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1)

Some formulae for the Brownian and Bessel spiders, II

The technique used to obtain the formulae in I is essentially excursion theory [in fact, starting from independent exponential time, and using scaling, to arrive to functionals taken at $T_* = T_{(x_1, \dots, x_k)}$]

Here, I will prove all the formulae in I, and more, using stochastic calculus for the spider - I look only at the Brownian case, the Bessel case being essentially similar.

1. Warm up: A proof of (9)

$$\boxed{P\left(B_{T_*} = x_i\right) = \frac{(1/a_i)}{\sum_\ell (1/a_\ell)}}$$

Define: $B_t^{(i)} = |B_t| 1_{(B_t \in I_i)}$; then, $\left(B_t^{(i)} - \frac{1}{k} L_t \,;\, t \geq 0\right)$ is a martingale. Hence, we have:

$$(9.a) \qquad E\left[B^{(i)}_{T_*}\right] = \frac{1}{k} E\left[L_{T_*}\right].$$

Call C the right-hand side, which does not depend on (i); so, we can write (9.a) in the form:

$$(9.a') \qquad a_i \, P\left(B_{T_*} = x_i\right) = C,$$

and, therefore:

$$(9.b) \qquad P\left(B_{T_*} = x_i\right) = \left(\frac{C}{a_i}\right).$$

To compute C, it remains to sum up the different quantities on the left-hand side of (9.b), and we obtain:

$$1 = C\left(\sum_i \frac{1}{a_i}\right).$$

Hence, $C = 1 \Big/ \sum_i \left(\frac{1}{a_i}\right)$, and, going back to (9.b), we obtain (9).

2. A proof of (7), $(7)_i$ and (10).

I use the same idea, but now with the semimartingale:

$$M_t^{(i)} \overset{\text{def}}{=} \sinh\left(\lambda B_t^{(i)}\right) \exp\left(-\frac{\lambda^2 t}{2}\right)$$

Itô's formula tells us that:

$$M_t^{(i)} = \lambda \int_0^t \cosh\left(\lambda B_s^{(i)}\right) e^{-\frac{\lambda^2 s}{2}} \, dB_s^{(i)} = \text{mart} + \left(\frac{\lambda}{k} \int_0^t e^{-\frac{\lambda^2 s}{2}} \, dL_s\right)$$

Hence, we obtain again:

(7.a) $$E\left[\sinh\left(\lambda B_{T_*}^{(i)}\right) \exp\left(-\frac{\lambda^2}{2} T_*\right)\right] = \frac{\lambda}{k} E\left[\int_0^{T_*} \exp\left(-\frac{\lambda^2 s}{2}\right) dL_s\right]$$

so that we find that the left-hand side does not depend on (i); call this common value C again; we have:

(7.a') $$\sinh(\lambda a_i) \; E\left[\exp\left(-\frac{\lambda^2 T_*}{2}\right) 1_{(B_{T_*} = x_i)}\right] = C$$

Now, in order to compute C, we remark that:

(7.b) $$E\left[\cosh\left(\lambda B_{T_*}\right) \exp\left(-\frac{\lambda^2}{2} T_*\right)\right] = 1.$$

Developing the left-hand side of (7.b), we get:

(7.c) $$\left(\sum_i \coth(\lambda a_i)\right) C = 1,$$

so that, now, we can plug this result in (7.a), and we get:

$(7)_i$ $$\boxed{E\left[\exp\left(-\frac{\lambda^2}{2} T_*\right) 1_{(B_{T_*} = x_i)}\right] = \frac{\left(1 / \sinh(\lambda a_i)\right)}{\sum_\ell \coth(\lambda a_\ell)}}$$

3. Further amplification and proof of (10) and (11).

Let $(z_u\ ;\ u \geq 0)$ be a bounded previsible process, and $g_t = \sup\{s < t : B_s = 0\}$

We have:

$$z_{g_t} M_t^{(i)} = z_{g_t} \left(\sinh(\lambda B_t^{(i)}) \exp - \frac{\lambda^2 t}{2} \right)$$

$$= \int_0^t z_{g_s}\, dM_s^{(i)} \quad \text{, from the balayage formula.}$$

$$= (\text{mart}) + \frac{\lambda}{k} \int_0^t z_s \exp\left(-\frac{\lambda^2 s}{2}\right) dL_s .$$

Hence, we have:

$$(12.a) \qquad E\left[z_{g_{T_*}} \sinh(\lambda B_{T_*}^{(i)})\, e^{-\frac{\lambda^2}{2} T_*} \right] = \frac{\lambda}{k} E\left[\int_0^{T_*} e^{-\frac{\lambda^2}{2} s} z_s\, dL_s \right],$$

and again this common value does not depend on (i).
Let me change z into $\tilde{z}_s = z_s \exp\left(-\frac{\lambda^2}{2} s\right)$, and write: $\tilde{T}_* = T_* - g_{T_*}$.

Now, we can write (12.a) as:

$$(12.a') \qquad E\left[\tilde{z}_{g_{T_*}} \sinh(\lambda a_i)\, e^{\left(-\frac{\lambda^2}{2} \tilde{T}_*\right)} 1_{(B_{T_*} = x_i)} \right] = \frac{\lambda}{k} E\left[\int_0^{T_*} \tilde{z}_s\, dL_s \right]$$

so that:

$$(12.a'') \qquad E\left[\tilde{z}_{g_{T_*}} e^{-\frac{\lambda^2}{2} \tilde{T}_*} 1_{(B_{T_*} = x_i)} \right] = \left(\frac{\lambda}{\sinh(\lambda a_i)} \right) \frac{1}{k} E\left[\int_0^{T_*} \tilde{z}_s\, dL_s \right]$$

From there, we obtain many results all at once; in fact, we see a good fraction of the assertions a) and b), on p. 2) of I.

Reading from (12. a''), we get:

a) $\mathcal{F}_{g_{T_*}}$ and the pair $\{\tilde{T}_*, B_{T_*}\}$ are independent;

b) $$E\left[\exp\left(-\frac{\lambda^2}{2}\tilde{T}_*\right)1(B_{T_*}=x_i)\right]=\frac{\lambda}{\sinh(\lambda a_i)}\left(\frac{1}{k}E[L_{T_*}]\right)$$

$$=\left(\frac{\lambda a_i}{\sinh(\lambda a_i)}\right)\left(\frac{(1/a_i)}{\sum_\ell (1/a_\ell)}\right),$$

from (9.a) and sequel.

Hence, we get both (9) and (10) this way:

(10) $$E\left[\exp\left(-\frac{\lambda^2}{2}\tilde{T}_*\right)\Big| B_{T_*}=x_i\right]=\frac{\lambda a_i}{\sinh(\lambda a_i)}$$

c) $$E\left[\tilde{z}_{g_{T_*}}\right]=\frac{\left\{\frac{1}{k}E\left[\int_0^{T_*}\tilde{z}_s\,dL_s\right]\right\}}{\left\{\frac{1}{k}E[L_{T_*}]\right\}}$$

in other words, the dual previsible projection of $1(g_{T_*}\le t)$ is $\left(\frac{\frac{1}{k}(L_{t\wedge T_*})}{\frac{1}{k}E[L_{T_*}]}\right)$.

[Note that the developments in this third part are sensibly different from those in the 2 first parts]

In particular, there is no exact computation of the constant C .... /.