Spider-martingales and symmetric functions of k variables.

This is a development of Section 6 of [L], Aug. 24th.
I keep p. 12 as in [L], but modify the last four lines as follows:

In order to present the underlying idea of this double recurrence, we first compute θ_1 and θ_2.

a) Since $(|B_t|^2 - t, t \geq 0)$ is a martingale, we have:

$$\theta_1(a_1, \dots, a_k) = E(T_*) = E(|B_{T_*}|^2) = \beta_2(a_1, \dots, a_k) = \frac{\left(\sum_j a_j\right)}{\sum_j (1/a_j)}$$

b) In order to compute θ_2, we look for a martingale of the form:

$$(1) \qquad |B_t|^4 + a\,t\,|B_t|^2 + b\,t^2 .$$

Since, on one hand: $|B_t|^4 = \text{mart.} + 6\int_0^t ds\,|B_s|^2$,

and, on the other hand: $t\,|B_t|^2 = \text{mart.} + \frac{t^2}{2} + \int_0^t ds\,|B_s|^2$,

we see that the process in (1) is a martingale iff:

$$6\int_0^t ds\,|B_s|^2 + a\int_0^t ds\,|B_s|^2 = 0, \text{ and: } \frac{a}{2} + b = 0, \text{ which gives: } a = -6\,;\ b = 3 .$$

Hence, we have obtained: $(1') \quad \left(|B_t|^4 - 6t\,|B_t|^2 + 3t^2;\ t \geq 0\right)$ is a martingale.
From this, we deduce:

$$E\left(|B_{T_*}|^4\right) - 6\,E\left(T_*\,|B_{T_*}|^2\right) + 3\,E\left((T_*)^2\right) = 0$$

Hence, the computation of θ_2 is reduced to that of:

In fact, we shall compute jointly the sequences $(\theta_p, p \in \mathbb{N})$ and $(\theta_p^{(-)}, p \in \mathbb{N})$, ~~whi~~ with the help of the Hermite polynomials of even degrees $(H_{2p}(x;t)\,;\, p \in \mathbb{N})$, which may be characterized as follows

Lemma: For any $p \in \mathbb{N}$, $p \geq 1$, there exists exactly one p-tuple of reals $(\alpha_{p,1}, \alpha_{p,2}, \dots, \alpha_{p,p})$ such that, if:

$$H_{2p}(x,t) \stackrel{def}{=} x^{2p} + \alpha_{p,1}\, t\, x^{2(p-1)} + \dots + \alpha_{p,p}\, t^{p},$$

then: $(H_{2p}(|B_t|, t), t \geq 0)$ is a martingale.

Proof: Immediate using Itô's formula, from which $\alpha_{p,1}$ is readily determined, then $\alpha_{p,2}, \dots$, and finally $\alpha_{p,p}$ $\square$

Note: A more general presentation of certain martingales of the form: $h(X_t, t)$, for a general Markov process (X_t) is given in Appendix C. //

Now, to proceed with the double recurrence, we use:

$$E\left[H_{2p}(|B_{T_*}|, T_*)\right] = 0, \quad p \geq 1$$

(to be continued as on p. 14 of [L]).