1^{st} of August, 1993.

Some remarks on random scaling (: 1^{er} essai).

In this Note, we are interested in the following question:
let $(V(t), t\geq 0)$ be a process (valued possibly in $\mathbb{R}^n$, or in an infinite dimensional vector space E), which has the scaling property:
for every $c>0$, $(V(ct), t\geq 0) \overset{(law)}{=} (c\,V(t), t\geq 0)$.
In particular, the distribution of: $\frac{1}{t}V(t)$ does not depend on t.
The question we are interested in is: under which condition on a random variable h, taking values in $\mathbb{R}_+\setminus\{0\}$, does one have:

$$(1)\qquad \frac{1}{t}V(t) \overset{(law)}{=} \frac{1}{h}V(h).$$

Our motivation to study this question comes from the following:
__Example 1.__ Take $V(t) \equiv A^+(t) \equiv \int_0^t ds\, 1_{(B_s>0)}$,

and $h \equiv \tau(u) \equiv \inf\{s:\ \ell_s > u\}$. Then:

$$(2)\qquad \frac{1}{t}A^+(t) \overset{(law)}{=} \frac{1}{\tau(u)}A^+(\tau(u)), \quad \text{for every } t, u>0.$$

In fact, instead of taking simply $V(t) = V_1(t) \equiv A^+(t)$, we can take:
$V(t) = V_2(t) \equiv (A^+(t), \ell_t^2)$ (*),
and we also have:

$$(3)\qquad \frac{1}{t}V_2(t) \overset{(law)}{=} \frac{1}{\tau(u)}V_2(\tau(u)), \quad \text{for every } t, u>0.$$

In this particular example, we would like to be able to replace $\tau(u)$ by a large number of random variables h.

(*) I preferred to restrict myself first to the 2-dimensional process, and a careful inspection of the proofs will show us how the results may be extended to our "full" process $(V(t), t\geq 0)$.

Throughout our discussion, we shall keep the following hypothesis:

Hypothesis ~~3(H)~~ $(\mathcal{H})$ (i) $(H_t, t \geq 0)$ is a continuous increasing process, such that, for every t, H_t is $V(t)$ measurable;

(ii) the pair (H, V) enjoys the scaling property:

$$\text{for every } c>0, \qquad (H_{ct}\,;\, V(ct);\, t \geq 0) \overset{(law)}{=} c\,(H_t; V(t); t \geq 0).$$

We now have the following:

Proposition 1: For every $f : \mathbb{R}_+ \times E \longrightarrow \mathbb{R}_+$, we have:

$$(4) \qquad E\left[\frac{dH_t}{H_t} f(H_t, V(t))\right] = \frac{dt}{t} E\left[f\left(\frac{t}{h_1}, t\,\frac{V(h_1)}{h_1}\right)\right],$$

where $h_u = \inf\{t : H_t > u\}$.

Proof: Consider $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times E \longrightarrow \mathbb{R}_+$; then, we have:

$$E\left[\int_0^\infty \frac{dH_t}{H_t}\varphi(t, H_t, V(t))\right] = E\left[\int_0^\infty \frac{dx}{x}\varphi(h_x, x, V(h_x))\right]$$

(by time-changing)

$$= \int_0^\infty \frac{dx}{x} E\left[\varphi(x h_1, x, x V(h_1))\right]$$

(by scaling).

$$= \int_0^\infty \frac{dt}{t} E\left[\varphi\left(t, \frac{t}{h_1}, t\,\frac{V(h_1)}{h_1}\right)\right]$$

(taking $t = x h_1$).

Taking now $\varphi(t, \alpha, \beta) = g(t) f(\alpha, \beta)$, for a generic g, we obtain (4) $\square$

<u>Corollary</u>: Under our hypothesis $(\mathcal{H})$, the two following properties (5) and (6) are equivalent:

(5) $$\left(H_1, V(1)\right) \overset{(\text{law})}{=} \left(\frac{1}{h_1} ; \frac{V(h_1)}{h_1}\right)$$

(6) $$E\left[dH_t \mid V(t)\right] = \frac{dt}{t} H_t .$$

<u>Proof</u>: (6) $\Rightarrow$ (5): As a consequence of (6) and (4), we get:

$$dt \text{ a.s.}, \qquad E\left[f\left(tH_1, tV(1)\right)\right] = E\left[f\left(\frac{t}{h_1}, t\frac{V(h_1)}{h_1}\right)\right]$$

from which we deduce (5).

(5) $\Rightarrow$ (6): We want to show:

(7) $$E\left[\int_0^\infty \frac{dH_t}{H_t}\varphi(t, V(t))\right] = E\left[\int_0^\infty \frac{dt}{t}\varphi(t, V(t))\right]$$

for every $\varphi : \mathbb{R}_+ \times E \longrightarrow \mathbb{R}_+$.

But, from (4), we know that the left-hand side of (7) is equal to:

$$E\left[\int_0^\infty \frac{dt}{t}\varphi\left(t, t\frac{V(h_1)}{h_1}\right)\right] = E\left[\int_0^\infty \frac{dt}{t}\varphi(t, tV(1))\right], \text{ from (5)}$$

$$= E\left[\int_0^\infty \frac{dt}{t}\varphi(t, V(t))\right],$$

using the scaling property of V. $\square$.

<u>Notation</u>:

If the properties (5) and/or (6) are satisfied, we will say that H is V-admissible, or that V is h-stable.

The following proposition gives a recipe for creating "new" V-admissible processes from "old" ones.

<u>Proposition 2</u>: Assume that n increasing processes $H^1, H^2, \dots, H^n$ satisfy the hypothesis $(\mathcal{H})$ with respect to V, and that moreover, each of them is V-admissible.

Assume moreover that:

for every c,
$$\left(H^1_{ct}, H^2_{ct}, \dots, H^n_{ct}; V(ct); t \geq 0\right)$$
$$\stackrel{(law)}{=} \left(c H^1_t, c H^2_t, \dots, c H^n_t; cV(t); t \geq 0\right)$$

Then, if $f : (\mathbb{R}_+)^n \longrightarrow \mathbb{R}_+$ is a C^1 function, which is homogeneous of order (1), i.e.:

$$(8) \qquad f(cx_1, \dots, cx_n) = c\, f(x_1, \dots, x_n),$$

then, the process:
$$H_t = f(H^1_t, H^2_t, \dots, H^n_t)$$ is V-admissible.

Proof: a) First, the hypothesis made about $(H^1, H^2, \dots, H^n, V)$ in this proposition and f ensures that (H, V) satisfies the hypothesis $(\mathcal{H})$.

b) We now want to prove that H satisfies (6).

We have:
$$dH_t = \sum_{i=1}^{n} f'_i(H^1_t, H^2_t, \dots, H^n_t)\, dH^i_t$$

and so:
$$E\left[dH_t \,|\, V(t)\right] = \sum_{i=1}^{n} f'_i(H^1_t, H^2_t, \dots, H^n_t)\, E\left[dH^i_t \,|\, V(t)\right]$$
$$= \left(\frac{dt}{t}\right)\left(\sum_{i=1}^{n} f'_i(H^1_t, H^2_t, \dots, H^n_t)\, H^i_t\right)$$

since each of the H^i's satisfies (6).

Now, from (8), we deduce that:

$$\sum_{i=1}^{n} f_i'(x_1, \dots, x_n)\, x_i = f(x_1, \dots, x_n),$$

hence, H satisfies (6) $\square$

Before we consider some specific applications, let us prove some ~~more~~ other general facts.

Proposition 3: 1) Let $(H_t, t \geq 0)$ be a continuous increasing process such that: for every $c > 0$, $(H_{ct}; t \geq 0) \overset{(law)}{=} (c H_t; t \geq 0)$

Then, H is H-admissible.

2) Let $(H_t, t \geq 0)$ and $(K_t, t \geq 0)$ be two continuous increasing processes such that:

for every $c > 0$, $(H_{ct}; K_{ct}; t \geq 0) \overset{(law)}{=} (c(H_t, K_t); t \geq 0)$

Define $V(t) = (H_t, K_t)$.

Then, H is V-admissible iff K is V-admissible.

I have difficulties to prove this.

3) Assume that $H_t = \int_0^t ds\, \theta(s)$, and that the process (H, V) satisfies the hypothesis $(\mathcal{H})$. Then, H is V-admissible iff:

$$(9) \qquad dt \text{ a.s.}, \qquad E[\theta(t) \mid V(t)] = \frac{1}{t} H_t.$$

Proof: 1) If $V = H$, and H satisfies the scaling property, then, (5), which now reduces to:

$$H_1 \overset{(law)}{=} \frac{1}{h_1} \quad \text{is satisfied.}$$

2) In this case, since H is V-admissible, we have from (5):

$$(10) \qquad (H_1, K_1) \overset{(law)}{=} \left(\frac{1}{h_1}; \frac{K_{h_1}}{h_1}\right),$$

and we want to prove: (11) $\quad (K_1, H_1) \overset{(law)}{=} \left(\frac{1}{k_1}, \frac{H_{k_1}}{k_1}\right)$.

Remark that (10), resp: (11), is equivalent to:

(10') $\left(H_1, \frac{K_1}{H_1}\right) \overset{(law)}{=} \left(\frac{1}{h_1}, K_{h_1}\right)$; (11') $\left(K_1, \frac{H_1}{K_1}\right) \overset{(law)}{=} \left(\frac{1}{k_1}, H_{k_1}\right)$

3) this is immediate as a consequence of the equivalence of (5) and (6)

Application: Let us consider again Example 1, with the 2-dimensional process $V \equiv V_2$.

a) The identity in law (3) tells us that ℓ^2 is V_2-admissible; hence, from Proposition 3, part 2), A^+ is also V_2-admissible.

[Important note: Even if the proof of Prop. 3, part 2) cannot be completed, or in particular, if this assertion is wrong !!, we can prove the result directly, [see Scaling I] what is less clear for me is that we can also replace V_2 by our original process $V(t) \equiv (V_1(t), V_2(t), \ldots\ldots)$.

b) Define $A^-(t) \equiv t - A^+(t)$; it is a consequence of Proposition 2 that A^- is V_2-admissible. (take: $H^1(t) \equiv t$, $H^2(t) \equiv A^+(t)$, and $f(x,y) = x - y$).

c) As an immediate consequence of the above discussion, we obtain ~~the states~~ the following examples (among many) of random times h for which V_2 is h-~~admissible~~ stable.

$$h = \inf\{t : H_t > 1\}, \qquad \text{with:}$$

(i) $H_t = aA_t^+ + bA_t^- + c\ell_t^2 \quad (a,b,c \geq 0)$; (ii) $H_t = (A_t^+ A_t^- \ell_t^2)^{1/3}$;

(iii) $H_t = (aA_t^+ + bA_t^-)^{1/2}\ell_t$ .....

$\boxed{\text{A negative example}}$. It may also be interesting to give some example of a 2-dimensional process $\left(V(t) \equiv (H(t), K(t)); t \geq 0\right)$ such that (H, V) satisfies $(\mathcal{H})$, but H is not V-admissible.

This is the case with: $H_t = S_t^2$; $K_t = \ell_t^2$.

Indeed, if K were V-admissible, we would have:

(11?) $$\frac{S^2_{\tau(u)}}{\tau(u)} \overset{(law)}{=} S_1^2, \qquad \tau(u) \equiv \inf\{t : \ell_t > u\}$$

or putting both sides upside down:

(12?) $$\frac{\tau(u)}{S^2_{\tau(u)}} \overset{(law)}{=} \frac{1}{S_1^2} \left(\overset{(law)}{=} \tau(1)\right).$$

[Well-known identity]

However, Knight's identity tells us precisely how wrong (12?) is!! Indeed, we have:

$\boxed{\text{Another negative example}}$: $V(t) \equiv (A^+(t),\ S_t^2)$

If $H_t \equiv S_t^2$ were V-admissible, we would have:

$$(13?) \qquad \frac{A^+(T_a)}{T_a} \overset{(law)}{=} A^+(1) \quad (\text{which is arc sine}).$$

or putting both sides upside down:

$$(14?) \qquad \frac{T_a}{A^+(T_a)} \overset{(law)}{=} \frac{1}{A^+(1)} \overset{(law)}{\equiv} (1+C^2).$$

$\boxed{\text{A sufficient condition for } H \text{ to be } V\text{-admissible}}$

Suppose that : $V = (H, W)$, where W is independent of H, and (H, W) enjoys the scaling property. ($\Longleftrightarrow (H, V)$ satisfies $(\mathcal{H})$).
Then, certainly:

$$(H_1, V(1)) \overset{(law)}{=} \left(\frac{1}{h_1}, \frac{V(h_1)}{h_1}\right)$$

<u>An interesting question seems to be</u>: Suppose that, under the general hypothesis $(\mathcal{H})$, H is V-admissible, then, does there exist a <u>skew-product decomposition</u> $V \equiv (H, W)$ of V with respect to H??

$\boxed{\text{On two-dimensional processes } (V(t) \equiv (H(t), K(t));\ t \geq 0)}$.

Assume that V satisfies the scaling property, and that H (and, therefore (?) K, by Proposition 3, 2)), is V-admissible.
Then, <u>is it</u> true that <u>every process</u> L <u>such that</u> (L, V) <u>satisfies</u> $(\mathcal{H})$ <u>is V-admissible</u>??

August 3^{rd}, 1993.

Some remarks on random scaling (: 2^{nd} essai).

Below, I show that Proposition 1 of [July 29^{th}], and Prop 1 of [Aug. 1^{st}] can be assembled together in order to give some better understanding of the laws of P^h of scaled Browmian motion:

$$\left(\frac{1}{\sqrt{h}} B_{sh} \, ; \, s \leq 1 \right) ,$$

where: $h \equiv h_1 \equiv \inf \{ u : H_u > 1 \}$, and $(H_t, t \geq 0)$ is a process which scales jointly with Brownian motion; more precisely:

$$(1) \qquad (H_{ct}, B_{ct} \, ; \, t \geq 0) \overset{(law)}{=} (c H_t, \sqrt{c} B_t \, ; \, t \geq 0) .$$

First, from the identity (2.c.2), the distribution P^h, which is given by:

$$E^h \left[F(X_s \, ; \, s \leq 1) \right] \overset{def}{=} E \left[F \left(\frac{1}{\sqrt{h}} B_{sh} \, ; \, s \leq 1 \right) \right]$$

satisfies:

$$(2) \qquad \left(\frac{dt}{t} \right) E^h \left[F(X_s, s \leq 1) \right] = E \left[\frac{dH_t}{H_t} F \left(\frac{1}{\sqrt{t}} B_{st} \, ; \, s \leq 1 \right) \right]$$

Now, let us assume furthermore that, for every t, H_t is $V(t)$ measurable, or, even better: $\mathcal{G}_t$ measurable, and that:

$$(3) \qquad E \left[dH_t \mid \mathcal{G}_t \right] = \frac{dt}{t} H_t .$$

Then, we deduce from (2) that: $\qquad (4) \quad P^h_{|\mathcal{G}_1} = P_{|\mathcal{G}_1} .$

We know that the hypothesis made on H is satisfied by:

$$H^+ = A^+ , \quad H^- = A^- , \text{ and } \quad H^\# = \ell^2 ,$$

and I shall denote the corresponding probabilities P^h by:

$$P^+ , P^- , \text{ and } P^\# .$$

Now, let us consider:

$$(5) \qquad H_t = f(A_t^+, A_t^-, \ell_t^2) ,$$

where f is an increasing continuous function in (x, y, z), which satisfies:

$$f(cx, cy, cz) = c f(x, y, z) .$$

Then, we have the following

<u>Theorem:</u> If <u>H is defined by</u> (5), <u>then</u>:

$$(4) \qquad P^h|_{\mathcal{G}_1} = P|_{\mathcal{G}_1} \quad \underline{\text{holds}} , \quad \underline{\text{and, furthermore}}:$$

$$(6) \qquad \boxed{P^h(\cdot \,|\, \mathcal{G}_1) = \alpha P^+(\cdot \,|\, \mathcal{G}_1) + \beta P^-(\cdot \,|\, \mathcal{G}_1) + \gamma P^\#(\cdot \,|\, \mathcal{G}_1)}$$

<u>where</u>: $\alpha = \dfrac{f_x'(A_1^+, A_1^-, \ell_1^2) A_1^+}{f(A_1^+, A_1^-, \ell_1^2)}$, $\beta = \dfrac{f_y'(A_1^+, A_1^-, \ell_1^2) A_1^-}{f(A_1^+, A_1^-, \ell_1^2)}$, $\gamma = \dfrac{f_z'(A_1^+, A_1^-, \ell_1^2) \ell_1^2}{f(A_1^+, A_1^-, \ell_1^2)}$

(Sequel to: 2nd essai / Aug. 3rd).

Put slightly differently, we have:

$$(7) \qquad \boxed{P^h = \alpha\cdot P^+ + \beta\cdot P^- + \gamma\cdot P^\#},$$

where, if δ is a ≥ 0 random variable, and Q a probability, $\delta\cdot Q$ indicates the measure: $\Gamma \longrightarrow \int_\Gamma \delta\, dQ$.

Comments: 1) P^+, P^- and $P^\#$ are carried by disjoint sets, namely: $(X_1>0)$, $(X_1<0)$, $X_1=0$.

2) As a particular case, we can write P, the Wiener measure in terms of P^+ and P^-:

$$(8) \qquad \boxed{P = A_1^+\cdot P^+ + A_1^-\cdot P^-}.$$

Note that, although it is also obviously true that: $(9) \quad P = A_1^+\cdot P + A_1^-\cdot P$,

~~it is not true that:~~ nonetheless we have: $A_1^+\cdot P \neq A_1^+\cdot P^+$, and $A_1^-\cdot P \neq A_1^-\cdot P^-$

To see (8) quickly, we should write, instead of (9):

$$P = 1_{(B_1>0)}\cdot P + 1_{(B_1<0)}\cdot P,$$

and then:

$$P(\cdot\,|\,\mathcal{G}_1) = P(B_1>0\,|\,\mathcal{G}_1)\,\frac{P((B_1>0)\cap\cdot\,|\,\mathcal{G}_1)}{P(B_1>0\,|\,\mathcal{G}_1)} + \left(\text{same with } <0 \text{ instead of } >0\right)$$

and we know that:

$$P(B_1\in\mathbb{R}^\pm\,|\,\mathcal{G}_1) = A_1^\pm, \quad \text{and} \quad P^\pm(\cdot\,|\,\mathcal{G}_1) = \frac{P((B_1\in\mathbb{R}^\pm)\cap\cdot\,|\,\mathcal{G}_1)}{A_1^\pm}$$

3) As a second particular case, we may assume the function f to be of the form:

$$f(x,y,z) \equiv \tilde{f}(x+y,z),$$

that is: $$H_t \equiv f(A_t^+, A_t^-, \ell_t^2) = \tilde{f}(t, \ell_t^2).$$

Now, since $\tilde{f}$ is homogeneous of degree 1, we have:

$$\tilde{f}(t,z) = t\,\varphi\left(\frac{z}{t}\right), \text{ for a certain function } \varphi,$$

and, therefore: $$\tilde{f}'_t(t,z) = \varphi\left(\frac{z}{t}\right) - \left(\frac{z}{t}\right)\varphi'\left(\frac{z}{t}\right).$$

From (6), (7) and (8), we obtain:

$$(9)\quad P^h(\cdot \mid \mathcal{G}_1) = \left[1 - (\ell_1^2)\left(\frac{\varphi'}{\varphi}\right)(\ell_1^2)\right] P(\cdot \mid \mathcal{G}_1) + (\ell_1^2)\left(\frac{\varphi'}{\varphi}\right)(\ell_1^2) P^{\#}(\cdot \mid \mathcal{G}_1)$$

It may be a better idea to write:

$$\tilde{f}(t,z) = z\,\psi\left(\frac{t}{z}\right), \quad \text{so that: } \tilde{f}'_t(t,z) = \psi'\left(\frac{t}{z}\right)$$

Now, ~~...~~ formula (9) can be written in the form:

$$(9')\quad P^h(\cdot \mid \mathcal{G}_1) = \left(\frac{\psi'}{\psi}\right)\left(\frac{1}{\ell_1^2}\right) \cdot P(\cdot \mid \mathcal{G}_1) + \left[1 - \left(\frac{\psi'}{\psi}\right)\left(\frac{1}{\ell_1^2}\right)\right] \cdot P^{\#}(\cdot \mid \mathcal{G}_1)$$

It may be interesting to look for a function ψ such that $g \equiv \sup\{t<1 : X_t = 0\}$ has a given law: $\gamma(t)\,dt$, maybe under P^h
(at least, $\gamma(t)\,dt$ would be the absolutely continuous component of the law of g)

Now, since, under P, we have: $\ell_1^{2} \overset{(law)}{=} g \cdot (2T)$,

with g arc sine distributed, and T exponential, we can write: for every $u : [0,1] \to \mathbb{R}_+$,

$$\int_0^1 dt\, \gamma(t)\, \cancel{f(t)}\, u(t) = \int_0^1 \frac{dt\, u(t)}{\pi\sqrt{t(1-t)}} \int_0^\infty dx\, e^{-x} \frac{\psi'}{\psi}\left(\frac{1}{2tx}\right), \text{ so that:}$$

(*)
$$\boxed{\gamma(t) = \frac{1}{\pi\sqrt{t(1-t)}} \int_0^\infty dx\, e^{-x} \left(\frac{\psi'}{\psi}\right)\left(\frac{1}{2tx}\right)}$$

$$= \frac{1}{2\pi\sqrt{t^3(1-t)}} \int_0^\infty dy\, e^{-(y/2t)} \left(\frac{\psi'}{\psi}\right)\left(\frac{1}{y}\right).$$

$$t = 1/u \implies \frac{1}{u^2}\gamma\left(\frac{1}{u}\right) = \frac{1}{2\pi\sqrt{u-1}} \int_0^\infty dy\, e^{-\left(\frac{yu}{2}\right)} \left(\frac{\psi'}{\psi}\right)\left(\frac{1}{y}\right).$$

$u = v+1$ /

$$2\pi \frac{\sqrt{v}}{(v+1)^2}\gamma\left(\frac{1}{v+1}\right) = \int_0^\infty dy\, \exp\left(-\frac{y}{2}(v+1)\right)\left(\frac{\psi'}{\psi}\left(\frac{1}{y}\right)\right).$$

(*) At least, starting from this formula, with a function ψ such that: $0 \le \frac{\psi'}{\psi} \le 1$, we obtain a number of distributions for g under P^h; may be look at some particular cases:

$$\left(\frac{\psi'}{\psi}\right)(\xi) = \frac{1}{1+\xi^\alpha}, \ldots\ldots /$$

Some remarks on random scaling (: 3^e essai). August 4th, 1993.

This is a succession of comments on the 1st essai [1st Aug.].

1. Comments on Proposition 1 and its Corollary.

Here, I remark that the identity (5) or (6) amounts to some "weak" Markov property for the triple:

$$\left(V(t), H_t, dH_t \right),$$

where: $V(t)$ plays the role of the entire past,
H_t ——— the present, (in particular, H_t is measurable with respect to $V(t)$)
dH_t ——— the future

The Corollary on p. 3 of [1st essai] may be presented as follows:

Proposition: Under the hypothesis ($\mathcal{H}$), the property:

$$(6) \qquad E\left[dH_t \mid V(t) \right] = \frac{dt}{t} H_t$$

is equivalent to:

$$(6') \qquad E\left[dH_t \mid V(t) \right] = E\left[dH_t \mid H_t \right].$$

Proof: Obviously, (6) implies (6'), and, ~~consequently~~ conversely, if (6') is satisfied, then the scaling property of the process $(H_t, t \geq 0)$ together with the general identity (4) implies:

$$E\left[\frac{dH_t}{H_t} f(H_t) \right] = \frac{dt}{t} E\left[f\left(\frac{t}{h_1} \right) \right] = \frac{dt}{t} E\left[f(H_t) \right]$$

so that, we always have: $E\left[dH_t \mid H_t\right] = \frac{dt}{t} H_t$.

Consequently, (6') implies (6) □

<u>Particular cases</u>: a) Assume that ($\mathcal{H}$) is satisfied, and that $H_t = \int_0^t ds\, \theta(s)$. Then, <u>(6), or (6')</u>, <u>is satisfied iff</u>:

(9') dt a.s., $\boxed{E\left[\theta(t) \mid V(t)\right] = E\left[\theta(t) \mid H_t\right]}$.

(and, again, we know a priori that: $E\left[\theta(t) \mid H_t\right] = \frac{1}{t} H_t$ is true under the scaling hypothesis for the process $(H_t, t \geq 0)$).

b) If, even more particularly, $\theta(t)$ is the indicator of a {random} set, which is the case in the arc sine study, then (9') is equivalent to:

(9'') dt a.s., $\boxed{\text{the triple } (V(t), H_t, \theta_t) \text{ is Markovian.}}$

<u>Remark</u>: The above characterization shows that the skew product idea on p. 8 of [1st essai] is somewhat naive, although the <u>scaling property</u> (of V) jointly with the <u>Markov property</u> (9'') may lead to <u>some independence result</u>; see, e.g., <u>Lamperti's representation of semistable Markov processes</u>.

c) Of course, we may also write (9'') in the following equivalent way: for every measurable f,

(9''') . $\boxed{E\left[f(V(t)) \mid H_t, \theta_t\right] = E\left[f(V(t)) \mid H_t\right]}$.

i.e., $V(t)$ and θ_t are independent conditionally on H_t.

This suggests the following questions:

Question 1: In the Brownian case, how difficult, or how simple is it to prove that:

$$P(B_t > 0 \mid V(t)) = P(B_t > 0 \mid A^+_t),$$

or, equivalently:

$$\boxed{E\left[f(V(t)) \mid A^+_t, (B_t > 0)\right] = E\left[f(V(t)) \mid A^+_t\right]}$$

(A simple answer to Question 1 should shed some light on analogous pb. for perturbed reflecting BM).

Question 2:

The preceding identity suggests strongly that one studies the distribution of $(B_u; u \leq t)$, given either the pair: $(A^+_t, (B_t > 0))$ or, only: A^+_t, so that:

(i) one may recover the above identity;

(ii) one may find some "maximal" σ-fields $\mathcal{G}^{max}$ such that:

$\mathcal{G}^{max}$ and $(B_1 > 0)$ are independent conditionally on A^+_1.

2. A closer look at the two "negative" examples.

(2.a) $\boxed{V(t) = (S_t^2 ; \ell_t^2)}$; (2.b) $\boxed{V(t) = (A^+(t), S_t^2)}$

(2.b) Introducing the notation $T_a = \inf\{t : B_t = a\}$, I will now ~~show~~ show that:

(13') $$\frac{A^+(T_1)}{T_1} \overset{(law)}{\neq} A^+(1),$$

or equivalently:

(14') $$\frac{T_1}{A^+(T_1)} \overset{(law)}{\neq} \frac{1}{A^+(1)} \left(\overset{(law)}{=} (1 + C^2) \right)$$

where C is a standard Cauchy variable.

Indeed, we have:

$$\frac{T_1}{A^+(T_1)} = 1 + \frac{A^-(T_1)}{A^+(T_1)},$$

and, in order to prove (14'), I need to show:

(15) $$\frac{A^-(T_1)}{A^+(T_1)} \overset{(law)}{\neq} C^2$$