[10 Février 1994]

1)

Some computations relative to Brownian motion with drift.

Consider $(X_t, t \geq 0)$ a Brownian motion with constant drift $\lambda > 0$, starting from 0, and let $T_a = \inf\{t : X_t = a\}$, $a > 0$.

For notational convenience, we will take (X_t) to be the coordinate process on the canonical space $C(\mathbb{R}_+, \mathbb{R})$, and we denote its law by P^λ.
We show the following

Proposition: 1) The distribution of T_a under P^λ is given by:

$$P^\lambda(T_a \in dt) = \frac{dt\ a}{\sqrt{2\pi t^3}} \exp\left(\lambda a - \frac{\lambda^2 t}{2} - \frac{a^2}{2t}\right).$$

Consequently, one has: $\quad E^\lambda\left[\exp\left(\frac{\lambda^2 T_a}{2}\right)\right] = \exp(\lambda a)$.

2) For every $z \geq 0$, one has:

$$E^\lambda\left[\int_0^{T_a} ds\, \exp(z X_s)\right] = \frac{\exp(za) - 1}{z\lambda + \frac{1}{2} z^2} \qquad (z \geq 0).$$

3) For every $n \in \mathbb{N}$, one has

$$E^\lambda\left[\int_0^{T_a} ds\, X_s^n\right] = \sum_{m=0}^{n} \frac{a^{m+1}}{(m+1)!} (-1)^{n-m} \frac{1}{\lambda(2\lambda)^{n-m}}$$

Proof: 1) The first formula follows from the Cameron-Martin relationship between P^λ and P^0, which implies:

$$P^\lambda(T_a \in dt) = \exp\left(\lambda a - \frac{\lambda^2 t}{2}\right) P^0(T_a \in dt),$$

and the result then follows from the well-known formula:

$$P^0(T_a \in dt) = \frac{dt\, a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right)$$

2) Thanks to the integrability properties of T_a under P^λ $(\lambda > 0)$, we can now deduce from Itô's formula, applied to $(\exp(zX_t), t \geq 0)$, that:

$$\exp(za) = 1 + z\lambda\, E^\lambda\left[\int_0^{T_a} ds \exp(zX_s)\right] + \frac{z^2}{2} E^\lambda\left[\int_0^{T_a} ds \exp(zX_s)\right]$$
$$= 1 + \left(z\lambda + \frac{z^2}{2}\right) E^\lambda\left[\int_0^{T_a} ds \exp(zX_s)\right],$$

which proves the second statement of the Proposition.

3) The third statement follows from the second by developing both sides as power series in z $\square$