$\boxed{\text{Some results on the Brownian spider}}$

[Exposé fait à Berkeley, le 24 Aout 199~~4~~3]

Introduction / Some motivations for the study of the Brownian spider.

- To get a better understanding of some still intriguing identities in law in the Brownian case, which are closely related to theta functions (hence : possibly develop multidimensional theta functions);

- The filtration of the Brownian spider has the one-dimensional representation property ; is it a Brownian filtation?
  If so, it has a very strange property at $g_1 = \sup\{s < 1 : B_s = 0\}$

- The state space of this Markov process is very simple in comparison with current studies on graphs, fractals, ~~a~~ trees, and so on, and it seems that a clear understanding may help ~~a~~ in turn to understand the behavior of these more complicated processes on trees, ...

Aug. 24th, 1993.

A summary of results on the Brownian spider.

0. Some notations.

$\left(B_t = (B_t^{(1)}, \dots, B_t^{(k)}); t \geq 0\right)$ shall denote the (k-legged) Brownian spider, or: Walsh's Brownian motion.

We denote:

$$T_t^{(i)} = \int_0^t ds\, 1_{(B_s \in I_i)} \equiv \int_0^t ds\, 1_{(B_s^{(i)} > 0)} .$$

$$s_t^{(i)} = \sup_{s \leq t} \left(B_s^{(i)}\right) ; \qquad s_0^{(i)} = \sup_{s \leq 1} \left(b^{(i)}(s)\right),$$

if $(b(s), s \leq 1)$ denotes the pinned spider.

$(\ell_t; t \geq 0)$ is the local time of $(B_t; t \geq 0)$ at 0; in order to avoid any confusion, recall that $(|B_t|, t \geq 0)$ is a reflecting Brownian motion, and $(\ell_t, t \geq 0)$ is chosen such that: $(|B_t| - \ell_t, t \geq 0)$ is a martingale.

Finally, we define: $\tau_\ell = \inf\{u : \ell_u > \ell\}$, $\ell \geq 0$, the inverse of $(\ell_u, u \geq 0)$

1. The joint law of the times spent in the k rays.

Barlow-Pitman-Yor (1989) obtained the following joint law, thus extending Paul Lévy's arc sine distribution:

$$(1) \quad \frac{1}{t}\left(T_t^{(i)}; i \leq k\right) \overset{(law)}{=} \frac{1}{\tau_\ell}\left(T_{\tau_\ell}^{(i)}; i \leq k\right) \overset{(law)}{=} \left(\frac{\sigma_i}{\sigma_1 + \dots + \sigma_k}; i \leq k\right)$$

for all $t > 0$, and $\ell > 0$,

where $(\sigma_i \,;\, 1 \leq i \leq k)$ denote k independent stable $\left(\frac{1}{2}\right)$ random variables.

The identity in law between fixed times t, and τ_ℓ, motivated further work by Pitman-Yor [ ], and Perman-Pitman-Yor [ ].

2. The joint law of the supremums on the different rays.

Let $\tilde{T}$ denote an $\exp\left(\frac{1}{2}\right)$ random variable, which is independent of $(B_t \,;\, t \geq 0)$.
It is not difficult, using excursion theory, to show:

$$(2) \quad P\left(s^{(1)}_{g_{\tilde{T}}} \leq a_1 \,; \cdots\cdots \,; s^{(k)}_{g_{\tilde{T}}} \leq a_k\right) = \frac{k}{\sum_{i=1}^{k} \coth(a_i)}$$

$$(3) \quad P\left(s^{(1)}_{\tilde{T}} \leq a_1 \,; \cdots\cdots \,; s^{(k)}_{\tilde{T}} \leq a_k\right) = 1 - \frac{1}{\left(\sum_{i=1}^{k} \coth(a_i)\right)} \sum_{i=1}^{k} \frac{1}{(\sinh(a_i))}$$

Proof: Use: $E\left[F(B_u; u \leq g_{\tilde{T}})\right] = \int_0^\infty d\ell \; E\left[F(B_u; u \leq \tau_\ell) \exp\left(-\frac{\tau_\ell}{2}\right)\right]$,

and: $$\boxed{E\left[\exp\left(-\frac{\lambda^2}{2}\tau_\ell\right) 1_{\left(\sup_{u \leq \tau_\ell} |B_u| \leq a\right)}\right] = \exp-\left(\lambda \ell \coth(\lambda a)\right)}$$

We may then write the left-hand side of (2), resp: (3), as:

$$(2') \quad P\left(|N|\, s_0^{(1)} \leq a_1 \,; \cdots\cdots \,; |N|\, s_0^{(k)} \leq a_k\right)$$

$$(3') \quad P\left(\sqrt{\tilde{T}}\, s_1^{(1)} \leq a_1 \,; \cdots\cdots \,; \sqrt{\tilde{T}}\, s_1^{(k)} \leq a_k\right)$$

This leads easily to some beautiful, but not so well understood identities, for $k = 2$:

$$\left(\rho_0^{(1)} + \rho_0^{(2)}\right)^2 \overset{(law)}{=} m_0^2 + \tilde{m}_0^2$$

$$\rho_1^{(1)} + \rho_1^{(2)} \overset{(law)}{=} \int_0^1 \frac{ds}{R_s}$$

3. A reinterpretation of formula (3).

The left-hand side of (3) may also be written as:

$$P\left(\tilde{T} \leq T_{\{x_1,\dots,x_k\}}\right) = 1 - P\left(\tilde{T} \geq T_{\{x_1,\dots,x_k\}}\right).$$

Hence, from formula (3), we get:

$$E\left[\exp\left(-\frac{1}{2} T_{\{x_1,\dots,x_k\}}\right)\right] = \frac{\sum_j \frac{1}{\sinh(a_j)}}{\sum_j \coth(a_j)},$$

and this formula can, by scaling, be readily amplified into:

$$(4) \qquad E\left[\exp -\frac{\lambda^2}{2} T_{\{x_1,\dots,x_k\}}\right] = \frac{\sum_j \frac{1}{\sinh(\lambda a_j)}}{\sum_j \coth(\lambda a_j)}.$$

Introduce the notation: $T_* = T_{\{x_1,\dots,x_k\}}$.

In fact, it is not difficult to refine (4) as:

$$(4)_j \qquad E\left[\exp\left(-\frac{\lambda^2}{2} T_*\right) ; B_{T_*} = x_j\right] = \frac{\frac{1}{\sinh(\lambda a_j)}}{\sum_m \coth(\lambda a_m)}.$$

In particular, letting λ go to 0, we get:

$$(5)_j \qquad P\left(B_{T_*} = x_j\right) = \frac{1/a_j}{\sum_m (1/a_m)}$$

Now, looking jointly at $(4)_j$ and $(5)_j$, we may again ~~$\$$~~ get a better understanding of them by ~~j~~ remarking that:

$(B_t\,;\, t \le g_{T_*})$ and $(B_{g_{T_*}+u}\,;\, u \geq 0)$ are independent,

and, moreover, given $B_{T_*} = x_j$, $(B_{g_{T_*}+u}\,;\, u \le T_* - g_{T_*})$ is a

3D Bessel process which is independent of $(B_t\,;\, t \le g_{T_*})$.

[Introduce the notation: $T_*^{(-)} = g_{T_*}$; $T_*^{(+)} = T_* - g_{T_*}$.]

In fact, formulae $(4)_j$ and $(5)_j$ may be refined into:

$$(6) \qquad E\left[\exp\left(-\frac{\lambda^2}{2} T_*^{(-)}\right)\right] = \frac{\sum_m \frac{1}{(\cancel{\lambda} a_m)}}{\sum_m \coth(\lambda a_m)}$$

$$(7) \qquad E\left[\exp\left(-\frac{\lambda^2}{2} T_*^{(+)}\right) \Big| B_{T_*} = x_j\right] = \frac{\lambda a_j}{\sinh(\lambda a_j)} \; ; \; P(B_{T_*} = x_j) = \frac{1/a_j}{\sum_m (1/a_m)}$$

4. The definition and some stability properties of spider-martingales.

Let k be an integer, and $[\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P]$ be a filtered probability space.

Definition:

A process $\{M_t = (M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)}); t \geq 0\}$ taking values in $(\mathbb{R}_+)^k$, and such that:

a) $M_0 = 0$;

b) (M_t) has continuous trajectories, and is $(\mathcal{F}_t)$ adapted;

c) the $(k+1)$ random sets:

$$\Gamma_0 = \{(t,\omega);\ \forall j,\ M_t^{(j)}(\omega) = 0\};\quad \Gamma_i = \{(t,\omega);\ M_t^{(i)}(\omega) \neq 0\},\ i = 1, 2, \ldots, k$$

constitute a partition of $\mathbb{R}_+ \times \Omega$; in other words, for any given (t,ω), there is at most one of the $M_t^{(i)}(\omega)$ which is not 0.

d) for every $t \geq 0$, the family $\{|M_\tau|;\ \tau \text{ stopping time},\ \tau \leq t\}$ is uniformly integrable;

e) [Fundamental property] For every bounded stopping time τ, the k positive reals:

$$\{E[M_\tau^{(i)}],\ i = 1, 2, \ldots, k\}$$ are equal.

will be called a spider-martingale.

Before we give examples, we show some equivalent properties to e).

Proposition 1: Assume that $(M_t, t \geq 0)$ satisfies a), b), c), d).
Then, the following properties are equivalent:

(i) M satisfies the fundamental property e);

(ii) for every (i,j), $i \neq j$, $\left(M_t^{(i,j)} \overset{\text{def}}{=} M_t^{(i)} - M_t^{(j)}, t \geq 0\right)$ is a martingale;

(iii) there exists an increasing process $(A_t, t \geq 0)$ which is carried by $\Gamma_0 = \{(t,\omega) : \forall j, M_t^{(j)}(\omega) = 0\}$, and k martingales $\{(N_t^{(j)}, t \geq 0) ; j = 1, 2, \dots, k\}$ such that: $M_t^{(i)} = N_t^{(i)} + A_t$.

Proof: (i) $\Longrightarrow$ (ii), since a martingale may be characterized by:
$E[X_\tau] = Cte$, τ a bd. stopping time

(ii) $\Longrightarrow$ (iii).
We remark that: $M_t^{(i)} = (M_t^{(i,j)})^+$, and, from Tanaka's formula:

$$(8) \qquad (M_t^{(i,j)})^+ = \int_0^t 1_{(M_s^{(i,j)} > 0)}\, dM_s^{(i,j)} + \frac{1}{2} L_t^{(i,j)},$$

where $L_t^{(i,j)}$ is the local time of $(M_t^{(i,j)})$ at 0.

Note that (8) is the canonical decomposition of $(M_t^{(i)})$ as a submartingale; more precisely, we can write:

$$M_t^{(i)} = N_t^{(i)} + A_t^{(i)},$$

where:

$$N_t^{(i)} = \int_0^t 1_{(M_s^{(i)} > 0)}\, dM_s^{(i)} ; \quad A_t^{(i)} = \int_0^t 1_{(M_s^{(i)} = 0)}\, dM_s^{(i)} \equiv \frac{1}{2} L_t^{(i,j)}$$

Since $L_t^{(i,j)} = L_t^{(j,i)}$, we have obtained: $A_t^{(i)} = A_t^{(j)}$, and finally, we have proved (iii).

(iii) $\Longrightarrow$ (i) $\qquad E\left[M_\tau^{(i)}\right] = E\left[A_\tau\right]$ does not depend on i.

Notation: $\langle M^{(i)}\rangle_t$ is the increasing process of $M^{(i)}$, or of $N^{(i)}$;

$$\langle M\rangle_t = \sum_{i=1}^{k} \langle M^{(i)}\rangle_t ; \quad \text{note that: } \langle M^{(i)}\rangle_t = \int_0^t 1_{(M_s^{(i)}>0)}\, d\langle M\rangle_s .$$

Important comments: 1. The notion of a spider-martingale generalizes the decomposition of a real-valued martingale $M_t \equiv (M_t^+, M_t^-)$ into its positive and negative parts.

2. Walsh's Brownian motion is the prototype of spider-martingales, just as Brownian motion is the prototype of continuous martingales. Indeed, we have

Proposition 2: [Dubins-Schwarz representation].

Let $(M_t, t\geq 0)$ be a spider-martingale such that:

(*) for every i, $\langle M^{(i)}\rangle_\infty = \infty$, a.s.

Then, there exists a Walsh Brownian motion $(W(u), u\geq 0)$ such that

$$M_t = W(\langle M\rangle_t), \quad t\geq 0.$$

Note: From now on, we shall always assume that (*) is satisfied, so that $(M_t, t\geq 0)$ will eventually reach any point x_i on the i^{th} ray I_i.

Application of the fundamental property: Let M be a spider martingale. If we define: $T_* \equiv T_{\{x_1,\dots,x_k\}}(M)$, then, we have:

$$(9) \qquad P(M_{T_*} = x_i) = \frac{1/a_i}{\sum_m (1/a_m)}$$

Proof: (i) We know, from e), that:

$$E\left[M^{(i)}_{T_*} \right] = C, \text{ a constant which does not depend on } i;$$

we have, obviously: (i') $C = a_i P(M_{T_*} = x_i)$,

hence, $P(M_{T_*} = x_i) = \frac{C}{a_i}$, and, summing over i, we get:

(ii) $$1 = C\left(\sum_m \frac{1}{a_m}\right)$$

so, C is determined, and plugging the value of C in (i'), we obtain (9) □

We shall soon see that the same principle allows us to recover all the formulae in paragraphs 2 and 3.

5. Some stability properties of spider-martingales.

Given a spider-martingale, we can create from it (or associate with it) many martingales, and/or spider-martingales.

Proposition 3: Let $(M_t, t \geq 0)$ be a spider-martingale.

Let $f: \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a space-time harmonic function, i.e:

$$\frac{1}{2} f''_{x^2} + f'_t = 0.$$

Then: (i) If f satisfies ~~the~~ (H'_0) $f'_x(0,t) \equiv 0$,

then, for every i, $\left(f(M^{(i)}_t, \langle M^{(i)} \rangle_t), t \geq 0\right)$ is a local martingale, and these different local martingales are orthogonal;

(ii) If f satisfies: (H_0) $\quad f(x,t) \equiv 0$ iff $x = 0$,

then: $F = \left(F_t^{(i)} \equiv f\left(M_t^{(i)}; \langle M \rangle_t\right); t \geq 0\right); i = 1, 2, \dots, k)$ is a spider-martingale

Beware: In (i), the increasing process we consider is $\langle M^{(i)} \rangle$, while in (ii), we consider $\langle M \rangle$

Comments: 1. There are many possible extensions of this Proposition, with different functions f_i, and so on.

2. f satisfies (H_0') iff it is of the form:

$$f(x,t) = \int_{0+}^{\infty} \mu(d\lambda) \cosh(\lambda x) \exp\left(-\frac{\lambda^2 t}{2}\right).$$

3. f satisfies (H_0) iff it is of the form:

$$f(x,t) = \int_{0+}^{\infty} \mu(d\lambda) \sinh(\lambda x) \exp\left(-\frac{\lambda^2 t}{2}\right).$$

Examples and Applications For simplicity, we consider here $(B_t, t \geq 0)$ Walsh's Brownian motion. Then:

(i) for every $\lambda_1, \lambda_2, \dots, \lambda_k$, the process:

$$C_t \left(= C_t^{\langle\lambda\rangle}\right) \stackrel{def}{=} \prod_{j=1}^{k} \cosh\left(\lambda_j B_t^{(j)}\right) \exp\left(-\frac{\lambda_j^2}{2} T_t^{(j)}\right)$$

is a martingale;

(ii) for every $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the process: $S_t^{(\lambda)} = \left(S_t^{(\lambda),i}; i = 1, 2, \dots, k\right)$ defined by:

$$\left(S_t^{(\lambda),i} = \sinh\left(\lambda B_t^{(i)}\right) \exp\left(-\frac{\lambda^2 t}{2}\right), t \geq 0\right)$$

is a spider martingale.

These two assertions are immediate consequences of Proposition 3.
In order to exhibit even more symmetry between (i) and (ii) above, and to obtain some more general formulae below than in paragraphs 2 and 3 above, we shall prove that, if $\lambda_1, \dots, \lambda_k$ are k reals, $\lambda_i \neq 0$, then:

$$\left(S_t^{\langle\lambda\rangle, i} \overset{\text{def}}{=} \frac{\sinh(\lambda_i B_t^{(i)})}{\lambda_i} \exp\left(-\frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 T_t^{(j)}\right) ; i=1,2,\dots,k \right)$$

is a spider-martingale.
This is proven as for Proposition 3 (using essentially Itô's formula), after remarking ~~that~~ the identity:

$$S_t^{\langle\lambda\rangle, i} = \prod_{j\neq i}\left(\cosh(\lambda_j B_t^{(j)}) \exp\left(-\frac{\lambda_j^2}{2} T_t^{(j)}\right)\right)\left(\frac{\sinh(\lambda_i B_t^{(i)})}{\lambda_i} \exp\left(-\frac{\lambda_i^2}{2} T_t^{(i)}\right)\right)$$

We may now deduce from this the following refinements of formulae (4) and (6)

$$(4') \qquad E\left[\exp\left(-\frac{1}{2}\sum_j \lambda_j^2 T_*^{(j)}\right)\right] = \frac{\sum_j \dfrac{\lambda_j}{\sinh(\lambda_j a_j)}}{\sum_j \lambda_j \coth(\lambda_j a_j)}$$

$$(4')_i \qquad E\left[\exp\left(-\frac{1}{2}\sum_j \lambda_j^2 T_*^{(j)}\right) 1_{(B_{T_*} = a_i)}\right] = \frac{\lambda_i / \sinh(\lambda_i a_i)}{\left(\sum_j \lambda_j \coth(\lambda_j a_j)\right)}$$

$$(6') \qquad E\left[\exp\left(-\frac{1}{2}\sum_j \lambda_j^2 T_*^{(j),-}\right)\right] = \frac{\sum_j \left(\dfrac{1}{a_j}\right)}{\sum_j \lambda_j \coth(\lambda_j a_j)}$$

Proof of (4'): Since $(S_t^{\langle\lambda\rangle}, t \geq 0)$ is a spider-martingale, we have:

$$(i) \qquad E\left[S_{T_*}^{\langle\lambda\rangle, i}\right] = \xi, \text{ a constant which does not depend on } i.$$

This identity may be written as:

$$(i') \qquad \frac{\sinh(\lambda_i a_i)}{\lambda_i} E\left[\exp\left(-\frac{1}{2} T_*^{\langle\lambda\rangle}\right) 1_{(B_{T_*} = x_i)}\right] = \xi$$

Now, since $(C_t^{\langle\lambda\rangle}, t \geq 0)$ is a martingale, we also get:

$$E\left[C_{T_*}^{\langle\lambda\rangle}\right] = 1, \quad \text{that is:}$$

$$(ii) \qquad \sum_{i=1}^{k} \cosh(\lambda_i a_i) \, E\left[\exp\left(-\frac{1}{2} T_*^{\langle\lambda\rangle}\right) 1_{(B_{T_*} = x_i)}\right] = 1$$

Multiplying and dividing the i^{th} expectation which appears in (ii) by: $\dfrac{\sinh(\lambda_i a_i)}{\lambda_i}$,

we obtain, using (i') and (ii) jointly:

$$\left(\sum_{i=1}^{k} \lambda_i \coth(\lambda_i a_i)\right) \xi = 1 \quad ,$$

so that, we have determined the value of ξ, and now, plugging its value into (i'), we obtain $(4')_i$, and (4') follows □

To prove (6'), we may either refer to the independence properties already alluded to, or prove them directly using stochastic calculus and, in particular, the balayage formula.

In our context, this has the following consequence:

if $(z_t, t \geq 0)$ is a bounded previsible process, then:

$$\left(z_{g_t} \sinh(\lambda B_t^{(i)}) \exp\left(-\frac{\lambda^2}{2} T_t^{(i)}\right) ; t \geq 0\right)_{i=1,2,\dots,k}$$

is again a spider martingale. [To be developed].

6. Spider-martingales and symmetric functions of k variables.

Consider Walsh's Brownian motion, and for an integer p, define:

$$(\ )\qquad \theta_p(a_1, \dots, a_k) = E\left[(T_*)^p \right].$$

This function of $a_1, \dots, a_k$ is symmetric in the k variables $(a_1, \dots, a_k)$, and has the homogeneity property:

$$(\ )\qquad \theta_p(\lambda a_1, \dots, \lambda a_k) = \lambda^{2p}\, \theta_p(a_1, \dots, a_k).$$

Similarly, consider: $(\ )\quad \beta_p(a_1, \dots, a_k) = E\left[|B_{T_*}|^p \right]$,

another symmetric function of k variables $(a_1, \dots, a_k)$, which has the homogeneity property:

$$(\ )\qquad \beta_p(\lambda a_1, \dots, \lambda a_k) = \lambda^p\, \beta_p(a_1, \dots, a_k).$$

We would like to find some explicit expressions for θ_p and β_p.
Thanks to formula $(5)_j$, β_p is readily computed:

$$\beta_p(a_1, \dots, a_k) = \left(\sum_{j=1}^{k} a_j^{p-1} \right) \frac{1}{\sum_{j=1}^{k} (1/a_j)}.$$

Obtaining an explicit formula for θ_p is more complicated, but may be done thanks to the <u>double recurrence</u> which will be developed below; to establish this double recurrence, we make two remarks:

(i) If we write $T_* = T_*^{(-)} + T_*^{(+)}$, where $T_*^{(-)} = g_{T_*}$, and if we use the independence of $T_*^{(-)}$ and $T_*^{(+)}$, we obtain:

$$E\left[(T_*)^p\right] = \sum_{n=0}^{p} \binom{p}{n} E\left[(T_*^{(-)})^n\right] E\left[(T_*^{(+)})^{p-n}\right].$$

Hence, with obvious notation, we find that:

$$(\ ) \qquad \theta_p = \sum_{n=0}^{p} \binom{p}{n} \theta_n^{(-)} \theta_{p-n}^{(+)} .$$

The computation of $\theta_m^{(+)}$ for a given integer m is easy, since we have:

$$\begin{aligned}
\theta_m^{(+)}(a_1, \dots, a_k) &= \sum_{j=1}^{k} E\left[(T_*^{(+)})^m 1_{(B_{T_*} = x_j)}\right] \\
&= \sum_{j=1}^{k} E\left((T_{a_j}^{(3)})^m\right) P(B_{T_*} = x_j) \\
&= \sum_{j=1}^{k} a_j^{2m} c_m^{(3)} \frac{(1/a_j)}{\sum_{\ell=1}^{k} (1/a_\ell)} \\
&= c_m^{(3)} \frac{\left(\sum_{j=1}^{k} a_j^{2m-1}\right)}{\sum_{\ell=1}^{k} (1/a_\ell)} .
\end{aligned}$$

Therefore, from formula ( ), the computation of θ_p amounts to that of $(\theta_n^{(-)} ; n \leq p)$

(ii) In fact, we shall compute the sequence $(\theta_p, p \in \mathbb{N})$, jointly with $(\theta_p^{(-)}, p \in \mathbb{N})$ with the help of the Hermite polynomials of even degrees $(H_{2p}(x,t), p \in \mathbb{N})$, and we shall use the two properties:

$$H_{2p}(x,t) = x^{2p} + \alpha_{p,1}\, t\, x^{2(p-1)} + \alpha_{p,2}\, t^2 x^{2(p-2)} + \dots\dots + \alpha_{p,p}\, t^p$$

and $$E\left[H_{2p}\left(|B_{T_*}|, T_*\right)\right] = 0\,, \qquad p \geq 1\,.$$

Hence, from the last equality, we see that, in order to compute θ_p, we need to know:, for $q = 0, 1, \cdots, (p-1)$:

$$\theta_{q,p}(a_1, \dots, a_k) \stackrel{\text{def}}{=} E\left[(T_*)^{2q}\, |B_{T_*}|^{2(p-q)} \right].$$

We have :

$$\theta_{q,p}(a_1, \dots, a_k) = \sum_{j=1}^{k} a_j^{2(p-q)}\, E\left[(T_*)^{2q}\, 1_{(B_{T_*} = a_j)} \right],$$

and, finally, using a double recurrence on the sequences $(\theta_p, \theta_p^{(-)})$, we see that we can compute these functions recursively.

[We need to give explicit formulae for $p = 1, 2, 3$, say].