On the stability properties of spider-martingales.

This is an extension of Section 5 of [L], which was written in a "minimal" form, in order to get to the applications as soon as possible.
Since [L] is a summary of a number of previous notes, and, in particular, of [MT] ≡ More transforms of a spider-martingale, Aug. 19th, I will take my notations from [MT], and I will develop Example 2 in (2.2), and Sections 3 and 4 in [MT].

a) On example 2 in (2.2).

Since, by hypothesis, the $g_i^{(j)}$'s are space-time harmonic, we only have to study the first condition (11).
Define
$$\varphi_i^{(j)}(t_i) = g_i^{(j)}(0, t_i) \quad , \quad \text{and} \quad \varphi_i^{(j)}(t_i) = \frac{\partial}{\partial x_i}\left(g_i^{(j)}\right)(0, t_i) .$$

Then, condition (11) is:

(13)
$$\sum_i \left(\prod_{\ell \neq i} \varphi_\ell^{(j)}(t_\ell)\right) \varphi_i^{(j)}(t_i) \quad \text{does not depend on } j .$$

Here is a particular example where this condition is satisfied; assume that:

(14)
$$\varphi_i^{(j)}(t_i) \equiv \frac{\partial}{\partial x_i}\left(g_i^{(j)}\right)(0, t_i) = \delta_{ij}\, \varphi_i(t_i) \quad , \text{and} \quad \varphi_\ell^{(j)}(t_\ell) \equiv g_\ell^{(j)}(0, t_\ell) = \varphi_\ell(t_\ell) ,$$
for $\ell \neq j$.

Then, the expression in (13) becomes, for a given j:
$$\left(\prod_{\ell \neq j} \varphi_\ell(t_\ell)\right) \varphi_j(t_j) \equiv \prod_{i=1}^{k} \varphi_i(t_i) ,$$

which, obviously, does not depend on j.

The construction in Section 1 of [MT] is a particular example of this, with:

$$(15)\quad \begin{cases} g_\ell^{(j)}(x,t) = \cosh(\lambda_\ell x)\exp\left(-\frac{\lambda_\ell^2 t}{2}\right) & (\ell \neq j) \\ g_j^{(j)}(x,t) = \frac{\sinh(\lambda_j x)}{\lambda_j}\exp\left(-\frac{\lambda_j^2 t}{2}\right) \end{cases}$$

and the functions $\varphi_j(t)$ are: $(15')\quad \varphi_j(t) = \exp\left(-\frac{\lambda_j^2 t}{2}\right)$.

More generally, we may take:

$$(15)_\mu\quad \begin{cases} g_\ell^{(j)}(x,t) = \int_0^\infty \mu_\ell(d\lambda)\ \cosh(\lambda x)\exp\left(-\frac{\lambda^2 t}{2}\right) & (\ell \neq j) \\ g_j^{(j)}(x,t) = \int_{0+}^\infty \mu_j(d\lambda)\ \frac{\sinh(\lambda x)}{\lambda}\exp\left(-\frac{\lambda^2 t}{2}\right) \end{cases}$$

This gives us already a vast class of such transforms, and here, we have:

$$(15')_\mu \qquad \varphi_j(t) = \int_0^\infty \mu_j(d\lambda)\exp\left(-\frac{\lambda^2 t}{2}\right).$$

If, instead of considering k measures μ, we introduced $(k\times k)$ measures $\mu_\ell^{(j)}(d\lambda)$, and defined:

$$(15)^*_\mu\quad \begin{cases} g_\ell^{(j)}(x,t) = \int_0^\infty \mu_\ell^{(j)}(d\lambda)\ \cosh(\lambda x)\exp\left(-\frac{\lambda^2 t}{2}\right) & (\ell \neq j) \\ g_j^{(j)}(x,t) = \int_0^\infty \mu_j^{(j)}(d\lambda)\ \frac{\sinh(\lambda x)}{\lambda}\exp\left(-\frac{\lambda^2 t}{2}\right), \end{cases}$$

then, condition (13) would become:

$$(13)^*_\mu \quad \left\| \quad \left(\prod_{\ell\neq j}\int_0^\infty \mu_\ell^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t_\ell}{2}\right)\right)\left(\int_0^\infty \mu_j^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t_j}{2}\right)\right) \right.$$

<u>does not depend on j.</u>

which brings us back to our previous choice, where: $\mu_\ell^{(j)}(d\lambda) \equiv \mu_\ell(d\lambda)$.

More generally again, we could take $\left(g_\ell^{(j)}(x,t)\right)$ to be of the form:

$$g_\ell^{(j)}(x,t) = \int_0^\infty \mu_\ell^{(j)}(d\lambda)\; c_\lambda(x,t) + \int_0^\infty \nu_\ell^{(j)}(d\lambda)\; s_\lambda(x,t),$$

where:

$$c_\lambda(x,t) = \cosh(\lambda x)\exp\left(-\frac{\lambda^2 t}{2}\right), \text{ and } s_\lambda(x,t) = \frac{\sinh(\lambda x)}{\lambda}\exp\left(-\frac{\lambda^2 t}{2}\right)$$

Consequently, we obtain, in this "general" example:

$$\varphi_\ell^{(j)}(t) = \int_0^\infty \mu_\ell^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t}{2}\right), \text{ and } \varphi_\ell^{(j)}(t) = \int_0 \nu_\ell^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t}{2}\right),$$

and condition (13) now becomes:

$$\sum_i \prod_{\ell \neq i}\left(\int \mu_\ell^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t_\ell}{2}\right)\right)\int_0^\infty \nu_i^{(j)}(d\lambda)\exp\left(-\frac{\lambda^2 t_i}{2}\right)$$

does not depend on j,

so that, from the injectivity of the Laplace transform, we obtain the condition:

$$(16) \qquad \sum_i \left(\prod_{\ell\neq i} \mu_\ell^{(j)}(d\lambda_\ell)\right)\nu_i^{(j)}(d\lambda_i) \quad \underline{\text{does not depend on } j}.$$

Comments: a) We should try to solve this system completely (Start with the measures $\mu_\ell^{(j)}$, and $\nu_\ell^{(j)}$, having densities).

b) This reminds me of the Cauchy-Riemann equations...

b) On Section 3 of [MT]

(i) More generally than solving the problem which is posed in the title, one can study the following question:

(Q) Find all the spider-martingales which are on the same rays as B, for every time $t \geq 0$.

A closely related question is:

(Q') Find all the spider-martingales with a given local time at the origin, for instance, the local time of B.

Here are some very interesting examples: $S_t = (S_t^{(1)}, S_t^{(2)}, \dots, S_t^{(k)})$,

with : $$S_t^{(i)} = \frac{\sinh(\lambda B_t^{(i)})}{\lambda} \exp\left(-\frac{\lambda^2}{2}(t-g_t)\right) \quad , \text{ for } \lambda \neq 0.$$

This example is very closely related with the example in Section 1 of [MT], with a few differences: the "balayage argument" allows to multiply or to divide by $\exp\left(\frac{\lambda^2}{2} g_t\right)$. Moreover, we have obtained here a family of spider martingales which have the same local time at 0 as B_t.

(ii) In fact, the balayage argument also proves, somewhat to my astonishment, that, for every set $(\lambda_1, \dots, \lambda_k)$, of reals, with $\lambda_i \neq 0$, the process:

$$\tilde{S}_t^{<\lambda>} = (\tilde{S}_t^{(1)}; \dots; \tilde{S}_t^{(k)}) \quad , \quad \text{where:}$$

$$\tilde{S}_t^{(i)} = \frac{\sinh(\lambda_i B_t^{(i)})}{\lambda_i} \exp\left(-\frac{\lambda_i^2}{2}(t-g_t)\right) \tag{17}$$

defines a spider-martingale.

Proof: From the balayage formula, for every i, $1 \leq i \leq k$, the process $(\tilde{S}^{(i)}_t)$ is a submartingale with increasing process:

$$\int_0^t dA_s \cosh(\lambda_i B^{(i)}_s) = A_t \qquad \square$$

Beware: On the other hand, it is not true that, for different λ_i's, the vector valued process:

$$(18) \qquad S^{(i)}_t = \frac{\sinh(\lambda_i B^{(i)}_t)}{\lambda_i} \exp\left(-\frac{\lambda_i^2 t}{2}\right) \quad ; \; i = 1, 2, \dots, k$$

is a spider-martingale; for a fixed i, $(S^{(i)}_t)$ is still a submartingale, but its increasing process is:

$$\int_0^t dA_s \exp\left(-\frac{\lambda_i^2 s}{2}\right), \text{ which depends on } i, \text{ unless } \lambda_1 = \lambda_2 = \dots = \lambda_k$$

Remark: By taking $\lambda_1 = \lambda_2 = \dots = \lambda_p = 0$, $p < k$, we have created, with (17), for $i \geq p$ spider-martingales which differ from B, also they have the same first p legs; therefore, this solves the original question posed in this Section.

(iii) More generally, easy modifications of the arguments in my work with J. Azéma on: Martingales of BM which vanish on the zero set of BM seem to give us all the possible spider-martingales which evolve on the same i^{th} ray at the same time B does; they are given by:

$$S_t = (S^{(1)}_t, \dots, S^{(k)}_t), \quad \text{with:}$$

$$(19) \qquad S^{(i)}_t \equiv S^{(i)}_t[z; u^{(i)}] = z_{g_t} \exp\left(\int_{g_t}^t u^{(i)}_s (d\hat{B}^{(i)}_s) - \frac{1}{2}\int_{g_t}^t (u^{(i)}_s)^2 ds\right) B^{(i)}_t$$

where $(z_s, s \geq 0)$ and $(u^{(i)}_s; s \geq 0)$ are previsible processes, with $z_s > 0$, and $\quad d\hat{B}_s^{(i)} = dB_s^{(i)} - \dfrac{ds}{B_s^{(i)}} \equiv d|B_s| - \dfrac{ds}{|B_s|}$, since here,

we can consider B only on the random set $\Gamma^{(i)} = \{(t,\omega) ;\ B_t^{(i)}(\omega) > 0\}$.

The local time at 0 of S is given by : $\int_0^t \gamma_s \, dA_s$, where A is the local time at 0 of B. We may call (19) <u>the balayage representation of S</u>.

<u>Exercise:</u> Give the explicit balayage representation of the spidermartingales which have been constructed in the above paragraphs.