On the number of crossings between two spheres in $\mathbb{R}^k$ $(k \geq 3)$.

1st Case: $A = S(0, a)$, $C = S(d, c)$ are two spheres situated at a distance $d > a + c$.

The inversions with pole 0, power a^2 / pole d, power c^2 } are denoted respectively as: q_A, q_C.

Explicit formulae are:

(1) $$q_A(x) = \frac{a^2}{x} \quad ; \quad q_C(x) = d + \frac{c^2}{x-d}$$

(at least for points x situated on the segment Od).

We are interested in the sequence: $y_{n+1} = q_C(q_A(y_n))$; $y_1 = q_C(0)$

We deduce from formula (1):

(2) $$y_{n+1} = \frac{(d^2-c^2)\,y_n - da^2}{dy_n - a^2} \quad ; \quad y_1 = \frac{d^2-c^2}{d}.$$

We want to transform the sequence (y_n) in a sequence (ξ_n) which satisfies:

(3) $$\xi_{n+1} = 2\rho - \frac{1}{\xi_n},$$

which will allow the appearance of the polynomials $C_n(\rho)$.

For this purpose, we write formula (2) in the general form:

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}$$

which is equivalent to: $y_{n+1} = \frac{\alpha}{\gamma} + \frac{1}{\gamma}\,\frac{\beta\gamma-\alpha\delta}{\gamma y_n+\delta}$

Let $z_n = \gamma y_n + \delta$, and $\Delta = \beta\gamma - \alpha\delta$. We obtain:

$$z_{n+1} = (\alpha+\delta) + \frac{\Delta}{z_n}.$$

To write this in the form (3), let $z_n = u\cdot\xi_n$, then:

$$\xi_{n+1} = \frac{\alpha+\delta}{u} + \frac{\Delta}{u^2\,\xi_n}$$

So that we have the form (3) in letting: $u^2 = -\Delta$.

In our particular case, we have:

$\alpha = d^2 - r^2$; $\beta = -dR^2$; $\gamma = d$; $\delta = -R^2$;

$\Delta = -r^2R^2$; $u = rR$.

Hence: $\rho = \frac{\alpha+\delta}{2u} = \frac{d^2-r^2-R^2}{2rR}$

Sorry! $a = R$; $c = r$

The sequence (ξ_n) is now completely determined, and we get:

(4) $$\xi_n = \frac{C_n(\rho)}{C_{n-1}(\rho)}\ ; \quad \rho = \cancel{\frac{d^2-r^2-R^2}{2rR}}\ \frac{d^2-a^2-c^2}{2ac}$$

Finally, going back from (3) to (2), we get:

(5) $$\boxed{y_n = \frac{a}{d}\left(c\,\frac{C_n(\rho)}{C_{n-1}(\rho)} + a\right)}\ ; \quad \rho = \frac{d^2-a^2-c^2}{2ac}$$

2nd Case: $A = \mathcal{S}(0,a)$; $C = \mathcal{S}(-d,c)$, with $c \geq d+a$.

~~The main change~~ The only change in all formulae ~~in the~~ (at the start) is the change of d in $(-d)$.

We get: $\alpha = c^2 - d^2$; $\beta = -da^2$; $\gamma = d$; $\delta = a^2$

$$\Delta = -a^2c^2 \; ; \; u = ac.$$

And then:

$$(4') \qquad \xi_n = \frac{C_n(\rho)}{C_{n-1}(\rho)}, \quad \text{with } \rho = \frac{a^2+c^2-d^2}{2ac}$$

$$(5') \qquad y_n = \frac{a}{d}\left(c\,\frac{C_n(\rho)}{C_{n-1}(\rho)} - a\right); \quad \rho = \frac{a^2+c^2-d^2}{2ac}.$$

Finally, we can write (5) and (5') in the single formula:

$$\boxed{y_n = \frac{a}{d}\left(c\,\frac{C_n(\rho)}{C_{n-1}(\rho)} \pm a\right) \; ; \; \rho = \frac{|a^2+c^2-d^2|}{2ac}}$$

with the sign $+$ in the exterior case (Case 1), sign $-$ in the interior case (Case 2).

The probability of having at least ~~(n)~~ crossings between A and C is then:

$$\boxed{\mathbb{P}(T_A^n < \infty) = \left(\frac{d^{n-1}\, C_0(\rho) \cdots\cdots C_{n-2}(\rho)}{(c\,C_1(\rho) \pm a\,C_0(\rho)) \cdots\cdots (c\,C_{n-1}(\rho) \pm C_{n-2}(\rho))}\right)^{k-2}}$$

Geometric interpretation of ρ

1) $\rho \equiv \rho(A,C) = \dfrac{|d^2-a^2-c^2|}{2ac}$ is the metric invariant of A and C

in that : $\rho(A,C) = \rho(A',C')$ when A' and C' are the inverses of A and C in a given inversion.

2) It takes values >1 when the spheres do not intersect, and <1 when they do. In the latter case, we have:

$$\cos\theta = \rho(A,C)$$

with θ = angle between A and C.

3) In general, ρ appears as follows (and the statement 1) is deduced from there):

Better to discuss in terms of Circles

Suppose we are in the exterior case, and (Γ) is a sphere which is orthogonal to both (A) and (C), and cuts the circles (A) and (C) at A, B, C, D respectively. Then:

$$\frac{CA}{CB} : \frac{DA}{DB} = \frac{\rho+1}{\rho-1}.$$

In the interior case, we get:

$$\frac{CA}{CB} : \frac{DA}{DB} = \frac{\rho-1}{\rho+1}.$$

In any case, unfortunately, the formula which gives the probability of at least $(n+1)$ crossings does not involve only ρ, but also d, c, a, although in a simple manner.

The sphere and the hyperplane.

1) We may consider this case to be obtained as the limit of the exterior case, as $C \to \infty$, but remains in the same coaxal system, which it is now convenient to define by A and the radical axis ($\equiv$ hyperplane) at distance β from 0.

Then, we have:

(6) $\beta = \dfrac{d^2+a^2-c^2}{2d}$, and: (7) $\rho = \dfrac{d^2-a^2-c^2}{2ac}$

From (6), we deduce: $\dfrac{c}{d} \longrightarrow 1$, as $C \longrightarrow \infty$.

From (6) and (7), we have: $\rho = \left(\dfrac{d^2-c^2}{2ac}\right) - \dfrac{a}{2c} = \left(\dfrac{d}{ca}\right)\left(\dfrac{d^2-c^2}{2d}\right) - \dfrac{a}{2c}$

$$= \left(\frac{d}{ac}\right)\left(\beta - \frac{a^2}{2d}\right) - \frac{a}{2c},$$

and, finally, we obtain:

$$\rho \xrightarrow[(C\to\infty)]{} \left(\frac{\beta}{a}\right),$$

so that the probability of having at least $(n+1)$ crossings between A and C now becomes:

(8) $$\boxed{P(T_A^n < \infty) = \left(\frac{1}{C_{n-1}\left(\frac{\beta}{a}\right)}\right)^{k-2}}$$

2) We now wish to discuss the distribution of $N' = N_{AC} - 1$.
To simplify matters, we shall assume $k=3$, and $a=1$, which we can always do by Brownian scaling.

If we let $\beta = \frac{1}{2}\left(x + \frac{1}{x}\right)$, with $0<x<1$, then:

a) x is the limiting point inside A of the coaxal system determined by A and H;

b) the following simple enough formula :

(9) $$C_n(\beta) = \frac{1-x^{2(n+1)}}{x^n(1-x^2)}$$ holds.

Indeed, this is easily deduced from the fact that, if $x=e^{-t}$, then:

$$C_n(\mathrm{ch}\,t) = \frac{\mathrm{sh}(nt)}{\mathrm{sh}\,t}$$

Formula (9) allows to write (8) as:

(10) $$P(N' \geq n) = \frac{(1-x^2)\,x^n}{1-x^{2(n+1)}}.$$

Now, developing $\frac{1}{1-x^{2(n+1)}}$ as $\sum_{k=0}^{\infty} x^{2k(n+1)}$, we obtain:

$$P(N' \geq n) = \sum_{k=0}^{\infty} (1-x^2)\,x^{2k}\,x^{(2k+1)n},$$

so that we find the following identity in law:

(11) $$\boxed{N' \overset{(d)}{=} \left[\frac{M}{2V+1}\right]}$$

with ~~$P(V=k) = 1-x$~~ M and V two independent, geometrically distributed random variables with respective parameters x and x^2, that is: $P(M=m) = (1-x)\,x^m$; $P(V=k) = (1-x^2)\,x^{2k}$.

A remarkable identity.

We have: $Cap(A,C) = E(N')$, which compared with the Morse-Feshbach result gives the nice identity:

$$(12) \qquad \sum_{n=0}^{\infty} \frac{1}{C_n(\beta)} = 2\sqrt{\beta^2-1} \sum_{k=0}^{\infty} \frac{1}{\left(\beta+\sqrt{\beta^2-1}\right)^{2k+1}-1}$$

We now remark that the above representation (11) gives a very elementary proof of (12). By choosing an appropriate probability space, we may obviously assume that the a.s. identity: $N' = \left[\frac{M}{2V+1}\right]$ holds.

Now, the left-hand side of (12) is: $\sum_{n=0}^{\infty} P(N' \geq n) = E(N')$,

while, we find, by inspection, that the right hand side of (12) is:

$$\sum_{k=0}^{\infty} P(V=k)\, E(N' \mid V=k)$$

with, indeed, the equality holding for each term of order k in both series.

Remark: Obviously, we could detail more from (12), by remarking:

$$\sqrt{\beta^2-1} = \frac{1}{2}\left(\frac{1-x^2}{x}\right) \quad ; \quad \beta+\sqrt{\beta^2-1} = \frac{1}{x}$$

and reducing (12) to:

$$(12') \qquad \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{1-t^{2k+1}}.$$