ann. Clermont.

[3] Pitman-Yor: Bessel processes & Infinitely divisible laws – L.N 850. (Proceedings Durham.)

[4] Burkholder: Exit Times of BM and Hardy Spaces – Advances in Maths, 26, 182-205 (1977).

[5] Davis: On stopping times for n-dim. BM. Annals of Proba (1978), 6, 4, 631-659.

[6] Jeulin-Yor: Nouveaux résultats sur le grossit. des tribus. Ann. ENS (1980).

[7] Shepp: