Some examples of loss of information for Brownian motion

Warwick (11th of April, 1991)

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The aim of this lecture is to exhibit some examples of subfiltrations $(\mathcal{G}_t)$ of the Brownian filtration $(\mathcal{F}_t)$, which retain some Brownian characteristics, and ideally which are themselves Brownian filtrations.

The reason of my interest in this topic is the following

CONJECTURE: A (necessary and) sufficient condition for a filtration $(\mathcal{F}_t)$ to be the natural filtration of a real-valued BM $(B_t, t \geq 0)$ is that there exists a $(\mathcal{F}_t)$ $(\beta_t, t\geq 0)$ such that:

every $(\mathcal{F}_t)$-martingale $(M_t, t\geq 0)$ may be represented as: $M_t = c + \int_0^t \varphi_s \, d\beta_s$, for some predictable process

For the most developed discussion, up to now, of this conjecture, see: Barlow-Pitman-(Sém. XXIII)

To have a good grip on the conjecture, it seems important to have constructed many Brownian subfiltrations of a given Brownian filtration.

There are many such examples, and I shall discuss 3 apparently very different ones.

1. A Gaussian type example

(This is taken from joint work with Th. Jeulin, in Sém. XXIV)

The simplest way to construct some Brownian subfiltration is certainly to remain within the Gaussian space of a given 1-dimensional BM $(B_t, t \geq 0)$, and to look for functions $f : \mathbb{R}_+^2 \to \mathbb{R}$ such that:

$$(1.a) \qquad X_t = \int_0^t f(s,t)\, dB_s$$

is a Brownian motion. Here, we shall look at a very particular case.

Proposition 1. 1) The process $\beta_t = B_t - \int_0^t \frac{ds}{s} B_s \quad (t \geq$

is a 1-dimensional BM.

2) For every $t > 0$, $\mathcal{F}_t^\beta = \sigma\{B_u - \frac{u}{t} B_t ; u \leq t\}$

In particular, for every t, $(\beta_s, s \leq t)$ is independent of the variable B_t, hence of $(B_u, u \geq t)$.

We arrived to this example thanks to the enlargement of the original filtration BM with the variable B_1:

$$(1.b) \qquad B_t = \gamma_t + \int_0^{t \wedge 1} ds \frac{B_1 - B_s}{1-s},$$

and then we reverse time from $t = 1$.

Although, for each time $t > 0$, we have: $\mathcal{F}_t^\beta \subsetneq \mathcal{F}_t^B$, nonetheless the identity $\mathcal{F}_\infty^\beta = \mathcal{F}_\infty^B$ holds, as is shown in particular by the fol

Proposition 2. Consider all solutions of

$$X_t = \beta_t + \int_0^t \frac{ds}{s} X_s$$

They are given by

$$X_t = t \left(Y - \int_t^\infty \frac{d\beta_u}{u} \right) \quad \text{where } Y = \lim$$

In particular, $X_t = B_t$ satisfies $\frac{B_t}{t} = -\int_t^\infty \frac{dB_u}{u}$.

We shall come back later to Proposition 2;
for the moment, we consider the ergodic properties of the application:

$$\Omega_* \equiv C(\mathbb{R}_+, \mathbb{R}) \longrightarrow \Omega_*$$

$$T: \quad (X_t, t \geq 0) \longrightarrow \left(T(X)_t \equiv X_t - \int_0^t \frac{ds}{s} X_s, t \geq 0\right)$$

As we have seen above, $T(W) = W$, where W is the Wiener measure.
We can easily show the following ergodic properties

Theorem 1: For every t, $\bigcap_n (T^n)^{-1}(\mathcal{F}_t)$ is W-trivial,

and, therefore, the application $T: (\Omega_*, W) \longrightarrow (\Omega_*, W)$ is strongly mixing.

Proof: It is not difficult to show the following identity

$$T^n(X)_t = \int_0^t L_n\left(\log \frac{t}{s}\right) dX_s,$$

where $(L_n, n \in \mathbb{N})$ is the sequence of Laguerre polynomials.
Then, all we need to show is that $\bigcap_n G_n$ is trivial, where G_n is the Gaussian space generated by $(T^n(B)_s, s \leq 1)$.
However, this space is orthogonal to all the variables $\int_0^1 dB_s L_n(\ldots$
for $n \in \mathbb{N}$, hence it is trivial. □

We now come back to Proposition 2, which shall give a key link between the transformation T, and the family of space time harmonic functions.

First of all, we introduce the set $\mathcal{I}$ of all probabilities P on $\Omega_* \equiv C(\mathbb{R}_+, \ldots$
where $X_t(\omega) = \omega(t)$, $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$, such that:

(i) $\left(\tilde{X}_t = X_t - \int_0^t \frac{ds}{s} X_s, t \geq 0\right)$ is a real-valued B…

(ii) $\forall t$, X_t is P independent of $\sigma\{\tilde{X}_s, s \leq t\}$

Then, we have:

Theorem 2: The following properties are equivalent:

1) $P \in \mathcal{J}$;
2) P is the law of: $(B_t + Yt,\ t \geq 0)$; where Y is a random variable which is independent of $(B_t,\ t \geq 0)$.
3) There exists a function $h: \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+$ which is space-time harmo[nic]
(i.e. $h(t, X_t)$ is a W-martingale),
and such that:

$$P = W^h, \quad \text{where}: \quad W^h_{|\mathcal{F}_t} = h(t, X_t) \cdot W_{|\mathcal{F}_t}$$

2. Tsirel'son's equation

We now come back to the CONJECTURE, which, of course, may be wrong in its fu[ll] generality; however, I feel that in the following subcase it is certainly right:

Consider: $$Q_{|\mathcal{F}_t} \equiv \exp\left(\int_0^t f(s,\omega)\, dX_s - \frac{1}{2}\int_0^t f^2(s,\omega)\, ds\right) \cdot W_{|\mathcal{F}_t}$$

for some predictable and bounded process f.
Under Q, $X^f_t \equiv X_t - \int_0^t ds\, f(s,\omega)$ is a BM;
however, it may well be that,

(2.a) $$\mathcal{F}^f_t \subsetneq \mathcal{F}_t$$

Nonetheless, it is easy to show and it is well-known that, under $Q \equiv W^f$, the BM $(X^f_t,\ t \geq 0)$ possesses the Brownian representation property with resp[ect to] $(\mathcal{F}_t)$. Hence, if the conjecture holds true, there exists a BM $(Y_t, t \geq 0)$ under Q such that $\sigma\{Y_s, s \leq t\} = \mathcal{F}_t$.

My feeling for the truth of the CONJECTURE in this case is that, because P and Q are locally equivalent, $(\mathcal{F}_t)$ has the "same characteristics" under P and under Q.

I shall now develop Tsirel'son's example, for which (2.a) holds.

Tsirel'son [ ] defined $f = f_*$ as follows:

$$f_*(s,\omega) = \sum_{k \in -\mathbb{N}} \left\{ \frac{\omega(t_k) - \omega(t_{k-1})}{t_k - t_{k-1}} \right\} 1_{]t_k, t_{k+1}]}(s)$$

where $t_k \downarrow 0$, as $k \to -\infty$, and $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

Then, we have the following

Theorem 3: 1) Under Q, for any k, $\left\{ \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right\}$ is independent of the Q-Brownian motion X^{f_*}, and uniformly distributed on

2) $\bigcap_{\varepsilon > 0} \mathcal{F}_\varepsilon$ is trivial under P and Q, but nonetheless:

$$\forall \varepsilon < t, \quad \mathcal{F}_t = \mathcal{F}_\varepsilon \vee \mathcal{F}_t^{f_*} \quad (\not\Rightarrow \; \mathcal{F}_t = \mathcal{F}_t^{f_*})$$

From the comments in Rogers-Williams, I was driven to think about the role of Brownian motion in this question.

~~Thus~~, Now, you may remark that fundamentally, the problem is a discrete time one, since, if we write, $\beta_t = X_t^{f_*}$, we have:

$$X_t - X_s = (\beta_t - \beta_s) + \int_s^t du\, f_*(u,\omega)$$

Hence,

$$\frac{X_{t_{k+1}} - X_{t_k}}{t_{k+1} - t_k} = \left(\frac{\beta_{t_{k+1}} - \beta_{t_k}}{t_{k+1} - t_k} \right) + \left\{ \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right\}$$

We may then consider a more general equation

$$(2.b) \qquad X_k = Y_k + \{ X_{k-1} \}$$

where the $(Y_k, k \in -\mathbb{N})$ has given laws μ_k

We furthermore assume that: Y_k is independent of $\mathcal{F}_{k-1} = \sigma\{X_n, n \le k-$

In particular, the Y_k's are independent.

Let $\mu = (\mu_k ; k \in -\mathbb{N})$, and define:

$$\mathcal{P}_\mu = \left\{ P \text{ on } \mathbb{R}^{-\mathbb{N}} \ni : \forall k,\ X_k - \{X_{k-1}\} \text{ has law } \mu_k, \text{ and is independent of } \mathcal{F}_{k-1} \right\}$$

We also introduce the notation $\mathcal{E}_k = \sigma\{Y_j ; j \le k\}$

The following statement establishes the existence of at least one solution for any sequ

Theorem 4. There exists a unique probability P^*_μ in $\mathcal{P}_\mu$ such that

(*) under P^*_μ, for any $k \in -\mathbb{N}$, $\{X_k\}$ is uniformly distributed on $[0,1]$

Then, in this case, for any $k \in -\mathbb{N}$, $\{X_k\}$ is independent of $\mathcal{E}_\infty$

Now, we are most interested in the problem of uniqueness in law, in this case,

we have: $\mathcal{P}_\mu = \{P^*_\mu\}$.

The solution to this problem, and, in fact, the description of all possible ca

shall appear clearly once we have stated the following

Lemma: Define $\varphi_j(p) = E\left[\exp(2i\pi p Y_j)\right]$ $(p \in \mathbb{Z}$

Call $\mathbb{Z}_+(\mu) = \{ p \in \mathbb{Z} \ni : \exists k$ sufficiently small with: $\prod_{j=-\infty}^{k} |\varphi_j(p)|$

Then, $\mathbb{Z}_+(\mu)$ is a subgroup of $\mathbb{Z}$ for the addition,

hence, there exists $p_\mu \in \mathbb{N}$ such that: $\mathbb{Z}_+(\mu) = p_\mu \mathbb{Z}$

We now have the following

Theorem 5: We shall say that a solution P $(\in \mathcal{P}_\mu)$ is strong if

$\mathcal{F}_k = \mathcal{E}_k$, for every k. (here and below, all notions are in relation with the relevant $P (\in \mathcal{P}_\mu)$)

Then:

1) There is uniqueness in law if and only if $p_\mu = 0$.

In this case, $\mathcal{P}_\mu = \{P^*_\mu\}$, P^*_μ is a weak solution since

$\mathcal{F}_{-\infty}$ is trivial, $\mathcal{F}_k = \mathcal{E}_k \vee \sigma(\{X_k\}) \equiv \mathcal{E}_k \vee \sigma(\{X_j\})$, for any $j \leq k$

and $\{X_j\}$ is independent from $\mathcal{E}_\infty$.

2) There exists some strong solution if and only if $p_\mu = 1$.

- In this case, there are several strong solutions;
- For any k, $\{X_k\}$ is measurable with respect to $\mathcal{F}_{-\infty} \vee \mathcal{E}_k$
- For any $P \in \mathcal{P}_\mu$, $\mathcal{F}_k = \mathcal{F}_{-\infty} \vee \mathcal{E}_k$.

3) There is not uniqueness in law and there exist no strong solution if and only if

$p_\mu = p > 1$.

In this case, for any $P \in \mathcal{P}_\mu$, $[p\{X_k\}]$ is uniformly distributed on $(0, 1, \ldots$

it is independent of $\mathcal{F}_{-\infty} \vee \mathcal{E}_k$, and

$$\mathcal{F}_k = \mathcal{F}_{-\infty} \vee \mathcal{E}_k \vee \sigma\left([p\{X_k\}]\right)$$

There are many examples of uniqueness in law, as we see from the following

Corollary. 1) There is uniqueness in law for $\mathcal{P}_\mu$ iff

$$(C_0) \quad \forall k \in -\mathbb{N}, \ \forall p \in \mathbb{Z}^*, \quad \prod_{j=-\infty}^{k} |\varphi_j(p)| = 0$$

2) Assume that, $\quad Y_j \overset{(law)}{=} c_j Y$,

where c_j is a constant. Then, if $\mathcal{L}(Y)$ admits a density, and $|c_j| \underset{(j \to \infty)}{\longrightarrow} \infty$,

then: $\varphi_j(p) = \varphi(c_j p) \underset{j \to \infty}{\longrightarrow} 0$, hence, (C_0) is satisfied.

(However, there are many other examples for which (C_0) is satisfied.)

Only a part de page 8

3. Girsanov's transform and principal values.

(Valeurs principales des temps locaux browniens) [Yamada] Ph. Biane, M. Yor

We consider a strictly positive martingale density $(D_t, t \geq 0)$ with respect to the Wiener measure W, that is : Q is a probability on $C(\mathbb{R}_+, \mathbb{R})$ which is locally equivalent to W, with $Q|_{\mathcal{F}_t} = D_t \cdot W|_{\mathcal{F}_t}$.

Then, as we have just seen in the previous paragraph (Tsirel'son's counterexample), there exists some strictly positive Brownian martingale $(D_t, t \geq 0)$ such that the filtration of

$$(3.a) \qquad B^D_t = B_t - \int_0^t \frac{d\langle B, D\rangle_s}{D_s}, \quad t \geq 0,$$

is strictly smaller than that of B.

In this paragraph, we construct some different examples of loss of information from Brownian motion, as follows: now, $(D_t, t \geq 0)$ denotes a Brownian martingale with $D_0 = 0$, $(D_t, t \geq 0) \not\equiv 0$, and such that:

$$(3.b) \qquad \int_0^t \frac{d\langle B, D\rangle_s}{D_s} = \lim_{\varepsilon \downarrow 0} \int_0^t 1_{(|D_s| \geq \varepsilon)} \frac{d\langle B, D\rangle_s}{D_s}$$

exists.

Then, Ruiz de Chavez ([ ]) showed (in fact, in greater generality) that, for any $t > 0$,

$$E\left[D_t \mid B^D_s ; s \leq t\right] = 0,$$

so that the natural filtration of B^D is strictly smaller than that of B.

Hence, in order to get some examples of this situation, we need to find some martingales $(D_t, t \geq 0)$, with $D_0 = 0$, such that (3.b) is satisfied.

The first known example of such a martingale D is $\boxed{D = B}$ (see Biane - [...] for instance), in order to prove (3.b), one writes: — Example 0

$$\int_0^t \frac{ds}{B_s} 1_{(|B_s| \geq \varepsilon)} = \int_\varepsilon^\infty \frac{da}{a}\left(\ell^a_t - \ell^{-a}_t\right),$$

and the Hölder continuity of the Brownian local times $(\ell^a_t, t \geq 0)$ implies the co[...]

Taking some inspiration from this example, we consider martingales $(D_t, t \geq 0)$ which satisfy the following stochastic differential equations:

$$(3.c) \qquad D_t = \int_0^t \frac{dB_s}{f(D_s)},$$

with $f: \mathbb{R} \to \mathbb{R}$, and $|f(x)| \geq \varepsilon > 0$.

We assume a priori that the equation (3.c) admits a unique strong solution (we shall discuss this point later).

Then, we have:

$$\int_0^t \frac{1(|D_s| \geq \varepsilon) \, d\langle B, D\rangle_s}{D_s} = \int_0^t \frac{1(|D_s| \geq \varepsilon) f(D_s) \, d\langle D, D\rangle_s}{D_s} = \int_\varepsilon^\infty \frac{da}{a} \left(f(a) \ell_t^a - f(-a)\ell_t^{-a} \right)$$

where, here, $(\ell_t^a ; a \in \mathbb{R})$ are the local times of $(D_t, t \geq 0)$.

We now write:

$$(3.d) \quad \int_\varepsilon^\infty \frac{da}{a} \left| f(a)\ell_t^a - f(-a)\ell_t^{-a} \right|$$

$$\leq \int_\varepsilon^\infty \frac{da}{a} |f(a) - f(0)| \ell_t^a + \int_\varepsilon^\infty \frac{da}{a} |f(-a) - f(0)| \ell_t^{-a} + |f(0)| \int_\varepsilon^\infty \frac{da}{a} |\ell_t^a - \ell_t^{-a}|$$

and we find that the integral in (3.d) converges, as $\varepsilon \to 0$, under the hypothesis:

$$(3.e) \qquad \int_{-1}^1 \frac{da}{a} |f(a) - f(0)| < \infty.$$

(here, we assume as well that f is locally bounded).

Now, combining condition (3.e) with some other conditions on f which ensure that the equation (3.c) enjoys the pathwise uniqueness property, we obtain the following

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ satisfy the following condition:

$$(3.e) \qquad \int_{-1}^{1} \frac{da}{a} \left| f(a) - f(0) \right| < \infty ,$$

and, moreover, either (A) or (B):

$$(A) \quad \begin{cases} (i) & f(x) \geq \varepsilon \quad \text{for some } \varepsilon > 0 ; \\ (ii) & |f(x) - f(y)|^2 \leq |k(x) - k(y)|, \text{ for some increasing function } k \end{cases}$$

$$(B) \quad \begin{cases} (i) & |f(x)| \geq \varepsilon , \text{ for some } \varepsilon > 0 ; \\ (ii) & |f(x) - f(y)|^2 \leq \rho(|x-y|) , \text{ for some function } \rho :]0,\infty[\to \\ & \text{such that } \int_0 \frac{da}{\rho(a)} = \infty \end{cases}$$

Then, the equation (3.c) admits a unique strong solution $(D_t, t \geq 0)$, and the filtration of B^D is strongly contained in the Brownian filtration.

Proof: That either (A) or (B) implies that the equation (3.c) admits a uniq strong solution is proved (for instance) in Theorem (3.5), p. 360 of Revuz-Yor [ ].

Remark: The condition (3.e) is satisfied:

- under (A), (ii), if we assume that: $\int_{-1}^{1} \frac{da}{a} \sqrt{|k(a) - k(0)|} < \infty$

- under (B), (ii), if $\int_0^1 \frac{da}{a} \sqrt{\rho(a)} < \infty$. $\square$

It is also of some interest to remark that the limit (3.b), as $\varepsilon \to 0$, does not necessarily exist for any Brownian martingale $(D_t, t \geq 0)$, with $D_0 = 0$. Indeed, if $f(0+)$ and $f(0-)$ both exist, but are not equal, then:

$$\int_\varepsilon^\infty \frac{da}{a} \left(f(a) \, \ell_t^a - f(-a) \, \ell_t^{-a} \right) \text{ cannot converge, as } \varepsilon$$

More generally than the solutions of (3.c), we may consider martingales:

$$D_t = \int_0^t \frac{dB_s}{f_s(\omega)},$$

where $(f_s(\omega), s \geq 0)$ is a predictable process with respect to the Brownian filtration, with $\forall s, |f_s(\omega)| \geq \varepsilon > 0$. Then, we have:

$$\int_0^t 1_{(|D_s| \geq \varepsilon)} \frac{d\langle B, D\rangle_s}{D_s} = \int_0^t 1_{(|D_s| \geq \varepsilon)} f_s(\omega) \frac{d\langle D, D\rangle_s}{D_s}$$

$$= \int_\varepsilon^\infty \frac{da}{a} \int_0^t f_s(\omega) (d\ell_s^a - d\ell_s^{-a})$$

In the case where $(f_s(\omega); s \geq 0)$ is a semimartingale, we have:

$$\Lambda_t^a \stackrel{\text{def}}{=} \int_0^t f_s(\omega)\, d\ell_s^a = f_t(\omega)\, \ell_t^a - \int_0^t \ell_s^a(\omega)\, df_s(\omega)$$

and it is not difficult to show that $(\Lambda_t^a ; a \in \mathbb{R})$ is locally Hölder continuous of order $\left(\frac{1}{2} - \varepsilon\right)$, hence the limit (3.b) exists, and the natural filtration of B^D is then strictly smaller than that of B.

Yet another family of martingales $(D_t, t \geq 0)$ is obtained when looking at:

$$(D_t = h(B_t, t) \quad , t \geq 0)$$

where $h: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a space-time harmonic function such that: $h(0,0$

We then remark that:

$$\frac{d\langle B, D\rangle_s}{D_s} = \frac{h'_x(B_s, s)}{h(B_s, s)}\, ds$$

Define: $\varphi(x, s) \equiv \dfrac{h'_x(x, s)}{h(x, s)}$. Then, from the general result of Ruiz

$$\boxed{\text{if } \int_0^t 1_{(|h(B_s,s)|\geq \varepsilon)}\,\varphi(B_s,s)\,ds \text{ converges as } \varepsilon \to 0,}$$ When is this condition satisfied?

then there is loss of information, in that the filtration of

$$B_t^{(\varphi)} = B_t - \int_0^t ds\, \varphi(B_s,s) \qquad (t \geq 0)$$

is strictly smaller than that of B.

Here are two interesting examples:

Example 1: $\quad h_1(x,t) \equiv \sinh(\alpha x)\exp\left(-\frac{\alpha^2 t}{2}\right);$

Then, we have: $\quad \varphi_1(x,t) = \alpha \coth(\alpha x),$

and the convergence hypothesis is satisfied, so that:

$$\boxed{\left(B_t - \alpha\int_0^t \coth(\alpha B_s)\,ds,\ t \geq 0\right)}$$ has a strictly smaller filtration th

Example 2: $\quad h_2(x,t) \equiv \sin(\alpha x)\exp\left(\frac{\alpha^2 t}{2}\right);$

Then, we have: $\quad \varphi_2(x,t) = \alpha \cotg(\alpha x)$, and, again, the convergence hypothesis is satisfied, so that:

$$\boxed{\left(B_t - \alpha\int_0^t \cotg(\alpha B_s)\,ds,\ t \geq 0\right)}$$ has a strictly smaller filtration tha

The above development now raises several interesting questions.

Question 1: Given a function $\varphi(x,t)$, does there exist a space-time harmonic f

h such that: $\quad \boxed{h'_x(x,t) = \varphi(x,t)\,h(x,t)}$

Hence, h satisfies the system:

$$(3\text{-}\beta)\quad \begin{cases} (i) & h'_x(x,t) = \varphi(x,t)\,h(x,t) \\ (ii) & h'_t(x,t) + \frac{1}{2} h''_{x^2}(x,t) = 0. \end{cases}$$

/ We should also note that h satisfies $h(0,0) = 0$ but we consider in gene

<u>Lemma</u>: <u>Let φ be given</u>

<u>A necessary condition that there exists a function h which satisfies (3.f) is th</u>

(3.g) $$\boxed{\varphi'_t + \frac{1}{2}\varphi''_{x^2} + \varphi'_x \varphi = 0.}$$

<u>Proof</u>: From (3.f), (i), we deduce:

$$h''_{x^2}(x,t) = \varphi'_x(x,t)\,h(x,t) + \varphi(x,t)\,h'_x(x,t)$$

$$= \left(\varphi'_x(x,t) + \varphi^2(x,t)\right)h(x,t)$$

Hence, the system (3.f) is equivalent to:

$$(3.f')\quad \begin{cases} (i') & h'_x = \varphi h \\ (ii') & h'_t = -\frac{1}{2}\left(\varphi'_x + \varphi^2\right)h. \end{cases}$$

We now differentiate h'_x with respect to t, resp. h'_t with respect to x; we obtain:

$$h''_{xt} = \varphi'_t h + \varphi h'_t \quad ; \quad h''_{tx} = -\frac{1}{2}\left(\varphi'_x + \varphi^2\right)'_x h - \frac{1}{2}\left(\varphi'_x +\right.$$

Hence, we obtain: $\varphi'_t h = -\frac{1}{2}\left(\varphi'_x + \varphi^2\right)'_x h$,

so that, finally, (3.g) is satisfied $\square$.

We may now ask the following questions:

<u>Question 2</u>: <u>Is the equation (3.g) sufficient to ensure the existence of a space</u> <u>harmonic function h such that (3.f) is satisfied?</u>

<u>Question 3</u>: <u>In the particular case where $\varphi'_t = 0$, which functions φ</u> <u>one obtain from (3.g)?</u>

We now <u>answer partially</u> Question 3. In the case $\varphi'_t = 0$, we deduce from (

that: (3.h) $\quad \varphi' + \varphi^2 = C$, for some constant C

We now remark that the 3 cases considered above:

$$\varphi_0(x) = \frac{1}{x} \quad ; \quad \varphi_1(x) = \alpha \coth(\alpha x) \quad ; \quad \varphi_2(x) = \alpha \cotg(\alpha x)$$

correspond respectively to the constants:

$$C_0 = 0 \quad ; \quad C_1 = \alpha^2 \quad ; \quad C_2 = -\alpha^2.$$

We now show that Question 2 admits a positive answer.

Proposition: If φ satisfies equation (3.g), then there exists a space-time harm[onic] function h such that:

$$(3.f)\ (i) \qquad h'_x = \varphi h \,,$$

that is: h satisfies (3.f).

Proof: From (3.f) (i), we deduce that there exists a function c_t, depending on t, such that:

$$(3.k) \qquad h(x,t) = c_t \exp\left(\int_a^x dy\, \varphi(y,t)\right)$$

We shall then show that, for a suitable choice of c_t, the corresponding function h is space-time harmonic.

From (3.k), we deduce:

$$h'_t = h\left(\int_a^x dy\, \varphi'_t(y,t)\right) + \frac{c'_t}{c_t} h$$

$$h'_x = h\varphi \quad ; \quad h''_{x^2} = h'_x \varphi + h\varphi'_x = h(\varphi^2 + \varphi'_x)$$

Hence, we have:

$$\frac{1}{2} h''_{x^2} + h'_t = h\left[\frac{1}{2}(\varphi^2 + \varphi'_x) + \int_a^x dy\, \varphi'_t(y,t) + \frac{c'_t}{c_t}\right]$$

However, since φ satisfies (3.g), we know that there exists a function $\theta(t)$ such tha[t]

$$\frac{1}{2}(\varphi^2 + \varphi'_x) + \int_a^x dy\, \varphi'_t(y,t) = \theta(t)$$

Hence, finally, to obtain a space-time harmonic function h, it remains to choose $(c$

We are now able to present, at least formally, the example of the 1^{st} paragraph, and the first example of the 3^{rd} paragraph ($\equiv$ Example 0), within the set up of the last Proposition and, in fact, to construct new examples from it.

First of all, remark that:

$$(3.\ell) \qquad h(x,t) \equiv \frac{1}{\sqrt{t}} \exp\left(\frac{x^2}{2t}\right)$$

is a space time harmonic function $(t>0)$.

Then, we have:

$$(3.\ell') \qquad h'_x(x,t) = \frac{x}{t^{3/2}} \exp\left(\frac{x^2}{2t}\right),$$

so that:

$$\varphi(x,t) \equiv \frac{h'_x}{h}(x,t) = \frac{x}{t}.$$

We now remark that:

$$(3.\ell') \qquad k(x,t) \equiv h'_x(x,t) = \frac{x}{t^{3/2}} \exp\left(\frac{x^2}{2t}\right)$$

is also a space-time harmonic function, which satisfies $k(0,t)=0$, so that:

$$(3.m) \qquad \psi(x,t) = \frac{k'_x}{k} = \frac{1}{x} + \frac{x}{t}$$

Hence, it seems very plausible that:

$$B_t - \int_0^t ds \left(\frac{1}{B_s} + \frac{B_s}{s}\right)$$

has a strictly smaller filtration than B, but I do not see how to modi[fy] the previous proofs to show this...

Quelques compléments à l'exposé (4 juin 1991)

a) Une fois que l'on a énoncé la conjecture, il me semble important de rappeler le problème de l'innovation, avec les références,

b) Dans le cas du changement absolument continu de probabilité [paragraphe 2], $(\mathcal{F}_t^f, Q)$ martingale est une $(\mathcal{F}_t, Q)$ martingale, et on a donc le

Lemme : Les deux propriétés suivantes sont équivalentes :

(i) il existe $t<\infty$ tel que : $\mathcal{F}_t^f \subsetneq \mathcal{F}_t$; (aux ensembles $(\mathcal{F}_\infty$, négligeables près)

(ii) $\mathcal{F}_\infty^f \subsetneq \mathcal{F}_\infty$

Ainsi, le problème est uniquement un problème à l'infini, et, en fait, il y a perte d'information, si et seulement si :

$$\sigma(X_t^f ; t\geq 0) \subsetneq \sigma(X_t ; t\geq 0).$$

Serait-il possible de choisir a priori f telle que, sous $Q \equiv W^f$, le processus $(X_t^f, t\geq 0)$ soit indépendant d'une certaine Z choisie elle aussi a priori ?

Dans le même ordre d'idées, remarquons qu'à partir de l'exemple de Tsirel'son, on construire une infinité de solutions avec perte d'information.

Il suffit de prendre pour nouvelle densité :

$$\exp\left(\int_0^t dX_u^f \, \varphi(X_s^f, s\leq u) - \frac{1}{2}\int_0^t du\, \varphi^2(X_s^f ; s\leq$$

Alors le nouveau mouvement brownien est :

$$X_t^f - \int_0^t \varphi(X_s^f ; s\leq u)\, du .$$

c) A propos du paragraphe 3. Pour vérifier si la propriété

$$(3\text{-}b) \quad \lim_{\varepsilon\to 0} \int_0^t 1_{(|D_s|\geq\varepsilon)} \frac{d\langle D, D\rangle_s}{D_s} \quad \text{existe}$$

est satisfait, il me semble intéressant d'introduire la notion de temps locaux relatifs de D (sous-entendu : relatifs à B), c'est-à-dire :
il existe une famille $\left(\ell_t^a(\omega)\right)$, mesurable en a, de processus à variation bornée telle

$$\int_0^t f(D_s(\omega))\, d\langle B, D\rangle_s = \int da\, f(a)\, \ell_t^a(\omega). \qquad (f \text{ borélienne bornée})$$

Démonstration :
On peut toujours écrire : $B_t = \int_0^t h_s\, dD_s + R_t$, décomposition orthogonale de B relativement à D.
On a alors : $\langle B, D\rangle_t = \int_0^t h_s(\omega)\, d\langle D\rangle_s$, et donc

$$\int_0^t f(D_s(\omega))\, d\langle B, D\rangle_s = \int_0^t f(D_s(\omega))\, h_s(\omega)\, d\langle D\rangle_s$$

$$= \int da\, f(a) \left(\int_0^t h_s(\omega)\, d\lambda_s^a(\omega)\right),$$

où (λ_s^a) est la famille bicontinue des temps locaux de D.
Il ne reste plus qu'à poser : $\ell_t^a(\omega) = \int_0^t h_s(\omega)\, d\lambda_s^a(\omega)$,
et on vérifie que : $\int da \int_0^t |h_s(\omega)|\, d\lambda_s^a(\omega) < \infty$, car cette quantité est égale à $\int_0^t |d\langle B, D\rangle_s| < \infty$.

En conséquence, une condition suffisante pour que la propriété (3.b) soit satisfaite est, d'après (*), que l'on ait :

$$(3.b') \qquad \int_0^\infty \frac{da}{a} \left| \ell_t^a - \ell_t^{-a} \right| < \infty,$$

et il s'agit maintenant de faire des hypothèses adéquates sur h (et donc sur D) pour que (3.b') soit satisfaite. □